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On Mittag-Leffler Function and Consequent Fractional Integral Operator Inequalities

Yonghong Liu [1], Ghulam Farid [2], Abaker A. Hassaballa [3], Jongsuk Ro [4,5,*], Mnahil M. Bashier [6] and B. A. Younis [7]

[1] School of Computer Science, Chengdu University, Chengdu, China; cdulyh@hotmail.com (Y.L.)
[2] Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock 43600, Pakistan; ghlmfarid@ciit-attock.edu.pk (G.F.)
[3] Department of Mathematics, Faculty of Science, Northern Border University, Arar, Saudi Arabia; abaker.abdalla@nbu.edu.sa
[4] School of Electrical and Electronics Engineering, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea
[5] Department of Intelligent Energy and Industry, Chung-Ang University, Dongjak-gu, Seoul 06974, Republic of Korea; jongsukro@gmail.com (J.R.)
[6] Department of Mathematics, Faculty of Science, Northern Border University, Arar, Saudi Arabia; mnahil.elradi@nbu.edu.sa
[7] Department of Mathematics, Faculty of Arts and Science, Elmagarda, King Khalid University, Saudi Arabia; byounis@kku.edu.sa
* Correspondence: jongsukro@gmail.com

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Abstract: The unified Mittag-Leffler (ML) function is important factor in formulation of compact form of fractional integrals. In this paper estimations of integral operators, with the unified ML function as kernel are given. The Hadamard inequality in the form of fractional integrals is proved by considering refinement of convexities. These estimations provide various inequalities as refinements of recently published results.

Keywords: Mittag-Leffler function; fractional integral operators; convex function

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1. Introduction

Fractional calculus composed on fractional derivatives, fractional anti-derivatives and special functions. The Mittag-Leffler (ML) function, gamma function, beta function and other such functions are used in defining mathematical models for real world problems. The exponential function is unique in the sense of its existence and properties, and it has very important place in the theory of differential equations. The ML function is a generalization of exponential function and is equally important in solving fractional differential equations. Gösta Mittag-Leffler introduced this function in [1]. There are plenty of mathematical concepts, equations, and models in different subjects of science which were extended and generalized with the help of this ML function. The ML function is given in the following equation:

$$E_v(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(nv+1)} \tag{1}$$

where $v, x \in \mathbb{C}$ and $\Re(v) > 0$, and $\Gamma(.)$ is the gamma function.
The ML function defined in (1) involves one parameter, and there also exist many extended and generalized ML functions in literature. For more information and detail about ML function one can see [2–4], and references therein. ML functions are frequently utilized in defining operators of fractional derivatives and fractional integration.
It is also presented in generalized form by using generalized beta and gamma functions along with pochhammer symbol. Currently, so called unified ML function is introduced in [5], and given in the following definition. Here we assume all the convergence conditions are satisfied and exclude the detail, one can see [5].

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Definition 1. *The unified ML function is defined by;*

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{a},\underline{b},\underline{c},p') = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_{p'}(b_i,a_i)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{n} B(c_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}, \tag{2}$$

where $\Gamma(\mu) = \int_0^\infty e^{-z} z^{\mu-1} dz$, $(\theta)_{kl} = \frac{\Gamma(\theta+lk)}{\Gamma(\theta)}$,

$$\beta_{p'}(\varrho,y) = \int_0^1 \tau^{\varrho-1}(1-\tau)^{y-1} e^{-\left(\frac{p'}{\tau(1-\tau)}\right)} d\tau, \tag{3}$$

and $\beta_{p'}$ *is the extension of well known beta function.*

The unified fractional integral operators containing the above ML function are defined as follows:

Definition 2. *(see[5]) Let* $\psi \in L_1[\xi_1,\xi_2]$*. Then* $\forall\ \varrho \in [\xi_1,\xi_2]$*, the fractional integral operator containing the unified ML function* $M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{a},\underline{b},\underline{c},p')$ *along with all the convergence conditions is defined by;*

$$\left(\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi\right)(\varrho;\underline{a},\underline{b},\underline{c},p') = \int_{\xi_1}^{\varrho} (\varrho-\tau)^{\alpha-1} M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\varrho-\tau)^{\mu};\underline{a},\underline{b},\underline{c},p')\psi(\tau)d\tau, \tag{4}$$

$$\left(\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi\right)(\varrho;\underline{a},\underline{b},\underline{c},p') = \int_{\varrho}^{\xi_2} (\tau-\varrho)^{\alpha-1} M^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\varrho-\tau)^{\mu};\underline{a},\underline{b},\underline{c},p')\psi(\tau)d\tau. \tag{5}$$

