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Abstract: Input reconstruction problem is usually encountered in the areas such as virtual sensing, image restoration, sensor linearization, and communications. The input reconstruction problem can be seen as an inverse problem. Given a nominal system, two types of approaches are commonly used for solving the inverse problem. The first one is to directly invert the nominal system, and the second one is to derive an inversion of the nominal system in an indirect way. However, for the first type of approaches, system inversion cannot be directly conducted under some conditions such as there exist nonminimum-phase zeros in the nominal system, while for the second type of approaches, simultaneously guaranteeing stability and causality of the obtained inversion is still not solved well. In order to avoid the drawbacks existing in the two types of approaches, an alternative approach is proposed for input reconstruction. In this approach, the input signal is first modeled as the output of a state-space model, afterwards two Kalman filters for the model resulted by combining the input signal model and the nominal system model are implemented alternatively such that the input signal can be sequentially reconstructed in an infinite horizon. The proposed input reconstruction approach is applied in two examples, and simulation results can illustrate the effectiveness of the proposed approach.

Keywords: Kalman filtering; input reconstruction; inverse modeling; inverse problems; virtual sensing

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1. Introduction

Inverse problems are important problems in both science and engineering [1,2], such as virtual sensing [3], image processing (e.g., image restoration [4]), sensor linearization [5], and digital predistortion for radio frequency communications [5]. In the point of view of system identification, there are two kinds of inverse problems [6]:

(i) Reconstruct the system inputs based on the system outputs and the inverse system model, which is also called the input reconstruction problem or the inverse system identification problem, see Figure 1(a).

(ii) Identify the forward system model based on the input-output data, which is a normal system identification problem, see Figure 1(b).

(a) Input reconstruction   **(b)** Model identification

Figure 1. Inverse problems (u: input; y: output; H: inverse system model; G: forward system model).

In this paper, the first type of inverse problems (i.e., the input reconstruction problem) is investigated. Generally, there are two types of approaches for solving the first kind of inverse problems:

(i) The first approach is to make a direct inversion of the nominal system firstly, and then input reconstruction can be conducted. Denote the transfer function of a discrete-time model as $G(z)$ of which a state-space realization is (A, B, C, D), when the inverse of the feedthrough term D does not exist, the direct inversion of the model $G(z)$ cannot be conducted [7,8]. In addition, if

there exist nonminimum-phase zeros in $G(z)$, an unstable inversion solution will be obtained [9]. So in practical applications of direct inversion approaches,

(ii) The second approach is to obtain an inverse system model of the nominal system model indirectly, and input reconstruction can then be realized. However, in order to obtain a stable inversion, there are a number of drawbacks in existing approaches:

  (a) For exact stable inversion approaches, an infinite preview of the desired output is needed [10–13], which is not applicable in practice.
  (b) For the approaches called pseudo-inversion, even though they do not need preview, however, they will encounter other problems such as the difficulty of choosing a suitable basis function [14,15].
  (c) For stable and unstable poles separation-based inversion methods, a finite preview is necessary [16,17].
  (d) For norm-based inversion methods, some of them suffer the problems such as input noise [5] and non-convex optimization [18,19] which often occur in system identification, while for H_∞ norm-based methods [9,20–23], a finite preview is needed.
  (e) For signal modelling-based inversion methods, the input signal, which is to be reconstructed, must be a periodic signal under stationary operating conditions [24,25].

As seen, for some indirect system inversion approaches, even though a stale inversion can be obtained, an infinite or a finite pre-actuation is still needed, which cannot be applied well in practice because sometimes the desired output in unknown. In order to solve the input reconstruction problem in a better way, in this paper an alternative approach is proposed. The presented approach can guarantee the stability of the input reconstructor, and simultaneously the proposed approach does not need any pre-actuation; Moreover, the approach can be applied to stable or unstable, proper or improper systems[1] with input to be reconstructed, and there is also no requirement for the type of input and output signals; Furthermore, it does not suffer non-convex or input noise problems.

