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Homogeneous Projective Coordinates for the Bondi-Metzner-Sachs Group

Giampiero Esposito [1,2] and Giuseppe Filiberto Vitale [1]

[1] Dipartimento di Fisica "Ettore Pancini", Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, 80126 Napoli, Italy; giuseppef.vitale@gmail.com
[2] INFN Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, 80126 Napoli, Italy
* Correspondence: gesposito@na.infn.it

Abstract: This paper studies the Bondi-Metzner-Sachs group in homogeneous projective coordinates, because it is then possible to write all transformations of such a group in a manifestly linear way. The 2-sphere metric, Bondi-Metzner-Sachs metric, asymptotic Killing vectors, generators of supertranslations, as well as boosts and rotations of Minkowski spacetime, are all re-expressed in homogeneous projective coordinates. Last, the integral curves of vector fields which generate supertranslations are evaluated in detail. This work prepares the ground for more advanced applications of the differential geometry of asymptotically flat spacetimes in projective coordinates.

Keywords: vector fields; Bondi-Metzner-Sachs group; homogeneous projective coordinates

1. Introduction

The Bondi-Metzner-Sachs [1–3] asymptotic symmetry group of asymptotically flat spacetime has received again much attention over the last decade by virtue of its relevance for black-hole physics [4–6], the group-theoretical structure of general relativity [7–20] and the infrared structure of fundamental interactions [21–24]. The appropriate geometric framework can be summarized as follows. In spacetime models for which null infinity can be defined, the cuts of null infinity are spacelike two-surfaces orthogonal to the generators of null infinity [25]. On using the familiar stereographic coordinate

$$\zeta = e^{i\varphi} \cot \frac{\theta}{2}, \tag{1}$$

the first half of Bondi-Metzner-Sachs transformations read as

$$\zeta' = f(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = f_\Lambda(\zeta), \tag{2}$$

where the matrix $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has unit determinant $(ad - bc) = 1$ and belongs therefore to the group $\mathrm{SL}(2,\mathbb{C})$. The resulting projective version of the special linear group can be defined as the space of pairs

$$\mathrm{PSL}(2,\mathbb{C}) = \{(f,\Lambda)|\ f : \zeta \in \mathbb{C} \rightarrow f_\Lambda(\zeta), \Lambda \in \mathrm{SL}(2,\mathbb{C})\}, \tag{3}$$

i.e., the group of fractional linear maps f_Λ according to Eq. (2) with the associated matrix Λ. Since

$$f_\Lambda(\zeta) = \frac{(a\zeta + b)}{(c\zeta + d)} = \frac{(-a\zeta - b)}{(-c\zeta - d)} = f_{-\Lambda}(\zeta), \tag{4}$$

one can write that $\mathrm{PSL}(2,\mathbb{C})$ is the quotient space $\mathrm{SL}(2,\mathbb{C})/\delta$, where δ is the homeomorphism defined by

$$\delta(a,b,c,d) = (-a,-b,-c,-d). \tag{5}$$

The fractional linear maps (2) can be defined for all values of ζ upon requiring that

$$f_\Lambda(\infty) = \frac{a}{c},\ f_\Lambda\left(-\frac{d}{c}\right) = \infty. \tag{6}$$

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Moreover, under fractional linear maps, lengths along the generators of null infinity scale according to

$$du' = K_\Lambda(\zeta) du, \tag{7}$$

where the conformal factor is given by [19,25]

$$K_\Lambda(\zeta) = \frac{1+|\zeta|^2}{|a\zeta+b|^2+|c\zeta+d|^2}. \tag{8}$$

By integration, Eq. (7) yields the second half of Bondi-Metzner-Sachs transformations:

$$u' = K_\Lambda(\zeta)\Big[u+\alpha(\zeta,\overline{\zeta})\Big]. \tag{9}$$

As was pointed out in Ref. [19], the complex homogeneous coordinates associated to the Bondi-Metzner-Sachs transformation (2) have modulus ≤ 1, which is the equation of a unit circle, and are

$$z_0 = e^{\frac{i\varphi}{2}}\cos\frac{\theta}{2},\ z_1 = e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}. \tag{10}$$

In other words, upon remarking that

$$\zeta = \frac{z_0}{z_1}, \tag{11}$$

Eq. (2) is equivalent to the linear transformation law

$$\begin{pmatrix} z_0' \\ z_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix}. \tag{12}$$

The next step of the program initiated in Ref. [19] consists in realizing that, much in the same way as the affine transformations in the Euclidean plane

$$x' = x+a,\ y' = y+b, \tag{13}$$

can be re-expressed with the help of a 3×3 matrix in the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix}, \tag{14}$$

one can further re-express Eq. (12) with the help of a 3×3 matrix in the form

$$\begin{pmatrix} w_0' \\ w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix}, \tag{15}$$

with the understanding that Eq. (12) is the restriction to the unit circle Γ of the map (15), upon defining

$$w_0|_\Gamma = 1,\ w_1|_\Gamma = z_0,\ w_2|_\Gamma = z_1. \tag{16}$$

The work in Ref. [19] has outlined the resulting geometric picture, where (w_0, w_1, w_2) are viewed as homogeneous coordinates in a complex projective plane. In our paper we have instead a less abstract and more concrete task: since the Bondi-Metzner-Sachs transformation (2) becomes linear when expressed in terms of z_0 and z_1, we are aiming to develop the Bondi-Metzner-Sachs formalism with the associated Killing vector fields by using the pair of variables (z_0, z_1) instead of $(\zeta, \overline{\zeta})$. For this purpose, the homogeneous projective coordinates for the 2-sphere are studied in Sect. 2, while the

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Bondi-Sachs metric in homogeneous coordinates is considered in Sect. 3. Asymptotic Killing fields for supertranslations are evaluated in Sect. 4, while their flow is investigated in Sect. 5. Concluding remarks and open problems are presented in Sect. 6, while technical details are provided in the Appendices.

2. Homogeneous Coordinates on the 2-Sphere

It is useful, as an instrument to develop the BMS formalism in homogeneous coordinates, to re-write the 2-sphere metric in the desired coordinates. By using the definition (10), we get

$$z_0 z_1 = \sin(\theta/2)\cos(\theta/2) = \frac{\sin(\theta)}{2} \Rightarrow \theta = \sin^{-1}(2 z_0 z_1), \tag{17}$$

while for φ we obtain

$$\frac{z_0}{z_1} = e^{i\varphi}\cot(\theta/2) \Rightarrow \varphi = -i\,\log\left(\tan(\theta/2)\frac{z_0}{z_1}\right). \tag{18}$$

By virtue of the identity

$$\tan(\theta/2) == \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \frac{\sin(\theta)}{1+\sqrt{1-\sin^2(\theta)}}, \tag{19}$$

we obtain for φ the more convenient expression

$$\varphi = -i\,\log\left(\frac{2 z_0^2}{1+\sqrt{1-4 z_0^2 z_1^2}}\right).$$

In order to re-express the 2-sphere metric, let us evaluate

$$\begin{aligned} d\theta^2 &= \frac{d\theta}{dz_0}\frac{d\theta}{dz_0}dz_0^2 + \frac{d\theta}{dz_1}\frac{d\theta}{dz_1}dz_1^2 + 2\frac{d\theta}{dz_0}\frac{d\theta}{dz_1}dz_0 dz_1 \\ &= \frac{4z_1^2}{1-4z_0^2z_1^2}dz_0^2 + \frac{4z_0^2}{1-4z_0^2z_1^2}dz_1^2 + \frac{8z_0z_1}{1-4z_0^2z_1^2}dz_0dz_1, \end{aligned} \tag{20}$$

while

$$\begin{aligned} \sin^2(\theta)\,d\varphi^2 &= 4z_0^2z_1^2\,d\varphi^2 = 4z_0^2z_1^2\left\{\frac{d\varphi}{dz_0}\frac{d\varphi}{dz_0}dz_0^2 + \frac{d\varphi}{dz_1}\frac{d\varphi}{dz_1}dz_1^2 + 2\frac{d\varphi}{dz_0}\frac{d\varphi}{dz_1}dz_0dz_1\right\} \\ &= \frac{-16z_1^2\left(1-2z_0^2z_1^2+\sqrt{1-4z_0^2z_1^2}\right)^2}{\left(1-4z_0^2z_1^2+\sqrt{1-4z_0^2z_1^2}\right)^2}dz_0^2 - \frac{64z_0^6z_1^4}{\left(1-4z_0^2z_1^2+\sqrt{1-4z_0^2z_1^2}\right)^2}dz_1^2 \\ &- \frac{32z_0^3z_1^3}{1-4z_0^2z_1^2}dz_0dz_1. \end{aligned} \tag{21}$$

Eventually, we obtain the metric for the 2-sphere in homogeneous coordinates

$$\begin{aligned} \Omega_2 &= d\theta^2 + \sin^2(\theta)d\varphi^2 = \sum_{\mu,\nu=0}^{1} g_{\mu\nu}dz^\mu dz^\nu \\ &- 4z_1^2\left(\frac{1-4z_0^2z_1^2+2\sqrt{1-4z_0^2z_1^2}}{1-4z_0^2z_1^2}\right)dz_0^2 + 8z_0z_1dz_0dz_1 \\ &- 4z_0^2\left(\frac{1-4z_0^2z_1^2-2\sqrt{1-4z_0^2z_1^2}}{1-4z_0^2z_1^2}\right)dz_1^2. \end{aligned} \tag{22}$$