By setting $a_i = l$, $p' = 0$ and $\Re(p') > 0$ in above definitions, one can get the generalized Q function

$$Q^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z;\underline{a},\underline{b}) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} \beta(b_i,l)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{n} \beta(a_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)},$$

is the generalized Q function defined in [6] and the fractional integral operators

$$\left({}_Q\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi\right)(\varrho;\underline{a},\underline{b}) = \int_{\xi_1}^{\varrho} (\varrho-\tau)^{\alpha-1} Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\varrho-\tau)^{\mu};\underline{a},\underline{b})\psi(\tau)d\tau, \tag{6}$$

$$\left({}_Q\Upsilon^{\omega,\lambda,\rho,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi\right)(\varrho;\underline{a},\underline{b}) = \int_{\varrho}^{\xi_2} (\tau-\varrho)^{\alpha-1} Q^{\lambda,\rho,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\tau-\varrho)^{\mu};\underline{a},\underline{b})\psi(\tau)d\tau \tag{7}$$

as given in [7]. Next, we give the following definition of integral operators.

Definition 3. *[8] Let* $\phi \in L_1[\xi_1,\xi_2]$*,* $0 < \xi_1,\xi_2 < \infty$ *be a positive function and let* $\Theta : [\xi_1,\xi_2] \to \mathbb{R}$ *be a differentiable and strictly increasing function. Also let* $\frac{\phi}{\varrho}$ *be an increasing function on* $[\xi_1,\infty)$ *and* $\varrho \in [\xi_1,\xi_2]$*. Then the unified integral operator is given by;*

$$({}^{\phi}_{\Theta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi)(\varrho;p') = \int_u^{\varrho} \Lambda^{\tau}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Theta;\phi)\psi(\tau)d(\Theta(\tau)), \tag{8}$$

$$({}^{\phi}_{\Theta}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\psi)(\varrho;p') = \int_{\varrho}^{v} \Lambda^{\varrho}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Theta;\phi)\psi(\tau)d(\Theta(\tau)), \tag{9}$$

where

$$\Lambda^{\tau}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu},\Theta;\phi) = \frac{\phi(\Theta(\varrho)-\Theta(\tau))}{\Theta(\varrho)-\Theta(\tau)} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\Theta(\varrho)-\Theta(\tau))^{\mu};\underline{a},\underline{b},\underline{c},p'). \tag{10}$$

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By setting $a_i = l$, $p' = 0$ and $\Re(p') > 0$ in (8) and (9), one can get the integral operator associated with generalized Q function given in [7]:

$$\left({}_{Q}^{\Theta}\Upsilon_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\phi,\omega,\lambda,\rho,\theta,k,n}\psi\right)(\varrho;\underline{a},\underline{b}) = \int_{\xi_1}^{\varrho}\Lambda_{\varrho}^{y}(Q_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,k,n},\Theta;\phi)\psi(\tau)d(\Theta(\tau)), \tag{11}$$

$$\left({}_{Q}^{\Theta}\Upsilon_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}^{\phi,\omega,\lambda,\rho,\theta,k,n}\psi\right)(\varrho;\underline{a},\underline{b}) = \int_{\varrho}^{b}\Lambda_{\varrho}^{y}(Q_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,k,n},\Theta;\phi)\psi(\tau)d(\Theta(\tau)), \tag{12}$$

where
$\Lambda_{\varrho}^{\tau}(Q_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi) = \dfrac{\phi(\Theta(\varrho)-\Theta(\tau))}{\Theta(\varrho)-\Theta(\tau)}Q_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(\omega(\Theta(\varrho)-\Theta(\tau))^{\mu},\underline{a},\underline{b},p')$. If Θ and $\frac{\phi}{\varrho}$ are increasing functions, one can note that for $u<\tau<v, u,v\in[\xi_1,\xi_2]$, the kernel $\Lambda_{\tau}^{u}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)$ satisfies the forthcoming inequality:

$$\Lambda_{\tau}^{u}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\tau) \le \Lambda_{v}^{u}(M_{\alpha,\beta,\gamma,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\tau). \tag{13}$$