The remainder of the paper is organized as follows. In Section 2, the modeling of signals with finite-length is introduced, based on which in Section 3 an alternative recursive Kalman filter-based input reconstruction approach is proposed. In Section 4, the performance of the proposed approach is verified and analyzed, and finally conclusions and future perspectives are given in Section 5.

2. Limited-Length Signal Modeling

In this section, the modeling of limited-length signals is illustrated. With the idea of Limited-length signal modeling, an alternative input reconstruction approach is proposed in Section 3.

Given a discrete-time signal $u(k) \in \mathbb{R}$ with length N and the sampling period T_s in seconds, now assume that the limited-length signal $u(k)$ is a whole period of a periodic signal $u_p(k) \in \mathbb{R}$, so within the length N, the signal $u(k)$ can be represented as the output of the following state-space model [24], i.e.,

$$\begin{cases} x_u(k+1) = A_u x_u(k), \\ u(k) = C_u x_u(k) + v_u(k), \end{cases} \tag{1}$$

where $x_u(k) \in \mathbb{R}^{n_u}$ denotes the state vector, the term $v_u(k)$ denotes the modeling error which is induced by the limited dimension of the matrix A_u, and the matrices A_u and C_u can be represented as

$$A_u = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & A_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{n_u} \end{pmatrix} \tag{2}$$

---

[1] The system is proper when the degree of the numerator does not exceed the degree of the denominator of its transfer function, otherwise the system is improper.

and

$$C_{\mathrm{u}} = \begin{pmatrix} 1 & | & C_1 & \cdots & C_{n_{\mathrm{u}}} \end{pmatrix}, \tag{3}$$

respectively, where $\mathbf{0}$ denotes a zero vector or a zero matrix, and the individual block entries in these block matrices are

$$A_i = \begin{pmatrix} \cos(2\pi i f_{\mathrm{b}} T_{\mathrm{s}}) & \sin(2\pi i f_{\mathrm{b}} T_{\mathrm{s}}) \\ -\sin(2\pi i f_{\mathrm{b}} T_{\mathrm{s}}) & \cos(2\pi i f_{\mathrm{b}} T_{\mathrm{s}}) \end{pmatrix} \tag{4}$$

and

$$C_i = \begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{5}$$

where $f_{\mathrm{b}} = \frac{1}{NT_{\mathrm{s}}}$, for $i = 1, 2, \ldots, n_{\mathrm{u}}$.

In the end of this section, the following assumption is made.

Assumption 1. *The sequence $\{v_{\mathrm{u}}(k)\}$ is assumed to be a white-noise sequence. The covariance function of the white-noise sequence $\{v_{\mathrm{u}}(k)\}$ is $\sigma^2 \delta_{kj}$ where σ^2 is a positive and constant scalar value, and δ_{kj} is the Kronecker Delta function depending on two integral numbers k and j [26]:*

$$\delta_{kj} = \begin{cases} 0, & if \quad k \neq j, \\ 1, & if \quad k = j. \end{cases} \tag{6}$$

3. Input Reconstruction Approach

In this section, an approach solving the reconstruction problem of an infinite-length input is derived.

Consider the following discrete-time, linear, time-invariant model which is minimal-realized[2] and proper[3]:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k) + Du(k) + v(k), \end{cases} \tag{7}$$

where $u(k) \in \mathbb{R}$, $y(k) \in \mathbb{R}^p$, and $x(k) \in \mathbb{R}^n$ denote the input, the output, and the state vector, respectively. The matrices A, B, C, D are the state matrix, the input matrix, the output matrix, and the feedthrough matrix, respectively. The term $v(k) \in \mathbb{R}^q$ represents a noise term.

The sampling period of the discrete-time model (2) is the same as the sampling period of the model (1), i.e., T_{s} in seconds. The model (7) can be stable or unstable.

