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Consensus for Linear Time-Varying Multi-Agent Systems Via Event-Triggered Communication

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Abstract: This paper represents the first investigation into the consensus problem of linear time-varying multi-agent systems utilizing an event-triggered communication scheme. First, a general event-triggered consensus control scheme is proposed for a general category of linear time-varying multi-agent systems. Under some suitable assumptions, it is demonstrated that all agents' states will converge exponentially, with Zeno behaviour being ruled out. Second, the consensus problem in a network of linear time-varying multi-agent systems with a spanning tree is investigated using the proposed control strategy. It demonstrates that the consensus issue for the specified system can be reformulated as a stabilization problem for an error system through a time-varying linear transformation. Then, the event-triggered consensus problem is just a special instance of the general event-triggered consensus problem mentioned above. Finally, to illustrate the efficacy of the event-triggered method proposed in this study, simulation results are shown.

Keywords: consensus; time-varying linear multi-agent systems; event-triggered communication; exponential convergence; Zeno behavior; distributed cooperative adaptation

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1. Introduction

The event-triggered communication and control strategy requires systems to communicate each other and then update the control law only at the necessary instants instead of continuously. This strategy has been the subject of increasing interest among researchers and engineers due to its advantages in saving the energy or reducing the computation loads [1,2]. Recently, this approach has quickly become a main point of attention within the multi-agent systems field and is used to overcome the consensus problem in multi-agent systems. However, from our understanding, almost all controllers that are triggered by events are made to handle the time-invariant multi-agent systems. This work aims to address the consensus problem in linear time-varying multi-agent systems by means of an event-triggered communication strategy. It is not a simple extension and needs some novel analysis method developed by us.

In the early stage of control theory, analog control equipments require that controllers are executed continuously. Thus, the field of control systems design and analysis is primarily concerned with the continuous-time systems [3]. As computer technologies advance swiftly, the implement manner of controllers are changed to be in digital platforms instead of analog platforms, where the controller is executed periodically at fixed sampling instants. A significant challenge is to identify an appropriate sampling period. In general, as stated in [4], the selection of such a period is predicated on a worst-case scenario, with the objective of ensuring the efficacy of the control task across the full range of operational conditions. Consequently, the control task is executed at a uniform rate, irrespective of the state of the plant. This control scheme is called as a time-triggered control scheme. Its merit lies in the simplicity in analysis and design, but its drawback is also obvious, that is, this results in the unnecessary consumption of energy and the accelerated wear and tear of the actuators since frequent changes of the actuator state.

To overcome the above disadvantage of time-triggered control, as an alternative approach, there exist certain known strategies for event-trigger control [5–17]. In contrast to the time-triggered control strategy, the event-triggered control strategy engage the actuators only under specific conditions. Thus, one notable benefit of these schemes is their capacity to ensure both reliable operation and improve energy utilization efficiency across the target systems. Specifically, in [5], Tabuada proposes an event-triggered stabilizer based on Lyapunov for a particular category of nonlinear systems, where the continuous and centralized monitoring is needed but the inter-section time is more than a constant. Then, the continuous and decentralized monitoring scheme is further addressed in [6] and [7]. The findings of [5] are further extended upon in [4] to encompass the self-triggered case, and in [8] to address the tracking problem. For linear systems with continuous time, an event-trigger control approach that occurs periodically is displayed in [9], and in [10], discrete-time systems as a means of reducing the frequency of monitoring. To mitigate the impact of network characteristics like delay and quantization, networked systems are regulated using the event-triggered control technique in [11–14]. The discussion of deterministic equivalence within event-triggered control systems is addressed in [15]. The estimate of states and parameters is studied based on event-triggered scheme in [16] and [17], respectively. Up to now, the event-triggered control scheme is currently a focal point of interest within the control field. It have been extended to a variety of area, and lots of interesting results are emerging. Because multi-agent systems have so many applications in the military and business, there has been a lot of interest in this field. Up until this point, there have been many noteworthy achievements, see, e.g, [18–23], simply to mention a few. In general, multi-agent systems are characterized as distributed networks of interconnected agents. Thus, two issues should be considered, i.e., control and communication.