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At this stage, upon defining the real-valued function

$$\gamma(z_0, z_1) = \frac{2}{\sqrt{1 - 4z_0^2 z_1^2}} = \frac{2}{\cos\theta}, \tag{23}$$

we can write the matrix of metric components in the form

$$\gamma_{AB} = \begin{pmatrix} -4z_1^2(1+\gamma) & 4z_0 z_1 \\ 4z_0 z_1 & -4z_0^2(1-\gamma) \end{pmatrix}, \tag{24}$$

with non-vanishing determinant $-16z_0^2 z_1^2 \gamma^2$ and inverse matrix

$$\gamma^{AB} = \begin{pmatrix} \frac{1-\gamma}{4z_1^2\gamma^2} & \frac{1}{4z_0 z_1 \gamma^2} \\ \frac{1}{4z_0 z_1 \gamma^2} & \frac{1+\gamma}{4z_0^2\gamma^2} \end{pmatrix}. \tag{25}$$

We can see from (17) that the terms

$$2z_0 z_1 = \sin(\theta) \to 4z_0^2 z_1^2 = \sin^2(\theta) \to 1 - 4z_0^2 z_1^2 = \cos^2(\theta) \to 4z_0 z_1 = 2\sin(\theta),$$

are real-valued, whereas

$$z_0^2 = e^{i\varphi}\cos^2(\theta/2),\ z_1^2 = e^{-i\varphi}\sin^2(\theta/2)$$

are complex.

3. Bondi-Sachs Metric in Homogeneous Coordinates

We can now write the retarded Bondi-Sachs (hereafter BS) metric in homogeneous coordinates with the help of the previous formulae. For this purpose, let us first write the general BS metric in the form

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + h_{AB}\left(dx^A + \frac{1}{2}U^A du\right)\left(dx^B + \frac{1}{2}U^B du\right). \tag{26}$$

On passing from (θ, φ) to (z_0, z_1) coordinates, we find the metric components of (3.1) expressed as follows:

$$g_{uu} = -U + \frac{1}{4}h_{z_0z_0}(U^{z_0})^2 + \frac{1}{4}h_{z_1z_1}(U^{z_1})^2 + \frac{1}{2}h_{z_0z_1}U^{z_0}U^{z_1}, \tag{27}$$

$$g_{ur} = -e^{2\beta}, \tag{28}$$

$$g_{uz_0} = \frac{1}{2}(h_{z_0z_0}U^{z_0} + h_{z_0z_1}U^{z_1}), \tag{29}$$

$$g_{uz_1} = \frac{1}{2}(h_{z_0z_1}U^{z_0} + h_{z_1z_1}U^{z_1}), \tag{30}$$

$$g_{z_0z_0} = h_{z_0z_0},\ g_{z_0z_1} = h_{z_0z_1},\ g_{z_1z_1} = h_{z_1z_1}. \tag{31}$$

The Bondi gauge $\partial_r \det(r^{-2}g_{AB}) = 0$ implies that [26] $\gamma^{AB}C_{AB} = 0$, where γ^{AB} is given in Eq. (2.9). With our coordinates, this relation reads as

$$\gamma^{AB}C_{AB} = 0 \Leftrightarrow g^{z_0z_0}C_{z_0z_0} + g^{z_1z_1}C_{z_1z_1} + 2g^{z_0z_1}C_{z_0z_1} = 0.$$

We no longer have the simple result $C_{z\bar{z}} = 0$ for the mixed component as in the stereographic coordinates, because in homogeneous coordinates we obtain

$$\frac{1-\gamma}{4z_1^2\gamma^2}C_{z_0z_0} + \frac{1+\gamma}{4z_0^2\gamma^2}C_{z_1z_1} + \frac{1}{2z_0z_1\gamma^2}C_{z_0z_1} = 0, \tag{32}$$

which implies that

$$C_{z_0z_1} = -\frac{1}{2}(1-\gamma)\frac{z_0}{z_1}C_{z_0z_0} - \frac{1}{2}(1+\gamma)\frac{z_1}{z_0}C_{z_1z_1}. \tag{33}$$

The angular components of the metric are

$$g_{z_0z_0} = r^2\gamma_{z_0z_0} + rC_{z_0z_0} + \mathcal{O}(r),\ g_{z_1z_1} = r^2\gamma_{z_1z_1} + rC_{z_1z_1} + \mathcal{O}(r),$$

$$\begin{aligned} g_{z_0z_1} &= r^2\gamma_{z_0z_1} + rC_{z_0z_1} + \mathcal{O}(r) \\ &= r^2\gamma_{z_0z_1} - r\left(\frac{z_0}{z_1}\frac{(1-\gamma)}{2}C_{z_0z_0} + \frac{z_1}{z_0}\frac{(1+\gamma)}{2}C_{z_1z_1}\right) + \mathcal{O}(r), \end{aligned}$$

where, of course, γ_{AB} is given in Eq. (2.8). These formulae, jointly with the falloff conditions

$$\begin{aligned} U(u,r,x^A) &= 1 - \frac{2m(u,r,x^A)}{r} + \frac{U_2(u,x^A)}{r^2} + \mathcal{O}(r^{-3}) \\ \beta(u,r,x^A) &= \frac{\beta_1(u,x^A)}{r} + \frac{\beta_2(u,x^A)}{r^2} + \mathcal{O}(r^{-3}) \\ U^A(u,r,x^B) &= \frac{U_2^A(u,x^B)}{r^2} + \frac{U_3^A(u,x^B)}{r^3} + \mathcal{O}(r^{-4}) \\ g_{AB}(u,r,x^A) &= r^2\gamma_{AB}(x^A) + rC_{AB}(u,x^A) + D_{AB}(u,x^A) + \mathcal{O}(r^{-1}), \end{aligned} \tag{34}$$

help to rewrite

$$g_{uu} = -\left(1 - \frac{2m}{r}\right) + \mathcal{O}(r^{-2}). \tag{35}$$

Upon assuming that $\beta_1/r \ll 1$, we get

$$g_{ur} = -\exp\left(\frac{2\beta_1}{r} + \mathcal{O}(r^{-2})\right) = -1 - \frac{2\beta_1}{r} + \mathcal{O}(r^{-2}), \tag{36}$$

while for g_{uz_0} and g_{uz_1} we find

$$\begin{aligned} g_{uz_0} &= \frac{1}{2}\left(r^2\gamma_{z_0z_0} + rC_{z_0z_0}\right)\left(\frac{U_2^{z_0}}{r^2} + \frac{U_3^{z_0}}{r^3}\right) + \frac{1}{2}\left\{r^2\gamma_{z_0z_1} - r\left[\frac{z_0}{z_1}\frac{(1-\gamma)}{2}C_{z_0z_0}\right.\right. \\ &\left.\left. + \frac{z_1}{z_0}\frac{(1+\gamma)}{2}C_{z_1z_1}\right]\right\}\left(\frac{U_2^{z_1}}{r^2} + \frac{U_3^{z_1}}{r^3}\right) \\ &= \frac{\gamma_{z_0z_0}}{2}U_2^{z_0} + \frac{\gamma_{z_0z_1}}{2}U_2^{z_1} + \frac{1}{r}\left[\frac{\gamma_{z_0z_0}}{2}U_3^{z_0} + \frac{C_{z_0z_0}}{2}U_2^{z_0} + \frac{\gamma_{z_0z_1}}{2}U_3^{z_1}\right. \\ &\left. - \frac{z_0}{z_1}\frac{(1-\gamma)}{4}C_{z_0z_0}U_2^{z_1} - \frac{z_1}{z_0}\frac{(1+\gamma)}{4}C_{z_1z_1}U_2^{z_1}\right] + \mathcal{O}(r^{-2}) \end{aligned} \tag{37}$$

and

$$\begin{aligned} g_{uz_1} &= \frac{1}{2}\left(r^2\gamma_{z_1z_1} + rC_{z_1z_1}\right)\left(\frac{U_2^{z_1}}{r^2} + \frac{U_3^{z_1}}{r^3}\right) + \frac{1}{2}\left\{r^2\gamma_{z_0z_1} - r\left[\frac{z_0}{z_1}\frac{(1-\gamma)}{2}C_{z_0z_0}\right.\right. \\ &\left.\left. + \frac{z_1}{z_0}\frac{(1+\gamma)}{2}C_{z_1z_1}\right]\right\}\left(\frac{U_2^{z_0}}{r^2} + \frac{U_3^{z_0}}{r^3}\right) \\ &= \frac{\gamma_{z_1z_1}}{2}U_2^{z_1} + \frac{\gamma_{z_0z_1}}{2}U_2^{z_0} + \frac{1}{r}\left[\frac{\gamma_{z_1z_1}}{2}U_3^{z_1} + \frac{C_{z_1z_1}}{2}U_2^{z_1} + \frac{\gamma_{z_0z_1}}{2}U_3^{z_0}\right. \\ &\left. - \frac{z_0}{z_1}\frac{(1-\gamma)}{4}C_{z_0z_0}U_2^{z_0} - \frac{z_1}{z_0}\frac{(1+\gamma)}{4}C_{z_1z_1}U_2^{z_0}\right] + \mathcal{O}(r^{-2}), \end{aligned} \tag{38}$$

where use has been made of (3.8). Eventually, we get the matrix of Bondi metric components