In the model (7), the input signal $u(k)$ is with length N, then $u(k)$ can be modeled, see the modeling process in Section 2. By augmenting the state vector $x_{\mathrm{u}}(k)$ of the model (1) with the state vector $x(k)$ of the model (7), we can obtain the following augmented model:

$$\begin{cases} x_{\mathrm{a}}(k+1) = A_{\mathrm{a}} x_{\mathrm{a}}(k) + v_1(k), \\ y(k) = C_{\mathrm{a}} x_{\mathrm{a}}(k) + v_2(k), \end{cases} \tag{8}$$

where the state vector

$$x_{\mathrm{a}}(k) = \begin{pmatrix} x_{\mathrm{u}}(k) \\ x(k) \end{pmatrix} \in \mathbb{R}^{n_{\mathrm{a}}}, \tag{9}$$

and the matrices

$$A_{\mathrm{a}} = \begin{pmatrix} A_{\mathrm{u}} & 0 \\ BC_{\mathrm{u}} & A \end{pmatrix} \tag{10}$$

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[2] A system is minimal-realized if and only if it is both controllable and observable.
[3] Actually, the proposed input reconstruction approach is not limited to proper systems, the approach can also be used for improper systems by replacing the present input $u(k)$ by future input in (7).

and

$$C_{\mathrm{a}} = \begin{pmatrix} DC_{\mathrm{u}} & C \end{pmatrix}, \tag{11}$$

and the noise terms

$$v_1(k) = \begin{pmatrix} \mathbf{0} \\ Bv_{\mathrm{u}}(k) \end{pmatrix} \tag{12}$$

and

$$v_2(k) = Dv_{\mathrm{u}}(k) + v(k). \tag{13}$$

The following assumptions are made for the above noise terms.

Assumption 2. *The distribution of the initial state vector $x_{\mathrm{a}}(0)$ is Gaussian.*

Assumption 3. *The sequence $\{v(k)\}$ is assumed to be a white-noise sequence, and the covariance matrix of the white noise sequence $\{v(k)\}$ is $R\delta_{kj}$ with R a positive and constant matrix.*

Assumption 4. *The sequences $\{v_1(k)\}$ and $\{v_2(k)\}$ are uncorrelated with the initial state vector $x_{\mathrm{a}}(0)$. The covariance matrices of $\{v_1(k)\}$ and $\{v_2(k)\}$ are denoted as $Q_{\mathrm{a}}\delta_{kj}$ and $R_{\mathrm{a}}\delta_{kj}$, respectively. With the values of σ^2 and R, the values of Q_{a} and R_{a} can be calculated according to (12) and (13).*

Below Section 3.1 and Section 3.2 are used to demonstrate the idea of the proposed input reconstruction approach. Section 3.1 illustrates how a limited-length input can be reconstructed, based on which a recursive reconstruction approach of an finite-length input is derived.

3.1. Limited-Length Input Reconstruction

Based on the above analysis and assumptions, the Kalman filter for the model (8) can be implemented [26]. Denote the conceptual time-varying transfer operator of the Kalman filter for the model (8) as $K_{\mathrm{f}}(q^{-1}, k)$, and then

$$\hat{x}_{\mathrm{a}}(k) = K_{\mathrm{f}}(q^{-1}, k)y(k), \tag{14}$$

where the notation "ˆ" denotes the estimate, and then

$$\hat{u}(k) = C_{\mathrm{r}}\hat{x}_{\mathrm{a}}(k), \tag{15}$$

where $C_{\mathrm{r}} = \begin{pmatrix} C_{\mathrm{u}} & \mathbf{0} \end{pmatrix}$, for $k = 0, 1, \dots, N$.

According to (9), the dimension of the state vector $x_{\mathrm{a}}(k)$ is

$$n_{\mathrm{a}} = n + 2f_{\mathrm{m}}NT_{\mathrm{s}} + 1, \tag{16}$$

where f_{m} in Hz denotes the frequency corresponding to the largest frequency component number (i.e., n_{u}) of the signal which is denoted as

$$u_{\mathrm{m}}(k) = u(k) - v_{\mathrm{u}}(k). \tag{17}$$

3.2. Recursive Input Reconstruction Algorithm

In Section 3.1, the Kaman filter for the model (8) can be successfully used for the reconstruction of the input signal $u(k)$ with length N, and it is clear that the drawbacks in existing indirect system inversion approaches mentioned in Section 1 can be avoided, however, the Kalman filter is merely effective in a finite-time horizon. Below a reconstruction approach of the input signal $u(k)$ with an infinite length is derived by involving another Kalman filter for the model (8), i.e., based on a collaboration of two Kalman filters, the reconstruction problem of an infinite-length signal can be solved.