The study of the event-triggered consensus problem in multi-agent systems has been spurred due to the applicability of applying event-triggered techniques to networked systems, and numerous intriguing findings have been made [24–44]. In [24], authors study the centralized/distributed event-triggered consensus problem for first-order multi-agent systems, in scenarios where distributed event-triggered is implemented, it is guaranteed that no Zero behavior occurs for a minimum of one agent. The issue of consensus in distributed event-triggered systems is further explored in [25], specifically for multi-agent systems utilizing combinational measurements, with each agent autonomously deciding the moment of its event, and the Zeno behavior cannot appear before each agent reaches consensus. In [26,27], for a category of general linear multi-agent systems, two consensus protocols utilizing distributed event-triggered are presented, and it is assured that the bounded consensus error can rule out the Zeno behavior.

The event-triggered scheme in [27] has been expanded upon to the case of leader-following in [28]. In the context of general linear multi-agent systems, reference [29] presents a distributed observer-driven output-feedback event-triggered consensus framework. Furthermore, the output-feedback event-triggered consensus technique tailed for a passive multi-agent systems is also explored in reference [30]. To naturally prevent the Zeno behavior, the papers [31] and [32] present distributed event-triggered consensus schemes utilizing a sampling technique for both first-order multi-agent systems and general linear multi-agent systems. The similar idea is used in [33]. Different from [31–33], a decentralised event-triggered consensus scheme is formulated in [34] for first/second-order systems where the time between events is limited by a positive constant. The study presented in paper [42] examines the consensus problem among high-order multi-agent systems, focusing particularly on the impact of event-triggered control. Recently, the event-triggered consensus/synchronization scheme is further designed for nonlinear multi-agent systems in [31,45–49] and discrete-time multi-agent systems in [35–38,50]. However, the emphasis of these studies is on multi-agent systems that are not subject to time variation.

In practice, the system parameters or models might vary with different setting. Unfortunately, up to now, a limited number of studies have addressed the consensus issue in multi-agent systems that vary over time. The reason lies in that there are fewer methods and tools can be used to deal with such systems comparing with the linear time-invariant systems. The synchronization of outputs among a collection of linear and time-varying multi-agent systems is the subject of inquiry in [39]. However, the requirement of continuous communication limits its execution in practice. Thus, motivated by this observation and the development of event-triggered consensus of time-invariant multi-agent systems, we are particularly interested in researching the event-triggered consensus problem within a category of generic linear time-varying multi-agent systems. The main highlights of our contributions are outlined subsequently.

(i) We establish a general framework of event-triggered consensus control that is applicable to a broad category of linear time-varying multi-agent systems over networks. Exponential convergence of consensus errors is demonstrated, along with the prevention of any Zeno behavior within the system. Despite the existence of several studies concerning event-triggered consensus schemes of linear time-invariant multi-agent systems [24–33,35–37,39,51], in light of the information we have, this study marks the first instance of examining the event-triggered scheme within linear time-varying multi-agent systems.

(ii) We further apply the established general results to analyze the event-triggered consensus among a group of specific linear time-varying multi-agent systems by using the system transformation matrix that under assumption of the network with a spanning tree structure. It is assured that the consensus error of the systems will exponentially converge to zero.

This work's remaining content is presented in the following manner. A few introductions to the algebra of graph theory, linear time-varying systems and two essential lemmas are given in Section II. The scheme to consensus using event-triggered control for a general category of linear time-varying multi-agent systems is discussed in Section III. we discuss our proposed scheme to tackle the event-triggered consensus problem among a collection of linear time-varying multi-agent systems in Section IV. In Section V, we provide a summary of the research findings.