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2m}{r}\right) & -1-\frac{2\beta_1}{r} & g_{uz_0} & g_{uz_1} \\ -1-\frac{2\beta_1}{r} & 0 & 0 & 0 \\ g_{uz_0} & 0 & r^2\gamma_{z_0z_0}+rC_{z_0z_0} & r^2\gamma_{z_0z_1}+rC_{z_0z_1} \\ g_{uz_1} & 0 & r^2\gamma_{z_0z_1}+rC_{z_0z_1} & r^2\gamma_{z_1z_1}+rC_{z_1z_1} \end{pmatrix} + \mathcal{O}(r^{-2}). \tag{39}$$

The gauge condition $\det(g_{AB}/r^2) = 0$, instead of giving a solution for D_{AB} such as in stereographic coordinates, gives us a condition for C_{AB}

$$\det(g_{AB}) = \det\begin{pmatrix} r^2\gamma_{z_0z_0}+rC_{z_0z_0}+D_{z_0z_0} & r^2\gamma_{z_0z_1}+rC_{z_0z_1}+D_{z_0z_1} \\ r^2\gamma_{z_0z_1}+rC_{z_0z_1}+D_{z_0z_1} & r^2\gamma_{z_1z_1}+rC_{z_1z_1}+D_{z_1z_1} \end{pmatrix}$$

$$\begin{aligned} &= r^4\left(\gamma_{z_0z_0}\gamma_{z_1z_1}-\gamma^2_{z_0z_1}\right)+r^3\left(\gamma_{z_0z_0}C_{z_1z_1}+\gamma_{z_1z_1}C_{z_0z_0}-2r^3\gamma_{z_0z_1}C_{z_0z_1}\right) \\ &+r^2\left(\gamma_{z_0z_0}D_{z_1z_1}+C_{z_0z_0}C_{z_1z_1}+\gamma_{z_1z_1}D_{z_0z_0}-C^2_{z_0z_1}-2\gamma_{z_0z_1}D_{z_0z_1}\right)+\mathcal{O}(r), \end{aligned}$$

$$\begin{aligned} \det\left(\frac{g_{AB}}{r^2}\right) &\Rightarrow \gamma_{z_0z_0}D_{z_1z_1}+C_{z_0z_0}C_{z_1z_1}+\gamma_{z_1z_1}D_{z_0z_0}-C^2_{z_0z_1}-2\gamma_{z_0z_1}D_{z_0z_1} \stackrel{!}{=} 0 \\ &\Longrightarrow D_{z_0z_1} = \frac{\gamma_{z_0z_0}D_{z_1z_1}+C_{z_0z_0}C_{z_1z_1}+\gamma_{z_1z_1}D_{z_0z_0}-C^2_{z_0z_1}}{2\gamma_{z_0z_1}} \\ &C^2_{z_0z_1} = C_{z_0z_0}C_{z_1z_1}. \end{aligned}$$

In order to determine the various coefficients in the falloff conditions, we require that the Bondi metric should satisfy the Einstein equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

Upon restricting to the vacuum case $T = 0$, in the limit as r approaches ∞ in the Einstein tensor, first looking at G_{rr}, and neglecting the terms of order $\mathcal{O}(r^{-4})$, we get

$$G_{rr} = -\frac{4\beta_1}{r^3} + \mathcal{O}(r^{-4}) \stackrel{!}{=} 0 \Rightarrow \beta_1 \equiv 0.$$

Upon looking at G_{rz_0} and G_{rz_1} respectively, we get lengthy relations for $U_2^{z_1}$ and $U_2^{z_0}$, compared to the stereographic coordinates case, which depend on other coefficients. However, we still manage to solve directly for $U_2^{z_0}$ and $U_2^{z_1}$. On studying $G_{rA} = 0$ we find

$$U_2^{z_0} = \frac{2z_0z_1\left(C_{z_1z_1}U_2^{z_1}+\gamma_{z_0z_1}U_3^{z_0}+\gamma_{z_1z_1}U_3^{z_1}\right)}{z_1^2(1+\gamma)C_{z_1z_1}+z_0^2(1-\gamma)C_{z_0z_0}} = -\frac{C_{z_1z_1}U_2^{z_1}+2\gamma_{z_0z_1}U_3^{z_0}}{C_{z_0z_1}}, \tag{40}$$

and

$$U_2^{z_1} = \frac{2z_0z_1\left(C_{z_0z_0}U_2^{z_0}+\gamma_{z_0z_0}U_3^{z_0}+\gamma_{z_0z_1}U_3^{z_1}\right)}{z_1^2(1+\gamma)C_{z_1z_1}+z_0^2(1-\gamma)C_{z_0z_0}} = -\frac{C_{z_0z_0}U_2^{z_0}+2\gamma_{z_0z_1}U_3^{z_1}}{C_{z_0z_1}}, \tag{41}$$

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where we recall that $C_{z_0 z_1}$ is given in Eq. (3.8). By virtue of Eqs. (3.12) and (3.13) we find eventually the metric in the form

$$ds^2 = -du^2 - 2dudr + 2\left(r^2\gamma_{z_0z_1} + rC_{z_0z_1}\right)dz_0dz_1 + \frac{2m}{r}du^2 + \left(r^2\gamma_{z_0z_0} + rC_{z_0z_0}\right)dz_0^2$$
$$+\left(r^2\gamma_{z_1z_1} + rC_{z_1z_1}\right)dz_1^2$$
$$+\left[\frac{\gamma_{z_0z_0}}{2}U_2^{z_0} + \frac{\gamma_{z_0z_1}}{2}U_2^{z_1} + \frac{1}{r}\left(\frac{C_{z_0z_0}}{2}U_2^{z_0} + \frac{C_{z_0z_1}}{2}U_2^{z_1} + \frac{\gamma_{z_0z_0}}{2}U_3^{z_0} + \frac{\gamma_{z_0z_1}}{2}U_3^{z_1}\right)\right]dudz_0$$
$$+\left[\frac{\gamma_{z_0z_1}}{2}U_2^{z_0} + \frac{\gamma_{z_1z_1}}{2}U_2^{z_1} + \frac{1}{r}\left(\frac{C_{z_0z_1}}{2}U_2^{z_0} + \frac{C_{z_1z_1}}{2}U_2^{z_1} + \frac{\gamma_{z_0z_1}}{2}U_3^{z_0} + \frac{\gamma_{z_1z_1}}{2}U_3^{z_1}\right)\right]dudz_1$$
$$+\mathcal{O}(r^{-2}). \tag{42}$$

Now we are ready to evaluate the BMS generators in homogeneous coordinates in order to determine the supertranslations.

4. Asymptotic Killing Fields

After finding the most general Bondi metric in homogeneous coordinates satisfying the asymptotically flat spacetime falloffs, our aim is to find the most general vector fields ξ satisfying the Bondi gauge condition and the asymptotically flat spacetime falloffs. As is well known, the Killing vectors solve by definition the equations

$$\left(\mathcal{L}_\xi g\right)_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu\xi^\rho + g_{\nu\rho}\partial_\mu\xi^\rho = 0.$$

Moreover, the preservation of the Bondi gauge condition yields [26]

$$\left(\mathcal{L}_\xi g\right)_{rr} = 0, \ \left(\mathcal{L}_\xi g\right)_{rA} = 0 \ \text{ and } \ g^{AB}\left(\mathcal{L}_\xi g\right)_{AB} = 0. \tag{43}$$

From these relations one can calculate the four components of ξ^μ. At this stage, instead of repeating the detailed calculations already available, for example, in Ref. [26], we can compute the asymptotic Killing fields in homogeneous coordinates by using the familiar transformation law of vector fields. In other words, the work in Ref. [26] has defined the stereographic variable (we write ψ rather than z used in Ref. [26], in order to avoid confusion with our ζ in Eq. (1.1))

$$\psi = e^{i\varphi}\tan\frac{\theta}{2} = \frac{1}{\bar{\zeta}}, \tag{44}$$

and has found, in Bondi coordinates u, r, θ, φ, the asymptotic Killing fields ξ_T^+ where the components depend on a function f and on the Bondi coordinates. On denoting as usual by Y_l^m the spherical harmonics on the 2-sphere, one finds [26]

$$\xi_T^+\Big|_{f=Y_0^0} = \frac{\partial}{\partial u}, \tag{45}$$

$$\xi_T^+\Big|_{f=Y_1^0} = \frac{(1-\psi\bar{\psi})}{(1+\psi\bar{\psi})}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{\psi}{r}\frac{\partial}{\partial\psi} + \frac{\bar{\psi}}{r}\frac{\partial}{\partial\bar{\psi}}, \tag{46}$$

$$\xi_T^+\Big|_{f=Y_1^1} = \frac{\psi}{(1+\psi\bar{\psi})}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{\psi^2}{2r}\frac{\partial}{\partial\psi} - \frac{1}{2r}\frac{\partial}{\partial\bar{\psi}}, \tag{47}$$

$$\xi_T^+|_{f=Y_1^{-1}} = \frac{\bar{\psi}}{(1+\psi\bar{\psi})}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) - \frac{1}{2r}\frac{\partial}{\partial\psi} + \frac{\bar{\psi}^2}{2r}\frac{\partial}{\partial\bar{\psi}}. \tag{48}$$