The specific mechanism of how to use the two Kalman filters alternatively to solve the reconstruction problem of an infinite-length signal can be illustrated by one timing diagram in Figure 2 and three logic blocks in Figure 3:

(i) *Timing Diagram*

In Figure 2, the start and end time points of the two Kalman filters K_1 and K_2 are displayed. In more detail, for K_1, it starts from the step $k = 0$ and stops at the step $k = N - 1$, and then starts again from $k = N$ and stops at $k = 2N - 1$, and so forth. While K_2 starts from $k = c$ and stops at $k = N - 1 + c$, and then starts again from $k = N + c$ and stops at $k = 2N - 1 + c$, and so on. The two Kalman filters have overlapped working periods (e.g., the shading part in Figure 2), with the value c, the two Kalman filters K_1 and K_2 can be implemented alternatively by using the logic blocks shown in Figure 3 such that both transient and finite-length problem can be solved.

(ii) *Logic Block 1*

The logic block 1 is used for initializing the prediction process in K_1, i.e., at steps $k_1 N$, for $k_1 = 0, 1, 2, \ldots$, the vector $x_a(k)$ and the matrix $P(k)$ are forced to be $\mathbf{0}_{n_a}$ (i.e., an n_a-by-1 zero vector) and I_{n_a} (an n_a-by-n_a identity matrix) selected from the initial value bank, respectively.

(iii) *Logic Block 2*

The logic block 2 is used for the initialization of K_2, i.e., at steps $k_2 N + c$, for $k_2 = 0, 1, 2, \ldots$, the vector $x_a(k)$ and the matrix $P(k)$ are forced to be $\mathbf{0}_{n_a}$ and I_{n_a} selected from the initial value bank, respectively.

(iv) *Logic Block 3*

The logic block 3 is used for reconstructing the input signal $\hat{u}(k)$. As seen in (15), the reconstructed input signal $\hat{u}(k)$ can be calculated by using the estimate $\hat{x}_a(k)$, based on which the specific idea behind the logic block 3 is illustrated in (18):

$$\hat{u}(k) = \begin{cases} C_r \hat{x}_a^{(1)}(k), & \text{if} \quad k \in A_1, \\ C_r \hat{x}_a^{(2)}(k), & \text{if} \quad k \in A_2, \end{cases} \tag{18}$$

where $\hat{x}_a^{(1)}(k)$ denotes the state vector $x_a(k)$ estimated by K_1 while $\hat{x}_a^{(2)}(k)$ represents the state vector $x_a(k)$ estimated by K_2, the sets A_1 and A_2 are respectively represented as

$$A_1 = [0, N - 1 - d] \cup [k_3 N + c - d, (k_3 + 1)N - 1 - d], \tag{19}$$

for $k_3 = 1, 2, 3, \ldots$, and

$$A_2 = [k_4 N - d, k_4 N + c - 1 - d], \tag{20}$$

for $k_4 = 1, 2, 3, \ldots$, where d is a positive integer, the reason why d is involved is that in practice the part of not interest in $u(k)$ is usually unknown such that the signal model (1) is not accurately enough to represent the signal $u(k)$ with length N.
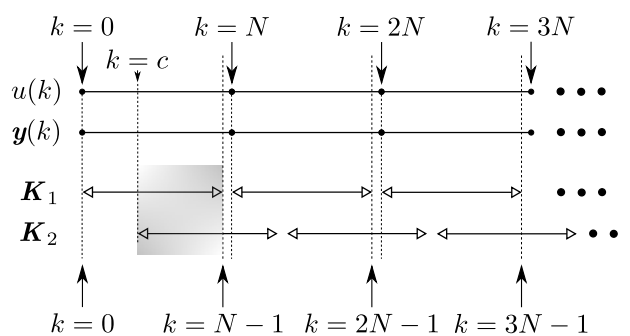