Moreover, the forthcoming inequalities hold which will be utilized to prove the results of this paper:

$$\Lambda_{\varrho}^{\tau}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\tau) \le \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\tau),\ \tau\in(\xi_1,\varrho), \tag{14}$$

$$\Lambda_{\tau}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\tau) \le \Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\tau),\ \tau\in(\varrho,\xi_2), \tag{15}$$

$$\Lambda_{\varrho}^{\xi_1}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\varrho) \le \Lambda_{\xi_2}^{\xi_1}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\varrho),\ \ \varrho\in(\xi_1,\xi_2), \tag{16}$$

$$\Lambda_{\xi_2}^{\varrho}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\varrho) \le \Lambda_{\xi_2}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Theta;\phi)\Theta'(\varrho),\ \ \varrho\in(\xi_1,\xi_2). \tag{17}$$

Convex functions have wide range of applications in various subjects of different fields including physics, mathematics, statistics, optimization, graph theory and economics. In mathematical analysis, integral and discrete versions of classical inequalities are studied very frequently. A celebrated Jensen's inequality is an analytic approach to define a convex function. Many classical inequalities are straightforward consequences of the Jensen inequality. Because of analytical representation of a convex function, the notion of convexity is extended and generalized in many terms. For instance m-, s-, h-, and many other convexities were defined by preserving an analytic inequality satisfied by a convex function, see [9–12].

New defined convexities along with fractional integrals have been utilized to obtain generalized and refined versions of classical inequalities by Hadamard, Ostrowski, Minkowski, Chebyshev, Grüss etc. One of the most extensively studied inequality is the Hadamard inequality. Many researchers have studied it for new classes of convex functions. For a detailed study on Hadamard inequalities for different types of convex functions, one can see [13–19]. Our aim in this paper is to establish some integral inequalities for refined $(\alpha,h-m)-p$-convex functions.

The definition of refined $(\alpha,h-m)-p$-convex function is given as follows.

Definition 4. *[20] Assume that $J\subseteq\mathbb{R}$, $I\subset(0,\infty)$ are intervals with $(0,1)\subset J$, and $h:J\to\mathbb{R}$ is a non-negative function. A function $\psi:I\to\mathbb{R}$ is called refined $(\alpha,h-m)-p$-convex function if the forthcoming inequality holds*

$$\psi((\tau\varrho^p+m(1-\tau)y^p)^{\frac{1}{p}}) \le h(\tau^{\alpha})h(1-\tau^{\alpha})(\psi(\varrho)+m\psi(y)), \tag{18}$$

where $p\in\mathbb{R}\backslash\{0\}$, $(\tau\varrho^p+m(1-\tau)y^p)^{\frac{1}{p}}\in I, \tau\in(0,1)$, $(\alpha,m)\in[0,1]^2$.

From the above definition, one can obtain the definitions of refined $(\alpha,h-m)$-, refined (p,h)-, refined (s,m)-, refined (α,m)-convexities along with many classes of refined convexities. Let we denote class of refined $(\alpha,h-m)$-p-convex functions defined over I by $R_{\alpha}^{h-m}C_p(I)$. A function $\psi\in R_{\alpha}^{h-m}C_p(I)$ satisfies the forthcoming inequalities which will be applied to establish the main results of this paper in the forthcoming section:

$$\psi(\tau^{\frac{1}{p}}) \le h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left(\psi(\xi_1^{\frac{1}{p}})+\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right), \tag{19}$$

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$$\psi(\tau^{\frac{1}{p}}) \leq h\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha} h\left(1-\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}\right)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right), \tag{20}$$

$$\psi(\varrho^{\frac{1}{p}}) \leq h\left(\frac{\varrho-\xi_1}{\xi_2-\xi 1}\right)^{\alpha} h\left(1-\left(\frac{\varrho-\xi_1}{\xi_2-\xi 1}\right)^{\alpha}\right)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right). \tag{21}$$

In upcoming Section 2, we prove Theorem 1 by applying the inequalities (19), (20), (14) and (15). In the same section Theorem 3 is established by using inequalities (21), (16), (17) and the Lemma 1, Theorem 2 is established by using inequalities (14) and (15). Consequences of each result are explained at the end of proofs. In Section 3, we give some Hadamard type inequalities. Throughout the paper we assume that all the notions described in Section 1 are valid.