Now by virtue of the basic identities

$$\frac{\partial}{\partial\psi} = \frac{\partial z_0}{\partial\psi}\frac{\partial}{\partial z_0} + \frac{\partial z_1}{\partial\psi}\frac{\partial}{\partial z_1}, \tag{49}$$

$$\frac{\partial}{\partial\bar{\psi}} = \frac{\partial z_0}{\partial\bar{\psi}}\frac{\partial}{\partial z_0} + \frac{\partial z_1}{\partial\bar{\psi}}\frac{\partial}{\partial z_1}, \tag{50}$$

and upon exploiting the formulae (A7)-(A10) in the Appendix, we find

$$\xi_T^+\Big|_{f=Y_1^0} = \frac{2}{\gamma}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) + \frac{z_0}{2r}\frac{(2-\gamma)}{\gamma}\frac{\partial}{\partial z_0} + \frac{z_1}{2r}\frac{(2+\gamma)}{\gamma}\frac{\partial}{\partial z_1}, \tag{51}$$

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$$\xi_T^+\Big|_{f=Y_1^1} = \frac{z_0}{2z_1}\frac{(\gamma-2)}{\gamma}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}\right)+\frac{1}{r}\frac{(z_0)^2}{z_1}\left(\frac{1}{4}-\frac{1}{\gamma(\gamma+2)}\right)\frac{\partial}{\partial z_0} + \frac{z_0}{2r}\left(\frac{1}{(\gamma+2)}-\frac{(\gamma+2)}{2\gamma}\right)\frac{\partial}{\partial z_1}, \quad (52)$$

$$\xi_T^+\Big|_{f=Y_1^{-1}} = \frac{z_1}{2z_0}\frac{(\gamma+2)}{\gamma}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}\right)-\frac{z_1}{2r}\left(\frac{1}{(\gamma-2)}+\frac{(\gamma-2)}{2\gamma}\right)\frac{\partial}{\partial z_0} + \frac{1}{r}\frac{(z_1)^2}{z_0}\left(\frac{1}{4}-\frac{1}{\gamma(\gamma-2)}\right)\frac{\partial}{\partial z_1}. \quad (53)$$

Now we denote by ξ_0,ξ_1,ξ_2,ξ_3 the vector fields (4.3), (4.9), (4.10) and (4.11), respectively. Nontrivial Lie brackets among them involve ξ_1,ξ_2,ξ_3 only. With our notation, we can re-write Eqs. (4.9)-(4.11) in the form

$$\xi_1 = A_{11}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}\right)+A_{12}\frac{\partial}{\partial z_0}+A_{13}\frac{\partial}{\partial z_1}, \quad (54)$$

$$\xi_2 = A_{21}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}\right)+A_{22}\frac{\partial}{\partial z_0}+A_{23}\frac{\partial}{\partial z_1}, \quad (55)$$

$$\xi_3 = A_{31}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}\right)+A_{32}\frac{\partial}{\partial z_0}+A_{33}\frac{\partial}{\partial z_1}, \quad (56)$$

where the values taken by the A_{ij} functions can be read off from (4.9)-(4.11). At this stage, a patient evaluation proves that such vector fields have vanishing Lie brackets:

$$[\xi_1,\xi_2]=[\xi_2,\xi_3]=[\xi_3,\xi_1]=0. \quad (57)$$

The result is simple, but the actual proof requires several details, for which we refer the reader to Appendix B.

5. Flow of Supertranslation Vector Fields

In order to appreciate that the familiar geometric constructions are feasible also in projective coordinates, we now consider the flow of supertranslation vector fields (4.9)-(4.11). For example, by virtue of (2.7), and defining $p=(u,r,z_0,z_1)$, the task of finding the flow of the supertranslation vector fields (4.9), (4.10) and (4.11) consists of solving a system of nonlinear and coupled differential equations. For this purpose, we denote by σ,Σ,χ, respectively, the appropriate flow, and define

$$\delta(W;\tau,p)=\sqrt{1-4\Big(W^3(\tau,p)W^4(\tau,p)\Big)^2}, \quad (58)$$

where $W=\sigma,\Sigma,\chi$, respectively, with components W^1,W^2,W^3,W^4. Hence we study the following coupled systems of nonlinear differential equations:

$$\frac{d\sigma^1}{d\tau}=\delta(\sigma;\tau,p), \quad (59)$$

$$\frac{d\sigma^2}{d\tau}=-\delta(\sigma;\tau,p), \quad (60)$$

$$\frac{d\sigma^3}{d\tau}=\frac{\sigma^3(\tau,p)}{2\sigma^2(\tau,p)}(\delta(\sigma;\tau,p)-1), \quad (61)$$

$$\frac{d\sigma^4}{d\tau}=\frac{\sigma^4(\tau,p)}{2\sigma^2(\tau,p)}(\delta(\sigma;\tau,p)+1), \quad (62)$$

$$\frac{d\Sigma^1}{d\tau}=\frac{\Sigma^3(\tau,p)}{2\Sigma^4(\tau,p)}(1-\delta(\Sigma;\tau,p)), \quad (63)$$

$$\frac{d\Sigma^2}{d\tau}=-\frac{\Sigma^3(\tau,p)}{2\Sigma^4(\tau,p)}(1-\delta(\Sigma;\tau,p)), \quad (64)$$

$$\frac{d\Sigma^3}{d\tau}=\frac{(\Sigma^3(\tau,p))^2}{4\Sigma^2(\tau,p)\Sigma^4(\tau,p)}\left[1-\delta(\Sigma;\tau,p)+\frac{\delta(\Sigma;\tau,p)}{(1+\delta(\Sigma;\tau,p))}\right], \quad (65)$$

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$$\frac{d\Sigma^4}{d\tau} = \frac{\Sigma^3(\tau,p)}{4\Sigma^2(\tau,p)}\left[\frac{\delta(\Sigma;\tau,p)}{(1+\delta(\Sigma;\tau,p))} - 1 - \delta(\Sigma;\tau,p)\right], \tag{66}$$

$$\frac{d\chi^1}{d\tau} = \frac{\chi^4(\tau,p)}{2\chi^3(\tau,p)}(1+\delta(\chi;\tau,p)), \tag{67}$$

$$\frac{d\chi^2}{d\tau} = -\frac{\chi^4(\tau,p)}{2\chi^3(\tau,p)}(1+\delta(\chi;\tau,p)), \tag{68}$$

$$\frac{d\chi^3}{d\tau} = -\frac{\chi^4(\tau,p)}{4\chi^2(\tau,p)}\left[\frac{\delta(\chi;\tau,p)}{(1-\delta(\chi;\tau,p)} + 1 - \delta(\chi;\tau,p)\right], \tag{69}$$

$$\frac{d\chi^4}{d\tau} = \frac{(\chi^4(\tau,p))^2}{4\chi^2(\tau,p)\chi^3(\tau,p)}\left[1+\delta(\chi;\tau,p) - \frac{\delta(\chi;\tau,p)}{(1-\delta(\chi;\tau,p))}\right], \tag{70}$$

with the initial conditions

$$W^1(0,p) = u,\ W^2(0,p) = r,\ W^3(0,p) = z_0,\ W^4(0,p) = z_1. \tag{71}$$

The resulting equations can only be solved numerically, to the best of our knowledge, and such solutions are displayed in Figures 1–9. Since the desired solutions are complex-valued, we have displayed both real and imaginary parts, with three choices of initial conditions.

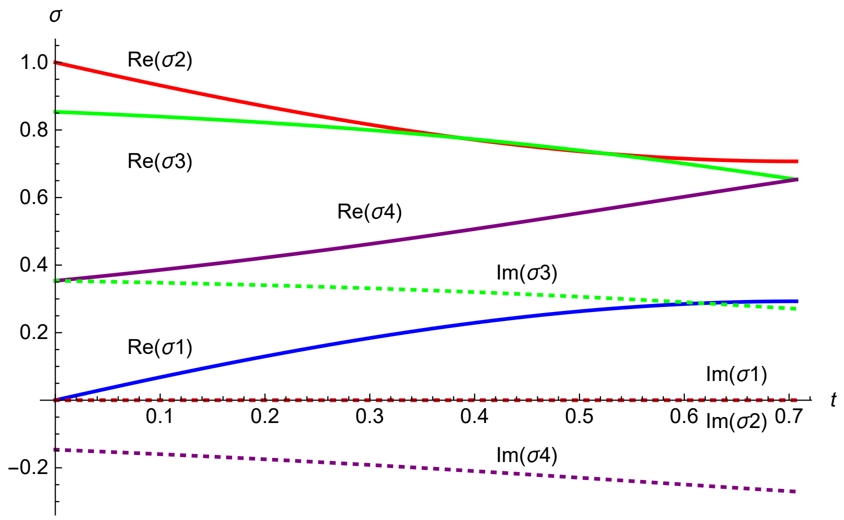

Figure 1. Numerical evaluation of the integral curve for the supertranslation vector field (4.9). The initial conditions (5.14) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{8}}\cos\frac{\pi}{8}, z_1 = e^{-i\frac{\pi}{8}}\sin\frac{\pi}{8}$.