Notations: Throughout this paper, R is used to represented the collection of real numbers; R^n stands for the set of $n \times 1$ real vectors; $R^{n\times n}$ refers to the set of $n \times n$ real matrices; $\mathbf{1}_N$ represents an $N \times 1$ column vector filled with ones; I denotes the identity matrix with the appropriate dimensions; A^{T} refers to the transpose of a matrix A; $\max\{\cdots\}$ indicates the maximum value among the elements; $\sup(\cdot)$ stands for the supremum, which refers to the least upper bound of a set; $\otimes$ represents the Kronecker product; $\sigma_{\max}(P)$ and $\sigma_{\min}(P)$ refer to the biggest and lowest eigenvalue of a positive definite matrix P; $\mathrm{diag}\{A_1, \cdots, A_m\}$, where $A_i, i = 1, 2, \cdots, m$, are $p_i \times q_i$ matrices, is a $\Sigma_{i=1}^{m} p_i \times \Sigma_{i=1}^{m} q_i$ block diagonal matrix; The symbol $|z|$ indicates the modulus of a real number z; $\|\cdot\|$ denotes the Euclidean norm; $A \geq B$ means that $A - B$ is a positive semi-definite matrix.

2. Preliminary

In this section, we give some knowledge on algebraic graph theory, linear time-varying systems, and some important lemmas. The subsequent research will be bulit upon these foundational elements.

2.1. Algebraic Graph Theory

We model a communication network among agents by means of a graph in this research. A digraph of order N is specified as a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \cdots, N\}$ signifies a finite and nonempty set of agents and a collection of ordered pairs of agents comprising the edges is indicated with $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. It is important to motion that $\mathcal{G}$ is considered undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$ for arbitrary $i \in \mathcal{V}$ and $j \in \mathcal{V}$. The neighbors of agent i are indicated with $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$, and $j \in \mathcal{N}_i$ signifies that node i has the ability to directly obtain the information from agent j. A sequence $i_0, i_1, \cdots, i_l$ forms a path in a digraph. In a directed tree inside a digraph, every node except the root has precisely one parent, which is the only node without a parent and is connected to every other node directly through pathways. A graph's directed spanning tree is a tree structure that uses its directed edges to span every node. The graph is considered to contain a directed spanning tree if a portion of a graph's edges can form one.

In the adjacency matrix $\mathcal{B} = [b_{ij}]_{N\times N}$ of the digraph $\mathcal{G}$, each entry b_{ij} is assigned a positive weight if the (v_j, v_i) belongs to the edge set $\mathcal{E}$; and $b_{ij} = 0$, otherwise. Assume that each node doesn't

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have its own edge, i.e., $b_{ii} = 0$. The Laplacian matrix, represented by $\mathcal{L} := [d_{ij}] \in R^{N\times N}$, is defined with elements such that $d_{ij} = -b_{ij}$ when $i \neq j$ and $d_{ii} = \sum_{j=1}^{N} b_{ij}$; $\mathcal{L}$ is a zero row sum matrix, that is, $\mathcal{L}1_N = 0$. Let $r \in R^{N\times 1}$ satisfy $r^{\mathrm{T}}\mathcal{L} = 0$ and $r^{\mathrm{T}}\mathbf{1}_N = 1$, then for a diagraph having a spanning tree, r exists and is unique. For the Laplacian matrix of a network having a spanning tree, the subsequent lemma is introduced to support our discussion.

Lemma 1. *[39]: Given that a graph $\mathcal{G}$ possesses a spanning tree. It can be seen that there is a symmetric positive definite matrix P that meets the condition of*

$$\left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right)^{\mathrm{T}} P + P\left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right) = -I.$$

The proof of this Lemma is similar to that of Lemma 1 in [37], and omitted here.

2.2. Linear Time-Varying System

We will analyze a linear time-varying system

$$\dot{x} = A(t)x + B(t)u, x(t_0) = x_0, \tag{1}$$

in this scenario, $x \in R^n$ denotes the state and $u \in R^m$ represents the control input; The matrices $A(t) \in R^{n\times n}$ and $B(t) \in R^{n\times m}$ are system matrices that depends on time t, t_0 is the initial time and $x_0 \in R^n$ is the initial state vector.