2. Estimations of operators with unified ML Kernels

Theorem 1. *Let $\psi \in R_{\alpha}^{h-m}C_p(I)$ be a positive and integrable over $[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in I$. Then for operators (8) and (9) we have:*

$$\begin{aligned}
&\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\varrho;p') + \left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\gamma,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\varrho;p') \\
&\leq \Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\varrho-\xi_1)\left(\phi(\xi_1^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)N_{\varrho}^{\xi_1}(r^{\alpha},h;\Theta') \\
&+\Lambda_{\xi_2}^{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\xi_2-\varrho)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)N_{\varrho}^{\xi_2}(z^{\alpha},h;\Theta'),
\end{aligned} \tag{22}$$

where $N_{\varrho}^{x}(u^{\alpha},h;\Theta') = \int_0^1 h(u^{\alpha})h(1-u^{\alpha})\Theta'(\varrho-u(\varrho-x))du$ and $\chi(t)=t^{\frac{1}{p}}$.

Proof. The following integral inequality can be obtained from inequalities (19) and (14):

$$\begin{aligned}
&\int_{\xi_1}^{\varrho}\Lambda_{\varrho}^{\tau}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)\psi(\tau^{\frac{1}{p}})d(\Theta(\tau)) \leq \Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi) \\
&\times\left(\psi(\xi_1^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)\int_{\xi_1}^{\varrho}h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)d\tau.
\end{aligned} \tag{23}$$

In above inequality by setting $r=\dfrac{\varrho-\tau}{\varrho-\xi_1}$ in right hand side while in the left hand side by using Definition 3, the forthcoming inequality is obtained

$$\begin{aligned}
&\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\varrho;p') \leq \Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\varrho-\xi_1) \\
&\times\left(\psi(\xi_1^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)\int_0^1 h(r^{\alpha})h(1-r^{\alpha})\Theta'(\varrho-r(\varrho-\xi_1))dr.
\end{aligned} \tag{24}$$

Hence the following estimate of left sided integral operator is yielded:

$$\begin{aligned}
&\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\varrho;p') \leq \Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi) \\
&\times(\varrho-\xi_1)\left(\psi(\xi_1^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)N_{\varrho}^{\xi_1}(r^{\alpha},h;\Theta').
\end{aligned} \tag{25}$$

On the other hand following integral inequality can be obtained from inequalities (20) and (15).

$$\begin{aligned}
&\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\varrho;p') \leq \Lambda_{\xi_2}^{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\xi_2-\varrho) \\
&\times\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)\int_0^1 h(z^{\alpha})h(1-z^{\alpha})\Theta'(\varrho+z(\xi_2-\varrho))dz.
\end{aligned} \tag{26}$$

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Hence the following estimate of right sided integral operator is yielded:

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\varrho;p') \leq \Lambda^{\varrho}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi) \times(\xi_2-\varrho)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)N^{\xi_2}_{\varrho}(z^{\alpha},h;\Theta'). \tag{27}$$

By adding the inequalities (25) and (27), one can get the required inequality (22). □

Remark 1. *By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$ in (22), then [21, Theorem 2.1] is obtained. If $0<h(t)<1$, then one can get [22, Theorem 1].*

Corollary 1. *By setting $\kappa=\vartheta$ in (22), the following result is obtained*

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\varrho;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\gamma,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\varrho;p')$$
$$\leq \Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\varrho-\xi_1)\left(\phi(\xi_1^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)N^{\xi_1}_{\varrho}(r^{\alpha},h;\Theta')$$
$$+\Lambda^{\varrho}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\xi_2-\varrho)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)N^{\xi_2}_{\varrho}(z^{\alpha},h;\Theta'),$$

Corollary 2. *By taking $h(\tau)=\tau$ in (22), we get the inequality for functions belong to the class $R^{I-m}_{\alpha}C_p(I)$:*

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\varrho;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\gamma,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\varrho;p')\leq(\varrho-\xi_1)$$
$$\times\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)\left(\phi(\xi_1^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)\int_0^1 r^{\alpha}(1-r^{\alpha})\Theta'(\varrho-r(\varrho-\xi_1))dr$$
$$+\Lambda^{\varrho}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\xi_2-\varrho)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right)\int_0^1 r^{\alpha})(1-r^{\alpha})\Theta'(\varrho-r(\varrho-\xi_1))dr.$$