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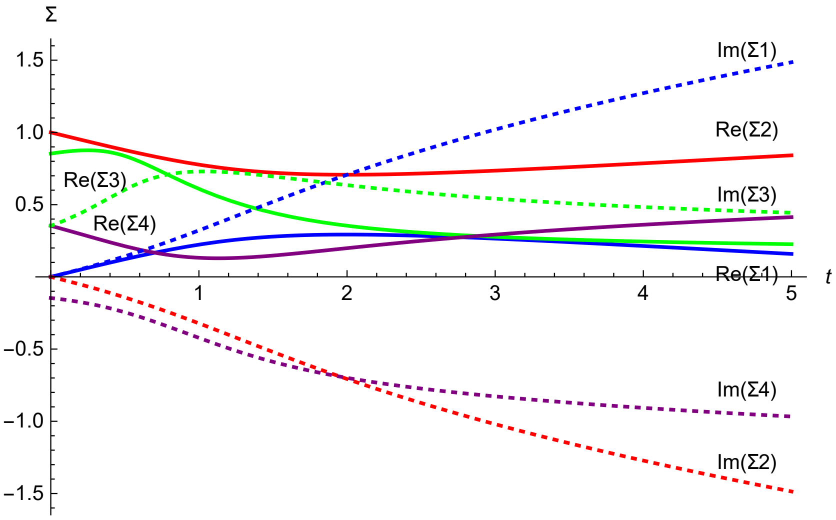

Figure 2. Numerical evaluation of the integral curve for the supertranslation vector field (4.10). The initial conditions (5.14) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{8}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{8}$.

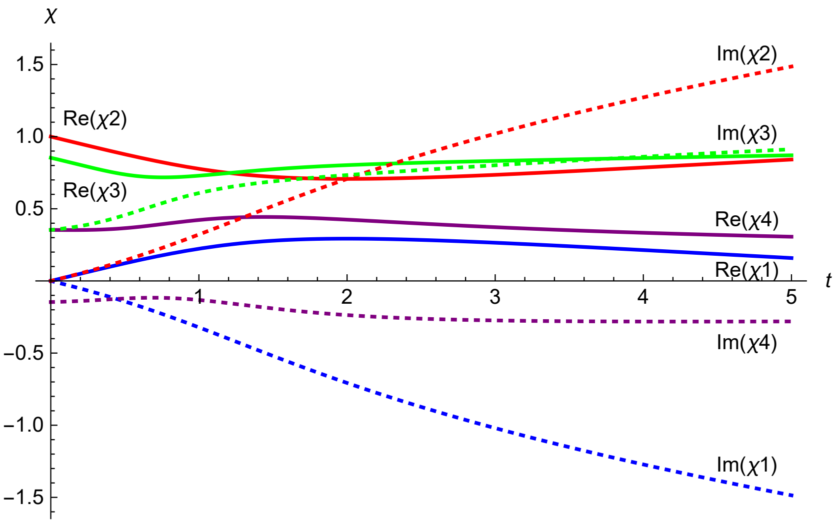

Figure 3. Numerical evaluation of the integral curve for the supertranslation vector field (4.11). The initial conditions (5.14) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{8}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{8}$.

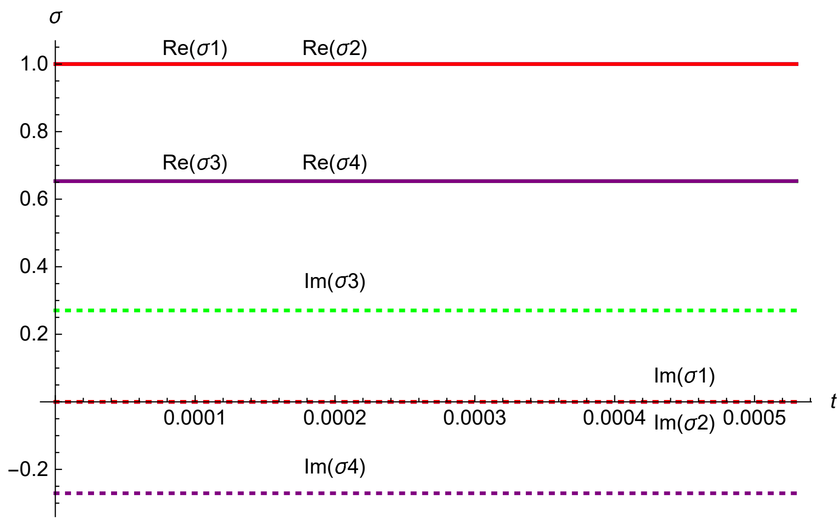

Figure 4. Numerical evaluation of the integral curve for the supertranslation vector field (4.9). The initial conditions (5.14) are taken to be $u = 1, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{4}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{4}$.

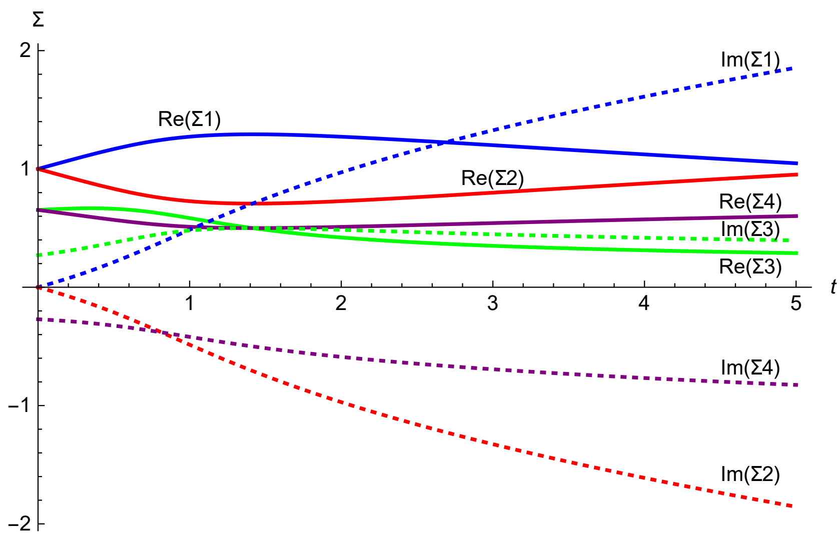

Figure 5. Numerical evaluation of the integral curve for the supertranslation vector field (4.10). The initial conditions (5.14) are taken to be $u = 1, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos\frac{\pi}{4}, z_1 = e^{-i\frac{\pi}{8}} \sin\frac{\pi}{4}$.

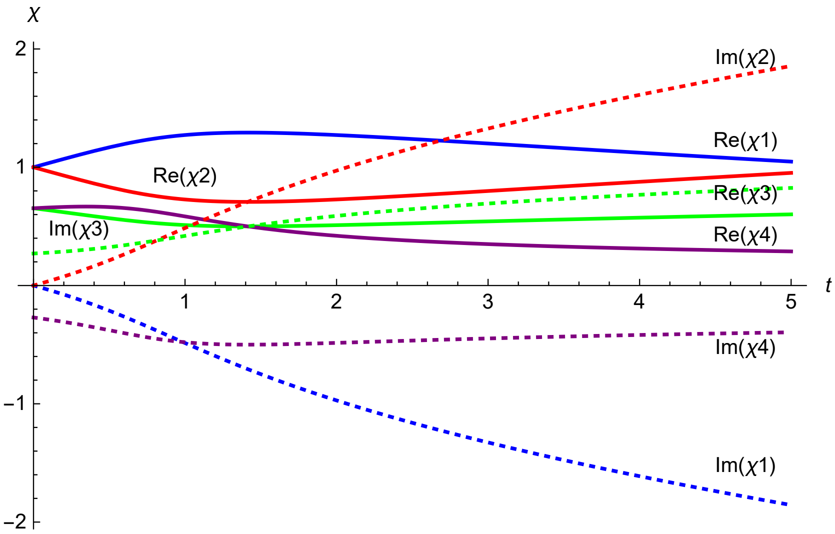

Figure 6. Numerical evaluation of the integral curve for the supertranslation vector field (4.11). The initial conditions (5.14) are taken to be $u = 1, r = 1, z_0 = e^{i\frac{\pi}{8}} \cos \frac{\pi}{4}, z_1 = e^{-i\frac{\pi}{8}} \sin \frac{\pi}{4}$.