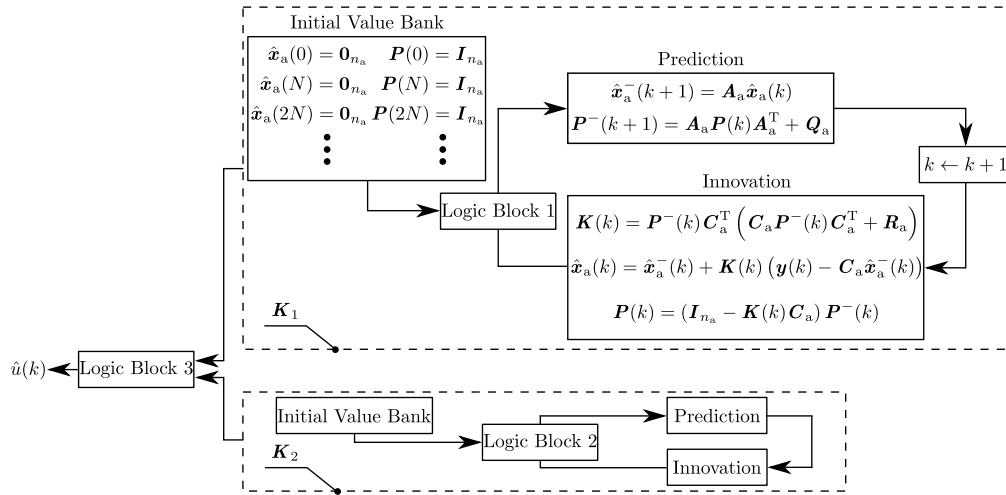**Figure 2.** Timing diagram (c and N are positive integers).

Figure 3. Input reconstruction (K_1: the first Kalman filter; K_2: the second Kalman filter; $\hat{x}_a^-(k)$: the a priori state estimate; $\hat{x}_a(k)$: a posteriori state estimate; $P^-(k)$: the a priori error variance; $P(k)$: the a posteriori error variance; $K(k)$: Kalman filter gain; The notation " $^{\mathrm{T}}$ ": transpose).

Based on the above analysis, the algorithm of reconstructing the input signal $u(k)$ with infinite length can be demonstrated in Algorithm 1, in which $K_1(q^{-1}, k)$ and $K_2(q^{-1}, k)$ denote the conceptual time-varying transfer operators of the Kalman filters K_1 and K_2, respectively, and the superscripts (1) and (2) are used for distinguishing between K_1 and K_2.

Algorithm 1: Algorithm of reconstructing input signal

    Initialization: $\hat{x}_a^{(1)}(0) = \mathbf{0}_{n_a}$, $P^{(1)}(0) = I_{n_a}$, $\hat{x}_a^{(2)}(0) = \mathbf{0}_{n_a}$, $P^{(2)}(0) = I_{n_a}$

1 $\hat{u}(0) = \hat{u}^{(1)}(0) = C_r \hat{x}_a^{(1)}(0)$

2 **for** $k = 0 \to \infty$ **do**

3   **if** $k = k_1 N$, for $k_1 = 0, 1, 2, \ldots$ **then**

4        $\hat{x}_a^{(1)}(k) = \mathbf{0}_{n_a}$, $P^{(1)}(k) = I_{n_a}$

5        $\hat{x}_a^{(1)}(k+1) = K_1(q^{-1}, k) y(k+1)$

6   **else**

7        $\hat{x}_a^{(1)}(k+1) = K_1(q^{-1}, k) y(k+1)$

8    $\hat{u}^{(1)}(k+1) = C_r \hat{x}_a^{(1)}(k+1)$

9   **if** $k = k_2 N + c$, for $k_2 = 0, 1, 2, \ldots$ **then**

10       $\hat{x}_a^{(2)}(k) = \mathbf{0}_{n_a}$, $P^{(2)}(k) = I_{n_a}$

11       $\hat{x}_a^{(2)}(k+1) = K_2(q^{-1}, k) y(k+1)$

12   **else**

13       $\hat{x}_a^{(2)}(k+1) = K_2(q^{-1}, k) y(k+1)$

14    $\hat{u}^{(2)}(k+1) = C_r \hat{x}_a^{(2)}(k+1)$

15   **if** $k + 1 \in A_1$ **then**

16       $\hat{u}(k+1) = \hat{u}^{(1)}(k+1)$

17   **if** $k + 1 \in A_2$ **then**

18       $\hat{u}(k+1) = \hat{u}^{(2)}(k+1)$

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4. Numerical Simulation and Analysis