Theorem 2. *Let ψ be differentiable function such that $|\psi'|\in R^{I-m}_{\alpha}C_p(I)$. Then for integral operators (8) and (9) we have:*

$$\left|\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}(\psi*\Lambda)\circ\chi\right)(\varrho;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}(\psi*\Lambda)\circ\chi\right)(\varrho;p')\right|$$
$$\leq(\varrho-\xi_1)\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)N^{\xi_1}_{\varrho}(r^{\alpha},h;\Theta')$$
$$+\Lambda^{\varrho}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu'},\Theta;\phi)(\xi_2-\varrho)\left(|\psi'(\xi_2^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)N^{\xi_2}_{\varrho}(r^{\alpha},h;\Theta'), \tag{28}$$

where

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}(\psi*\Lambda)\circ\chi\right)(\varrho;p'):=\int_a^{\varrho}\Lambda^{\tau}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,'},\Theta;\phi)\psi'\left((\tau)^{\frac{1}{p}}\right)d(\Theta(\tau)),$$
$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}(\psi*\Lambda)\circ\chi\right)(\varrho;p'):=\int_{\varrho}^{\xi_2}\Lambda^{\varrho}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,'},\Theta;\phi)\psi'\left((\tau)^{\frac{1}{p}}\right)d(\Theta(\tau)).$$

Proof. It is given that $|\psi'|$ is in the class $R^{I-m}_{\alpha}C_p(I)$, therefore one can have

$$|\psi'\left(\tau^{\frac{1}{p}}\right)|\leq h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right). \tag{29}$$

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The inequality (29) can takes the following form

$$-h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)\leq\psi'\left(\tau^{\frac{1}{p}}\right) \tag{30}$$
$$\leq h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right).$$

From the inequality (30), we have

$$\psi'\left(\tau^{\frac{1}{p}}\right)\leq h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right). \tag{31}$$

Integrating the product of inequalities (14) and (31) over $[\xi_1,\varrho]$, one yield

$$\int_{\xi_1}^{\varrho}\Lambda_{\varrho}^{\tau}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)\psi'\left((\tau)^{\frac{1}{p}}\right)d(\Theta(\tau))\leq\Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)$$
$$\times\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)\int_{\xi_1}^{\varrho}h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)d(\tau).$$

Which gives

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}(\psi*\Lambda)\circ\chi\right)(\varrho;p')\leq\Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)(\varrho-\xi_1) \tag{32}$$
$$\times\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)N_{\varrho}^{\xi_1}(r^{\alpha},h;\Theta').$$

Similarly, from other part of inequality (30), one can have

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}(\psi*\Lambda)\circ\chi\right)(\varrho;p')\geq-\Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)(\varrho-\xi_1) \tag{33}$$
$$\times\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)N_{\varrho}^{\xi_1}(r^{\alpha},h;\Theta').$$

From (32) and (33), forthcoming inequality is observed

$$\left|\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}(\psi*\Lambda)\circ\chi\right)(\varrho;p')\right|\leq\Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi) \tag{34}$$
$$\times(\varrho-\xi_1)\left(|\psi'(\xi_1^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)N_{\varrho}^{\xi_1}(r^{\alpha},h;\Theta').$$

Now, $|\psi'|\in R_{\alpha}^{I-m}C_p(I)$ hence we have

$$|\psi'\left(\tau^{\frac{1}{p}}\right)|\leq h\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}h\left(1-\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}\right)\left(|\psi'(\xi_2^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right). \tag{35}$$

Now, similarly for (14) and (29), from (15) and (35), one can have

$$\left|\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}(\psi*\Lambda)\circ\chi\right)(\varrho;p')\right|\leq\Lambda_{\xi_2}^{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)(\xi_2-\varrho) \tag{36}$$
$$\times\left(|\psi'(\xi_2^{\frac{1}{p}})|+m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)N_{\varrho}^{\xi_2}(r^{\alpha},h;\Theta').$$

The inequality (28) is the sum of (34) and (36). □

Remark 2. *By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$ in (28), one can get [21, Theorem 2.1]. If $0<h(t)<1$, one can get [22, Theorem 1].*

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Corollary 3. *By taking $\kappa = \vartheta$ in (28) one can get the forthcoming inequality:*

$$\left| \left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}(\psi * \Lambda) \circ \chi\right)(\varrho; p') + \left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}(\psi * \Lambda) \circ \chi\right)(\varrho; p') \right|$$
$$\leq (\varrho - \xi_1)\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi)\left(|\psi'(\xi_1^{\frac{1}{p}})| + m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right) N^{\xi_1}_{\varrho}(r^{\alpha}, h; \Theta')$$
$$+ \Lambda^{\varrho}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi)(\xi_2 - \varrho)\left(|\psi'(\xi_2^{\frac{1}{p}})| + m\left|\psi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right) N^{\xi_2}_{\varrho}(r^{\alpha}, h; \Theta').$$

Next, lemma is necessary for the proof of upcoming theorem.