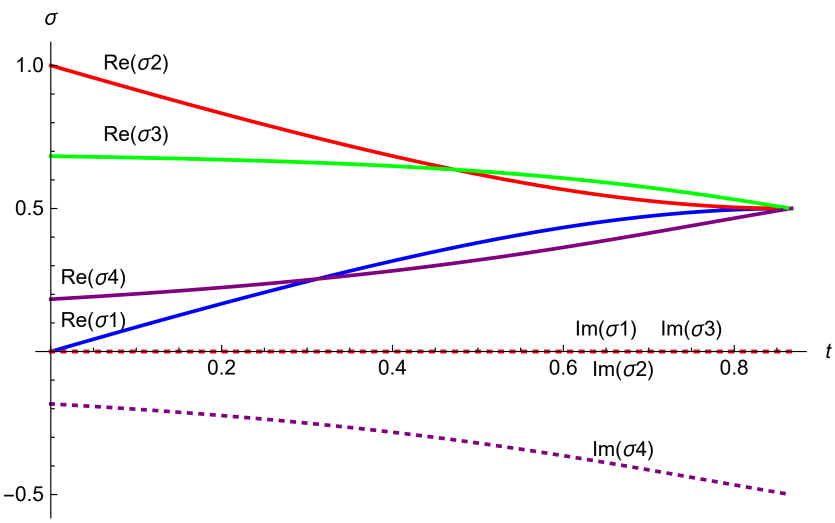

Figure 7. Numerical evaluation of the integral curve for the supertranslation vector field (4.9). The initial conditions (5.14) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{4}} \cos \frac{\pi}{12}, z_1 = e^{-i\frac{\pi}{4}} \sin \frac{\pi}{12}$. In this particular case, the real parts meet at a single point.

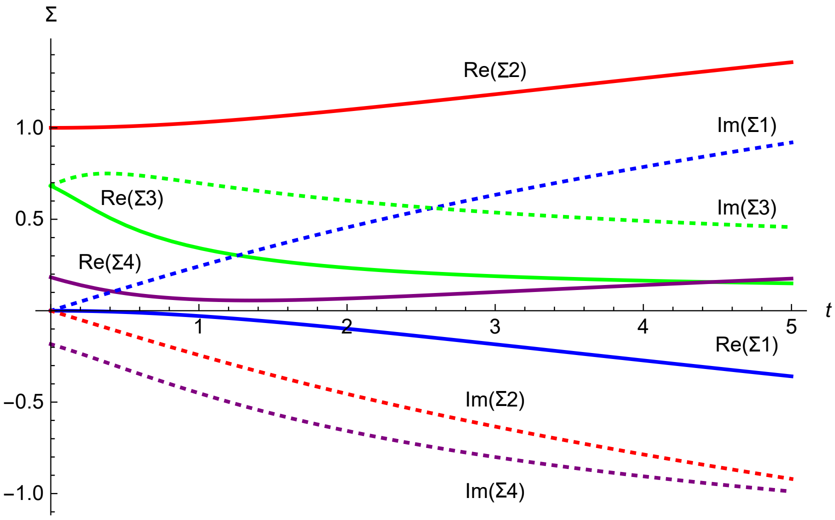

Figure 8. Numerical evaluation of the integral curve for the supertranslation vector field (4.10). The initial conditions (5.14) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{4}} \cos\frac{\pi}{12}, z_1 = e^{-i\frac{\pi}{4}} \sin\frac{\pi}{12}$.

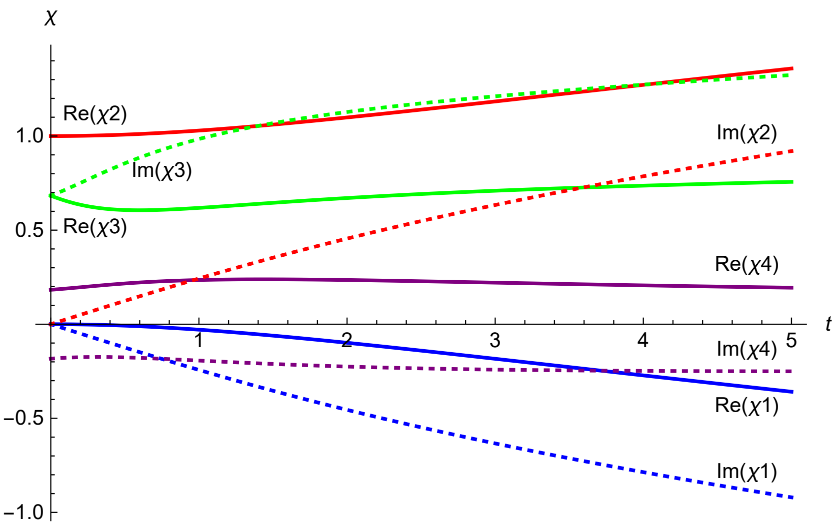

Figure 9. Numerical evaluation of the integral curve for the supertranslation vector field (4.11). The initial conditions (5.14) are taken to be $u = 0, r = 1, z_0 = e^{i\frac{\pi}{4}} \cos\frac{\pi}{12}, z_1 = e^{-i\frac{\pi}{4}} \sin\frac{\pi}{12}$.

6. Concluding Remarks and Open Problems

As far as we can see, the interest of our investigation lies in having shown that homogeneous projective coordinates lead to a fully computational scheme for all applications of the BMS group. This might pay off when more advanced properties will be studied. In particular, we have in mind the concept of superrotations [21,22,26] on the one hand, and the physical applications of the Segre manifold advocated in Ref. [19]. In other words, since our Eq. (1.15) contains Eq. (1.12), which in turn is just a re-expression of the BMS transformation (1.2), one might aim at embedding the study of BMS symmetries into the richer mathematical framework of complex analysis [27] and algebraic geometry. Our paper has tried to prepare the ground for such a synthesis.

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Appendix A. The Use of Homogeneous Coordinates

By virtue of Eqs. (1.10) and (2.7), we find

$$(z_0)^2 = e^{i\varphi}\frac{(1+\cos\theta)}{2} \Longrightarrow e^{i\varphi} = \frac{2(z_0)^2}{(1+\cos\theta)} = \frac{2(z_0)^2\gamma}{(\gamma+2)}, \tag{A1}$$

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and hence the variable ψ in Eq. (4.2) can be re-expressed in the form

$$\psi = \frac{2(z_0)^2\gamma}{(\gamma+2)}\frac{(1-\cos\theta)}{\sin\theta} = \frac{2(z_0)^2\gamma}{(\gamma+2)}\frac{\left(1-\frac{2}{\gamma}\right)}{2z_0z_1} = \frac{z_0}{z_1}\frac{(\gamma-2)}{(\gamma+2)}, \tag{A2}$$

while

$$\bar{\psi} = \frac{1}{\zeta} = \frac{z_1}{z_0}. \tag{A3}$$

Moreover, we need the identities

$$\psi\bar{\psi} = \frac{\left(1-\cos^2\frac{\theta}{2}\right)}{\cos^2\frac{\theta}{2}} \Longrightarrow \cos\frac{\theta}{2} = \frac{1}{\sqrt{(1+\psi\bar{\psi})}},\ \sin\frac{\theta}{2} = \sqrt{\frac{\psi\bar{\psi}}{(1+\psi\bar{\psi})}}, \tag{A4}$$

$$e^{i\frac{\varphi}{2}} = \left(\frac{\psi}{\bar{\psi}}\right)^{\frac{1}{4}},\ e^{-i\frac{\varphi}{2}} = \left(\frac{\bar{\psi}}{\psi}\right)^{\frac{1}{4}}, \tag{A5}$$

which, jointly with the definitions (1.10), lead to

$$z_0 = \left(\frac{\psi}{\bar{\psi}}\right)^{\frac{1}{4}}\frac{1}{\sqrt{1+\psi\bar{\psi}}},\ z_1 = \left(\frac{\bar{\psi}}{\psi}\right)^{\frac{1}{4}}\sqrt{\frac{\psi\bar{\psi}}{(1+\psi\bar{\psi})}}. \tag{A6}$$

At this stage, we can evaluate the partial derivatives occurring in Eqs. (4.7) and (4.8) by patient application of Eqs. (A2), (A3) and (A6), i.e.,

$$\frac{\partial z_0}{\partial\psi} = \frac{z_1}{2}\frac{(\gamma+2)}{\gamma(\gamma-2)}, \tag{A7}$$

$$\frac{\partial z_1}{\partial\psi} = \frac{1}{2}\frac{(z_1)^2}{z_0}\frac{(\gamma+2)}{\gamma(\gamma-2)}, \tag{A8}$$

$$\frac{\partial z_0}{\partial\bar{\psi}} = -\frac{(z_0)^2}{2z_1}\frac{(\gamma-1)}{\gamma}, \tag{A9}$$

$$\frac{\partial z_1}{\partial\bar{\psi}} = \frac{z_0}{2}\frac{(\gamma+1)}{\gamma}, \tag{A10}$$

and we find eventually the asymptotic Killing fields in the form (4.9)-(4.11). Our homogeneous projective coordinates z_0 and z_1 have also been considered in Ref. [28], but in that case, upon writing

$$\zeta = \frac{(x+iy)}{(1-z)}, \tag{A11}$$

one finds that the x, y, z coordinates for the embedding of the 2-sphere in three-dimensional Euclidean space are given by

$$x = \frac{2\mathrm{Re}(\zeta)}{(1+|\zeta|^2)} = \frac{(z_0\bar{z}_1+\bar{z}_0z_1)}{(|z_0|^2+|z_1|^2)}, \tag{A12}$$

$$y = \frac{2\mathrm{Im}(\zeta)}{(1+|\zeta|^2)} = \frac{(z_0\bar{z}_1-\bar{z}_0z_1)}{i(|z_0|^2+|z_1|^2)}, \tag{A13}$$

$$z = \frac{(|\zeta|^2-1)}{(|\zeta|^2+1)} = \frac{(|z_0|^2-|z_1|^2)}{(|z_0|^2+|z_1|^2)}. \tag{A14}$$