In this section, the proposed input reconstruction approach is validated using two examples. In the first example, a randomly generated system is used for validating the effectiveness of the proposed approach. In the second example, a mass-spring-damper system is used to validate the effectiveness

4.1. Input Reconstruction of a Randomly Generated System

Firstly, the system matrices for the model (7) are randomly generated:

$$A = \begin{pmatrix} -0.4055 & -0.5366 & 0.1323 & 0.2417 \\ 0.0904 & -0.3823 & 0.5593 & 0.2120 \\ -0.5795 & -0.2297 & -0.5002 & 0.1414 \\ -0.1410 & 0.1588 & -0.2798 & -0.2501 \end{pmatrix}, \tag{21}$$

$$B = \begin{pmatrix} -1.0164 & 2.9668 & 0.0826 & 0.0870 \end{pmatrix}^{\mathrm{T}}, \tag{22}$$

$$C = \begin{pmatrix} -0.3544 & 0.7639 & 0.9837 & 1.4688 \\ -1.3393 & 0.4579 & 0 & 0.1052 \end{pmatrix}, \tag{23}$$

and

$$D = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\mathrm{T}}. \tag{24}$$

Secondly, choose a white noise sequence with the covariance function $\sigma_{\mathrm{w}}^2 \delta_{kj}$ with

$$\sigma_{\mathrm{w}}^2 = 1 \times 10^{-2}, \tag{25}$$

and then pass it through a 7th-order Butterworth low-pass filter with a cutoff frequency of 300 Hz, afterwards choose the filtered white noise as the input $u(k)$. Additionally,

Thirdly, set $\sigma^2 = 1 \times 10^{-2}$, and set the value of R to be

$$\begin{pmatrix} 1 \times 10^{-6} & 0 \\ 0 & 1 \times 10^{-6} \end{pmatrix}.$$

Fourthly, choose the sampling period T_{s} as 1×10^{-4} seconds, and then based on the model (7), the simulated output $y(k)$ can be obtained.

Finally, with the simulated output $y(k)$, based on using Algorithm 1, in which $f_{\mathrm{m}} = 300$ Hz, $N = 1000$, $c = 200$, $d = 50$, and $T_{\mathrm{s}} = 1 \times 10^{-4}$ seconds, the reconstructed input signal $\hat{u}(k)$ can be obtained.

The input reconstruction results from the step 800 to the step 1400 based on using Algorithm 1 is illustrated in Figure 4. As seen in Figure 4, by comparing the difference between $u(k)$ and $\hat{u}(k)$, it can be concluded that the proposed input reconstruction approach is effective, and it can be used for a signal with infinite length.
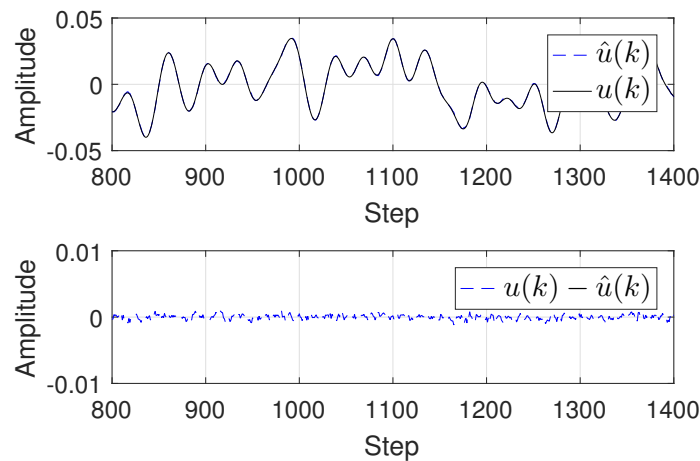

Figure 4. Reconstructed input.