Lemma 1. *[21] Let $\psi \in R^{I-m}_{\alpha}C_p(I)$, and $m \in (0,1]$, $0 < \xi_1 < m\xi_2$, where ξ_1, $\xi_2 \in I$. If*

$$\psi(\varrho^{\frac{1}{p}}) = \psi\left(\left(\frac{\xi_1^p + \xi_2^p - \varrho}{m}\right)^{\frac{1}{p}}\right), \tag{37}$$

then the forthcoming inequality is valid:

$$\psi\left(\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}}\right) \leq h\left(\frac{1}{2^{\alpha}}\right) h\left(\frac{2^{\alpha} - 1}{2^{\alpha}}\right)(1 + m)\psi(\varrho^{\frac{1}{p}}). \tag{38}$$

The following theorem gives the Hadamard inequality.

Theorem 3. *Let $\psi \in R^{h-m}_{\alpha}C_p(I)$ be a positive and integrable over $[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in I$, and (37) holds. Then for operators (8) and (9) we have:*

$$\frac{\psi\left(\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^{\alpha}}\right) h\left(\frac{2^{\alpha} - 1}{2^{\alpha}}\right)(m + 1)} \left(\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma\mu,\nu,\xi_2^-}1\right)(\xi_1; p') + \left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma\mu,\nu,\xi_1^+}1\right)(\xi_2; p')\right) \tag{39}$$
$$\leq \left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi \circ \chi\right)(\xi_2; p') + \left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi \circ \chi\right)(\xi_1; p') \leq (\xi_2 - \xi_1)$$
$$\times (\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi) + \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi))\left(\psi(\xi_2^{\frac{1}{p}}) + m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right) N^{\xi_1}_{\xi_2}(r^{\alpha}, h; \Theta').$$

Proof. The kernel (10) satisfies inequality (16), and $\psi \in R^{h-m}_{\alpha}C_p(I)$ satisfies the inequality (21). From inequalities (21) and (16), one can get the forthcoming inequality:

$$\int_{\xi_1}^{\xi_2} \Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi)\psi((\varrho)^{\frac{1}{p}})d(\Lambda(\varrho)) \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi)$$
$$\times \left(\psi(\xi_2^{\frac{1}{p}}) + m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right) \int_{\xi_1}^{\xi_2} h\left(\frac{\varrho - \xi_1}{\xi_2 - \xi_1}\right)^{\alpha} h\left(1 - \left(\frac{\varrho - \xi_1}{\xi_2 - \xi_1}\right)^{\alpha}\right) d\varrho.$$

By setting $r = \frac{\varrho - \xi_1}{\xi_2 - \xi_1}$, and using Definition 3, in the aforementioned inequality one can get

$$\left({}^{\phi}_{\Lambda}Y^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi \circ \chi\right)(\xi_1; p') \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Theta; \phi)(\xi_2 - \xi_1) \tag{40}$$
$$\times \left(\psi(\xi_2^{\frac{1}{p}}) + m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right) \int_0^1 h(r^{\alpha})h(1 - r^{\alpha})\Theta'(\xi_1 + r(\xi_2 - \xi_1))dr.$$

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From aforementioned inequality one can get the forthcoming inequality:

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\xi_1;p') \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi) \times(\xi_2-\xi_1)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right)N^{\xi_1}_{\xi_2}(r^{\alpha},h;\Theta'). \tag{41}$$

Similarly, the forthcoming inequality can be yielded from (17) and (21):

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\xi_1;p') \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi) \times(\xi_2-\xi_1)\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right)N^{\xi_1}_{\xi_2}(r^{\alpha},h;\Theta'). \tag{42}$$

By adding (41) and (42), we get the forthcoming inequality

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\xi_2;p')+\left({}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\xi_1;p') \leq (\xi_2-\xi_1) \times((\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)))\left(\psi(\xi_2^{\frac{1}{p}})+m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right) \times N^{\xi_1}_{\xi_2}(r^{\alpha},h;\Theta'). \tag{43}$$