The global spacetime translations of Minkowski spacetime can be first re-expressed in $u, r, \xi, \bar{\xi}$ coordinates, and read eventually, in terms of the asymptotic Killing fields (4.9)-(4.11),

$$X_0 = -\xi_T^+\Big|_{f=Y_0^0},\ X_1 = -\xi_T^+\Big|_{f=Y_1^1} - \xi_T^+\Big|_{f=Y_1^{-1}},$$
$$iX_2 = \xi_T^+\Big|_{f=Y_1^{-1}} - \xi_T^+\Big|_{f=Y_1^1},\ X_3 = -\xi_T^+\Big|_{f=Y_1^0}. \tag{A15}$$

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Explicitly, we find

$$\begin{aligned} X_1 &= \left(\frac{z_0}{2z_1}\frac{(\gamma-2)}{\gamma}+\frac{z_1}{2z_0}\frac{(\gamma+2)}{\gamma}\right)\left(-\frac{\partial}{\partial u}+\frac{\partial}{\partial r}\right) \\ &+ \frac{1}{2r}\left[\frac{(z_0)^2}{z_1}\left(-\frac{1}{2}-\frac{2}{\gamma(\gamma+2)}\right)+z_1\left(\frac{(\gamma-2)}{2\gamma}+\frac{1}{(\gamma-2)}\right)\right]\frac{\partial}{\partial z_0} \\ &+ \frac{1}{2r}\left[\frac{(z_1)^2}{z_0}\left(-\frac{1}{2}+\frac{2}{\gamma(\gamma-2)}\right)+z_0\left(\frac{(\gamma+2)}{2\gamma}-\frac{1}{(\gamma+2)}\right)\right]\frac{\partial}{\partial z_1}, \end{aligned} \tag{A16}$$

$$\begin{aligned} X_2 &= i\left(\frac{z_0}{2z_1}\frac{(\gamma-2)}{\gamma}-\frac{z_1}{2z_0}\frac{(\gamma+2)}{\gamma}\right)\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial r}\right) \\ &+ \frac{i}{2r}\left[\frac{(z_0)^2}{z_1}\left(\frac{1}{2}-\frac{2}{\gamma(\gamma+2)}\right)+z_1\left(\frac{(\gamma-2)}{2\gamma}+\frac{1}{(\gamma-2)}\right)\right]\frac{\partial}{\partial z_0} \\ &- \frac{i}{2r}\left[\frac{(z_1)^2}{z_0}\left(\frac{1}{2}-\frac{2}{\gamma(\gamma-2)}\right)+z_0\left(\frac{(\gamma+2)}{2\gamma}-\frac{1}{(\gamma+2)}\right)\right]\frac{\partial}{\partial z_1}, \end{aligned} \tag{A17}$$

$$X_3 = \frac{2}{\gamma}\left(-\frac{\partial}{\partial u}+\frac{\partial}{\partial r}\right)+\frac{1}{2r}\left(z_0\frac{\partial}{\partial z_0}-z_1\frac{(\gamma+2)}{\gamma}\frac{\partial}{\partial z_1}\right). \tag{A18}$$

The boost (K_i) and rotation (J_{ij}) vector fields for Lorentz transformations in Minkowski spacetime can be written in $u, r, \xi, \bar{\xi}$ coordinates as is shown, for example, in Refs. [21,26]. At that stage, by using again Eqs. (4.7), (4.8) and (A7)-(A10) we find

$$\begin{aligned} K_1 &= \frac{1}{2}\left(\frac{z_0}{z_1}\frac{(\gamma-2)}{\gamma}+\frac{z_1}{z_0}\frac{(\gamma+2)}{\gamma}\right)\left(-u\frac{\partial}{\partial u}+(u+r)\frac{\partial}{\partial r}\right) \\ &- \frac{(u+r)}{2r}\left[\frac{(z_0)^2}{z_1}\left(\frac{1}{2}-\frac{2}{\gamma(\gamma+2)}\right)-z_1\left(\frac{(\gamma-2)}{2\gamma}+\frac{1}{(\gamma-2)}\right)\right]\frac{\partial}{\partial z_0} \\ &+ \left[\frac{(z_1)^2}{z_0}\left(\frac{1}{2}-\frac{2}{\gamma(\gamma-2)}\right)+z_0\left(-\frac{(\gamma+2)}{2\gamma}+\frac{1}{(\gamma+2)}\right)\right]\frac{\partial}{\partial z_1}, \end{aligned} \tag{A19}$$

$$\begin{aligned} K_2 &= \frac{i}{2}\left(\frac{z_0}{z_1}\frac{(\gamma-2)}{\gamma}-\frac{z_1}{z_0}\frac{(\gamma+2)}{\gamma}\right)\left(u\frac{\partial}{\partial u}-(u+r)\frac{\partial}{\partial r}\right) \\ &+ i\frac{(u+r)}{2r}\left\{\left[\frac{(z_0)^2}{z_1}\left(\frac{1}{2}-\frac{2}{\gamma(\gamma+2)}\right)+z_1\left(\frac{(\gamma-2)}{2\gamma}+\frac{1}{(\gamma-2)}\right)\right]\frac{\partial}{\partial z_0}\right. \\ &+ \left.\left[\frac{(z_1)^2}{z_0}\left(-\frac{1}{2}+\frac{2}{\gamma(\gamma-2)}\right)+z_0\left(-\frac{(\gamma+2)}{2\gamma}+\frac{1}{(\gamma+2)}\right)\right]\frac{\partial}{\partial z_1}\right\}, \end{aligned} \tag{A20}$$

$$K_3 = \frac{2}{\gamma}\left(-u\frac{\partial}{\partial u}+(u+r)\frac{\partial}{\partial r}\right)+\frac{1}{2r}(u+r)\left(z_0\frac{\partial}{\partial z_0}-z_1\frac{(\gamma+2)}{\gamma}\frac{\partial}{\partial z_1}\right), \tag{A21}$$

$$J_{12} = \frac{i}{2}\left(z_0\frac{\partial}{\partial z_0}-z_1\frac{\partial}{\partial z_1}\right), \tag{A22}$$

$$\begin{aligned} J_{23} &= \frac{i}{2}\left[\left(-\frac{(z_0)^2}{2z_1}\frac{\gamma}{(\gamma+2)}+z_1\left(\frac{1}{2}-\frac{1}{(\gamma-2)}\right)\right)\right]\frac{\partial}{\partial z_0} \\ &+ \frac{i}{2}\left[\left(-\frac{(z_1)^2}{2z_0}\frac{\gamma}{(\gamma-2)}+z_0\left(\frac{1}{2}+\frac{1}{(\gamma+2)}\right)\right)\right]\frac{\partial}{\partial z_1}, \end{aligned} \tag{A23}$$

$$\begin{aligned} J_{31} &= -\frac{1}{2}\left[\left(\frac{(z_0)^2}{2z_1}\frac{\gamma}{(\gamma+2)}+z_1\left(\frac{1}{2}-\frac{1}{(\gamma-2)}\right)\right)\right]\frac{\partial}{\partial z_0} \\ &+ \frac{1}{2}\left[\left(\frac{(z_1)^2}{2z_0}\frac{\gamma}{(\gamma-2)}+z_0\left(\frac{1}{2}+\frac{1}{(\gamma+2)}\right)\right)\right]\frac{\partial}{\partial z_1}. \end{aligned} \tag{A24}$$

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Appendix B. Lie Brackets of Asymptotic Killing Fields