In Figure 5, it can be seen that without the logic block 3 in the proposed approach, merely using K_1 or K_2 cannot reconstruct the input signal well, especially near the step 1000 and the step 1200.
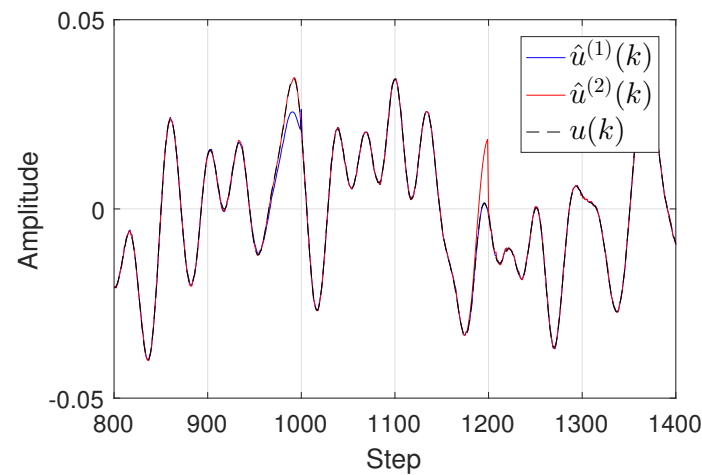

Figure 5. Input reconstruction results based on using one Kalman filter K_1 or K_2.

4.2. Virtual Input Force Sensing

Figure 6 displays a mass-spring-damper system which can be found in many mechanical systems. One common example is the suspension of a car. In this example, the input force u is reconstructed by using the displacement d_s of the mass m, i.e., a virtual input force sensor is formulated using the proposed input reconstruction approach and the data from the displacement sensor measuring the value of d_s.
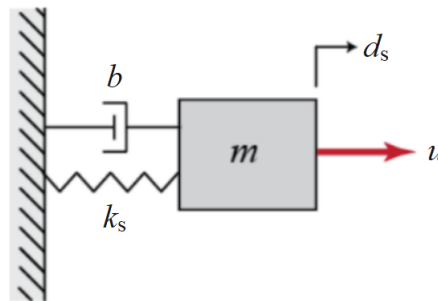

Figure 6. Mass-spring-damper system (k_s: the spring constant; b: the damping constant; m: the mass; d: the displacement; u: the input force).

The following numerical values to the variables m, k_s, and b in the mass-spring-damper system is shown in Table 1.

Table 1. Values of the variables.

Parameter	Value
m	1.0 kg
k_s	1.0 N/m
b	0.2 Ns/m

According to the above values, the continuous-time transfer function $G(s)$ between the input u and the output d_s can be obtained:

$$G(s) = \frac{1}{ms^2 + bs + k_s} = \frac{1}{s^2 + 0.2s + 1}. \tag{26}$$

Transform the continuous-time transfer function $G(s)$ into a discrete-time state-space model with the sampling period $T_s = 1 \times 10^{-4}$ seconds, then the system matrices of the state-space model can be obtained:

$$A = \begin{pmatrix} -0.2 & -1.0 \\ 1 & 0 \end{pmatrix}, \tag{27}$$

$$B = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}, \tag{28}$$

$$C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \tag{29}$$

and

$$D = 0. \tag{30}$$

Given the input force

$$u(k) = 10\sin(2\pi f_1 k T_s) + 5\sin(2\pi f_2 k T_s), \tag{31}$$

where the frequency $f_1 = 40$ Hz and the frequency $f_2 = 60$ Hz.

Set $\sigma^2 = 1 \times 10^{-2}$, and set $R = 1 \times 10^{-6}$, and choose the sampling period T_s as 1×10^{-4} seconds, and then based on the model (7), the simulated output $d_s(k)$ can be obtained. Finally, with the simulated output $d_s(k)$, based on using Algorithm 1, in which $f_m = 300$ Hz, $N = 1000$, $c = 200$, $d = 50$, and $T_s = 1 \times 10^{-4}$ seconds, the virtual input force sensor output $\hat{u}(k)$ can be obtained.