Now, we multiply inequality (38) on both sides by $\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)\Theta'(\varrho)$ and integrate over $[\xi_1,\xi_2]$, to obtain the upcoming inequality

$$\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)\int_{\xi_1}^{\xi_2}\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)d(\Theta(\varrho)) \leq (1+m) \times\left(h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\int_{\xi_1}^{\xi_2}\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)\psi(\varrho^{\frac{1}{p}})d(\Theta(\varrho)).$$

Definition 3 is used to get the forthcoming inequality:

$$\frac{\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\left({}^{\phi}_{\Lambda}\Upsilon^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p') \leq \left({}^{\phi}_{\Lambda}\Upsilon^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\xi_1;p'). \tag{44}$$

Now, multiplying both sides of (38) by $\Lambda^{\varrho}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)\Theta'(\varrho)$ and integrating the resulting inequality over $[\xi_1,\xi_2]$, we get

$$\frac{\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(m+1)}\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\alpha,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p') \leq \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\xi_2;p'). \tag{45}$$

From (43), (44) and (45), inequality (39) can be achieved. □

Remark 3. *By setting $n=1$, $b_1=\lambda+lk$, $a_1=\theta-\lambda$, $c_1=\lambda$, $\rho=\nu=0$ in (39), one can get [21, Theorem 2.4]. If $0<h(t)<1$, then one can get [22, Theorem 2].*

Corollary 4. *By setting $\kappa = \vartheta$ in (39), the forthcoming inequality is obtained:*

$$\frac{\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)(m+1)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma\mu,\nu,\xi_2^-}1\right)(\xi_1;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)$$
$$\leq \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\circ\chi\right)(\xi_2;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\circ\chi\right)(\xi_1;p')$$
$$\leq 2(\xi_2-\xi_1)\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Theta;\phi)\left(\psi(\xi_2^{\frac{1}{p}}) + m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right)N^{\xi_1}_{\xi_2}(r^\alpha,h;\Theta').$$

3. Some deduced results

We deduce some particular cases for classes $R_1^{h-m}C_p(I)$, $R_\alpha^{I-m}C_p(I)$, and $R_\alpha^{h-1}C_p(I)$.

Theorem 4. *Let ψ is in the class of functions $R_\alpha^{h-1}C_p(I)$. Then from (39) one can yield:*

$$\frac{\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{2h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)$$
$$\leq \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\right)(\xi_2;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\right)(\xi_1;p') \leq (\xi_2-\xi_1)$$
$$\times(\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Theta;\phi)+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Theta;\phi))\left(\psi(\xi_2^{\frac{1}{p}}) + \psi\left(\xi_1^{\frac{1}{p}}\right)\right)N^{\xi_1}_{\xi_2}(r^\alpha,h;\Theta').$$

Theorem 5. *Let ψ is in the class of functions $R_1^{h-m}C_p(I)$. Then from (39) one can yield:*

$$\frac{\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h^2\left(\frac{1}{2}\right)(1+m)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p') + \left({}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)$$
$$\leq \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\right)(\xi_2;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\right)(\xi_1;p') \leq (\xi_2-\xi_1)$$
$$\times(\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Theta;\phi)+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Theta;\phi))\left(\psi(\xi_2^{\frac{1}{p}}) + m\psi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\right)N^{\xi_1}_{\xi_2}(r,h;\Theta').$$

Theorem 6. *Let ψ is in the class of functions $R_\alpha^{I-m}C_p(I)$. Then from (39) one can yield:*

$$\frac{2^{2\alpha}\psi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{(2^\alpha-1)(m+1)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)$$
$$\leq \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\psi\right)(\xi_1;p') + \left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\psi\right)(\xi_2;p') \leq (\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Theta;\phi)$$
$$+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Theta;\phi))\int_0^1 r^\alpha(1-r^\alpha)\Psi'(\xi_2 - r(\xi_2-\xi_1))dr.$$

4. Conclusion

We investigated fractional inequalities for unified ML functions by utilizing fractional integral operators. These inequalities were analyzed for the class of functions denoted by $R_\alpha^{h-m}C_p(I)$. In particular cases, refinements of well known integral inequalities that had been published in recent past can be deducted. We presented some deductions for specific classes $R_1^{h-m}C_p(I)$, $R_\alpha^{I-m}C_p(I)$, and $R_\alpha^{h-1}C_p(I)$.

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