Given the vector fields (4.12) and (4.13), the evaluation of their Lie bracket shows that

$$[\xi_1, \xi_2] = \rho_1 \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r} \right) + \rho_2 \frac{\partial}{\partial z_0} + \rho_3 \frac{\partial}{\partial z_1}, \tag{B.1}$$

where, upon defining the functions

$$\alpha_1 = \frac{2(z_0)^3 z_1}{r} \gamma \left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)} \right), \tag{B.2}$$

$$\alpha_2 = \frac{2}{\gamma} \frac{1}{r^2} \frac{(z_0)^2}{z_1} \left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)} \right), \tag{B.3}$$

$$\alpha_3 = \frac{(z_0)^3 z_1}{r} \gamma \left[\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma} \right], \tag{B.4}$$

$$\alpha_4 = \frac{z_0}{\gamma} \frac{1}{r^2} \left[\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma} \right], \tag{B.5}$$

$$\alpha_5 = \frac{z_0}{2r} \left(\frac{2}{\gamma} - 1 \right) \left[\frac{1}{2z_1} \left(1 - \frac{2}{\gamma} \right) + (z_0)^2 z_1 \gamma \right], \tag{B.6}$$

$$\alpha_6 = \frac{(z_0)^2}{4 z_1 r^2} \left(\frac{2}{\gamma} - 1 \right)^2, \tag{B.7}$$

$$\alpha_7 = \frac{(z_0)^4 z_1 (8 + (\gamma-2)\gamma^2)}{4 r^2 (\gamma+2)^2}, \tag{B.8}$$

$$\alpha_8 = \frac{(z_0)^4 z_1 \gamma}{2r^2} \left[\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma} \right], \tag{B.9}$$

$$\alpha_9 = \frac{z_0}{4r^2} \left(\frac{2}{\gamma} - 1 \right) \left[\frac{1}{(\gamma+2)} - \frac{(\gamma+2)}{2\gamma} + \frac{4(z_0)^2 (z_1)^2 \gamma(\gamma+1)}{(\gamma+2)^2} \right], \tag{B.10}$$

$$\alpha_{10} = \frac{z_0 z_1}{4r} \left(\frac{2}{\gamma} + 1 \right) \left[-\frac{1}{(z_1)^2} \left(1 - \frac{2}{\gamma} \right) + 2(z_0)^2 \gamma \right], \tag{B.11}$$

$$\alpha_{11} = \frac{z_0}{4r^2} \left(\frac{4}{\gamma^2} - 1 \right), \tag{B.12}$$

$$\alpha_{12} = \frac{(z_0)^2 z_1}{2r^2} \left(\frac{2}{\gamma} + 1 \right) \left[-\frac{1}{(z_1)^2} \left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)} \right) + \frac{2(z_0)^2 \gamma(\gamma+1)}{(\gamma+2)^2} \right], \tag{B.13}$$

$$\alpha_{13} = \frac{(z_0)^3 (z_1)^2 \gamma}{r^2} \left(\frac{1}{4} - \frac{1}{\gamma(\gamma+2)} \right), \tag{B.14}$$

$$\alpha_{14} = \frac{z_0 (8 - (\gamma-6)\gamma^2)}{16 r^2 \gamma^2}, \tag{B.15}$$

we find that

$$\rho_1 = \alpha_1 + \alpha_3 + \alpha_5 + \alpha_{10} = 0, \tag{B.16}$$

$$\rho_2 = \alpha_2 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_{12} = 0, \tag{B.17}$$

$$\rho_3 = \alpha_4 + \alpha_9 + \alpha_{11} + \alpha_{13} + \alpha_{14} = 0. \tag{B.18}$$

In the course of performing the calculation, the definition (2.7) leads to the useful identity

$$\frac{1}{\gamma^2} = \frac{4(z_0)^2 (z_1)^2}{(\gamma^2 - 4)}. \tag{B.19}$$

An analogous procedure shows that

$$[\xi_2, \xi_3] = [\xi_3, \xi_1] = 0, \tag{B.20}$$

with the help of two additional sets of 14 nonvanishing functions, one set for each Lie bracket in (B20). For example, in the Lie bracket among ξ_2 and ξ_3, the coefficient of $\frac{\partial}{\partial z_0}$ is a function

$$\begin{aligned}
\rho &= -\frac{z_0}{4r^2} - \frac{(z_0)^2}{16r^2}\left(1 - \frac{4}{\gamma(\gamma+2)}\right)2z_0(z_1)^2\gamma\left(1 - \frac{\gamma^2}{(\gamma-2)^2}\right) \\
&+ \frac{1}{16r^2}\left(1 + \frac{4}{\gamma(\gamma-2)}\right)\left[2z_0\left(1 - \frac{4}{\gamma(\gamma+2)}\right) - 2(z_0)^3(z_1)^2\gamma\left(\frac{\gamma^2}{(\gamma+2)^2} - 1\right)\right] \\
&+ \frac{z_0}{8r^2}\left(1 - \frac{4}{\gamma(\gamma-2)}\right)\left[\left(1 - \frac{4}{\gamma(\gamma+2)}\right) + (z_0)^2(z_1)^2\gamma\left(\frac{\gamma^2}{(\gamma+2)^2} - 1\right)\right] \\
&+ \frac{z_0}{8r^2}\left(1 + \frac{4}{\gamma(\gamma+2)}\right)\left[\frac{1}{2}\left(1 + \frac{4}{\gamma(\gamma-2)}\right) + (z_0)^2(z_1)^2\gamma\left(1 - \frac{\gamma^2}{(\gamma-2)^2}\right)\right] \\
&= 0. \qquad \text{(B.21)}
\end{aligned}$$

References

1. H. Bondi, M.G.J. Van der Burg, A.W.K. Metzner, Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems, Proc. Roy. Soc. London A **269**, 21 (1962).
2. R.K. Sachs, Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time, Proc. Roy. Soc. London A **270**, 103 (1962).
3. R.K. Sachs, Asymptotic symmetries in gravitational theory, Phys. Rev. **128**, 2851 (1962).
4. S.W. Hawking, M.A. Perry, A. Strominger, Soft hair on black holes, Phys. Rev. Lett. **116**, 231301 (2016).
5. S.W. Hawking, M.A. Perry, A. Strominger, Superrotation charge and supertranslation hair on black holes, JHEP **2017** 5, 161 (2017).
6. S. Haco, S.W. Hawking, M.A. Perry, A. Strominger, Black hole entropy and soft hair, JHEP **2018** 12, 98 (2018).
7. P.J. McCarthy, Representations of the Bondi-Metzner-Sachs group I. Determination of the representations, Proc. Roy. Soc. London A **330**, 517 (1972).
8. P.J. McCarthy, Structure of the Bondi-Metzner-Sachs group, J. Math. Phys. **13**, 1837 (1972).
9. P.J. McCarthy, Representations of the Bondi-Metzner-Sachs group II. Properties and classification of representations, Proc. Roy. Soc. London A **333**, 317 (1973).
10. P.J. McCarthy, M. Crampin, Representations of the Bondi-Metzner-Sachs group III. Poincaré spin multiplicities and irreducibility, Proc. Roy. Soc. London A **335**, 321 (1973).
11. P.J. McCarthy, M. Crampin, Representations of the Bondi-Metzner-Sachs group IV. Cantoni representations are induced, Proc. Roy. Soc. London A **351**, 55 (1976).
12. G. Barnich, C. Troessaert, Finite BMS transformations, JHEP **2016** 3, 167 (2016).
13. F. Alessio, G. Esposito, On the structure and applications of the Bondi-Metzner-Sachs group, Int. J. Geom. Methods Mod. Phys. **15**, 1830002 (2018).
14. M. Henneaux, C. Troessaert, BMS group at spatial infinity. The Hamiltonian (ADM) approach, JHEP **2018** 3, 147 (2018).
15. G. Barnich, K. Nguyen, R. Ruzziconi, Geometric action of extended Bondi-Metzner-Sachs group in four dimensions, JHEP **2022** 12, 154 (2022).
16. C. Chowdhury, A. Anupam, A. Kundu, Generalized BMS algebra in higher even dimensions, Phys. Rev. D **106**, 126025 (2022).
17. O. Fuentealba, M. Henneaux, J. Matulich, C. Troessaert, Bondi-Metzner-Sachs group in five spacetime dimensions, Phys. Rev. Lett. **128**, 051103 (2022).
18. D. Prinz, A. Schmeding, Lie theory for asymptotic symmetries in general relativity: The BMS group, Class. Quantum Grav. **39**, 065004 (2022).
19. Z. Mirzaiyan, G. Esposito, On the nature of Bondi-Metzner-Sachs transformations, Symmetry **15**, 947 (2023).
20. D.A. Weiss, A microscopic analogue of the BMS group, JHEP **2023** 4, 136 (2023).
21. A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory* (Princeton University Press, Princeton, 2018).
22. S. Pasterski, Implications of superrotations, Phys. Rep. **829**, 1 (2019).
23. E. Himwich, Z. Mirzaiyan, S. Pasterski, A note on the subleading soft graviton, JHEP **2021** 4, 172 (2021).

doi:10.20944/preprints202406.0126.v1

24. G. Compère, S.E. Gralla, An asymptotic framework for gravitational scattering, Class. Quantum Grav. **40**, 205018 (2023).
25. J.M. Stewart, *Advanced General Relativity* (Cambridge University Press, Cambridge, 1990).
26. X. Kervyn, BMS Symmetries of Gravitational Scattering, https://arxiv.org/abs/2308.12979, gr-qc (2023)
27. G. Della Sala, A. Saracco, A. Simioniuc, G. Tomassini, *Lectures on Complex Analysis and Analytic Geometry* (Scuola Normale Superiore, Pisa, 2006).
28. R. Penrose, W. Rindler, *Spinors and Space-Time. 1: Two-Spinor Calculus and Relativistic fields* (Cambridge University Press, Cambridge, 1984).