The virtual input force sensing results from the step 800 to the step 1400 based on using Algorithm 1 is illustrated in Figure 7. As seen in Figure 7, by comparing the difference between $u(k)$ and $\hat{u}(k)$, it can be concluded that the proposed input reconstruction approach is effective in the application of virtual sensing.

In Figure 8, it can be seen that without the logic block 3 in the proposed approach, merely using K_1 or K_2 cannot reconstruct the input signal well, especially near the step 1000 (not obvious) and the step 1200.
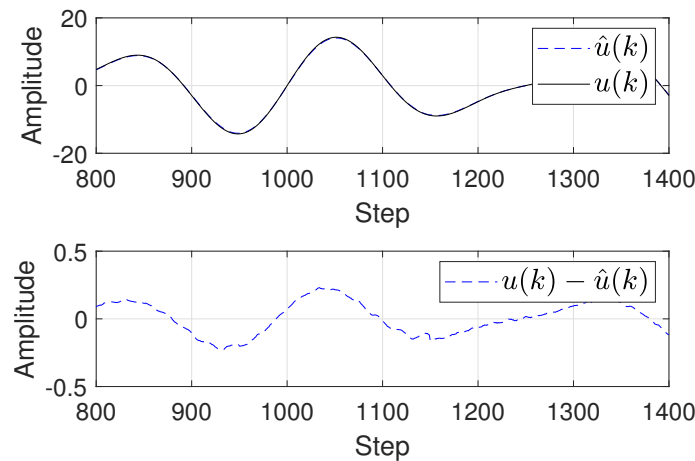

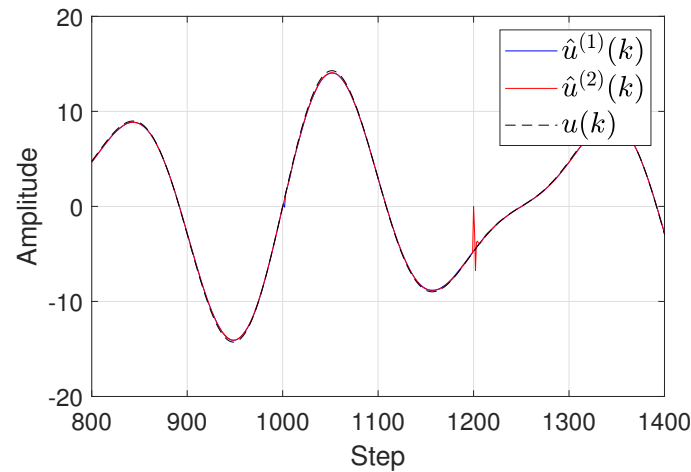**Figure 7.** Output of virtual input force sensor.

Figure 8. Virtual input force sensing based on using one Kalman filter K_1 or K_2.

According to the applications of Algorithm 1 in the above examples, direct inversion of the model (7) can be avoided, and only the detectability[4] of the model (8) should be guaranteed, and in addition, the proposed approach can avoid the drawbacks in existing indirect system inversion approaches mentioned in Section 1.

5. Conclusions and Perspectives

In this paper, an alternative approach for reconstructing infinite-length signal is proposed, and the proposed approach is prospective method for virtual sensing, sensor linearization and so on. The validation results from two examples can illustrate the effectiveness of the proposed approach. According to both theoretical derivation and numerical studies, it can be known that the proposed approach can avoid all the drawbacks existing in current input reconstruction approaches. However, because of the observability problem of the augmented model (8), the proposed approach can be only used for reconstructing a single input or multiple inputs without spectrum overlapping. In the future, the method will be extended to solve the multiple-input reconstruction problem and the problem of input reconstruction of nonlinear systems. Moreover, the effects from the parameters f_m, N, c, d, Q_a, and R_a, and the Kalman filter dimension problem will be explored.

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