We define $\Phi_A(t_0, t)$ as the state transition matrix associated with system (1), this represents the sole solution to the matrix differential equation $\dot{\Phi}_A(t, t_0) = A(t)\Phi_A(t, t_0)$ with $\Phi_A(t_0, t_0) = I_n$. The controllability syntax of pair $(A(t), B(t))$ is defined as

$$W_c(t_0, t) := \int_{t_0}^{t} \Phi_A(t_0, v)B(v)B(v)^{\mathrm{T}}\Phi_A(t_0, v)^{\mathrm{T}} dv.$$

Definition 1. *[52]: A pair $(A(t), B(t))$ is classified as uniformly controllable if it is possible to find a positive pair (ε, T) such that $\sigma_{\min}(W_c(t, t+T)) \geq \varepsilon$ for all $t \geq 0$.*

In the realm of time-varying systems, the persistently exciting (PE) condition is a pivotal element in stability analysis, defined as detailed below.

Definition 2. *(PE condition)[52]: To characterize a time-varying symmetric matrix $\Delta(t)$ as* PE, *it must satisfy the following condition: two positive constants T and ε, are presented such that*

$$\int_{t}^{t+T} \Delta(v)dv \geq \varepsilon I, \forall t \in [t_0, +\infty).$$

By adhering to the argumentation outlined in the proof of Theorem 1 from reference [37], we arrive at the lemma.

Lemma 2. *Given that $(A(t), B(t))$ is uniformly controllable, then $\Phi_A(t_0, t)B(t)B(v)^{\mathrm{T}}\Phi_A(t_0, t)^{\mathrm{T}}$ exhibits* PE, *implying the existence of a pair (T, ε) such that*

$$\sigma_{\min}\left(\int_{t}^{t+T} \Phi_A(t_0, v)B(v)B(v)^{\mathrm{T}}\Phi_A(t_0, v)^{\mathrm{T}} dv\right) \geq \varepsilon.$$

The cooperative PE is also an important concept which is used in DCA system identification shown in [39].

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Definition 3. *[53]: A series of matrix-valued functions $\Delta_i(t), i = 1, 2, \cdots, N$, is identified as satisfying the cooperative PE condition provided that two positive constants T and ε can be located to satisfy*

$$\int_t^{t+T} \sum_{i=1}^{N} \Delta_i(v)dv \geq \varepsilon I.$$

According to definition 3, the subsequent lemma is demonstrated within [53].

Lemma 3. *Let $\Delta(t) = \mathrm{diag}\{\Delta_1, \cdots, \Delta_N\}$. If $\Delta_i(t)$ is cooperative PE, and $\mathcal{L}$ corresponds to the Laplacian matrix of a graph that is both undirected and connected, then there exists a pair (T, ε) such that*

$$\int_t^{t+T} [\Delta(v) + \mathcal{L}]dv \geq \varepsilon I.$$

2.3. Several Inequalities

The subsequent inequalities are instrument in establishing the proof of primary theorems within this paper.

Lemma 4. *[54]: Given any two vectors $m \in R^n, n \in R^n$, along with a positive constant $\mu > 0$, we can establish that*

$$\|m + n\|^2 \geq \frac{\mu}{1+\mu}\|m\|^2 - \mu\|n\|^2.$$

Lemma 5. *[54]: (Cauchy-Schwartz inequality): For any pair of integrable vector-valued functions $f(t) \in R^n$ and $g(t) \in R^n$, the subsequent inequality is valid:*

$$\left(\int_t^{t+T} f(v)^{\mathrm{T}} g(v)dv\right)^2 \leq \int_t^{t+T} \|f(v)\|^2 dv \int_t^{t+T} \|g(v)\|^2 dv.$$

Lemma 6. *Given that the function $\varphi(t) : [t_0, +\infty) \to [0, +\infty)$ is non-negative and that $t_0 \geq 0$. Assume the following conditions hold for a real number $0 < \sigma < 1$ and positive constants T, ν, and γ:*

$$\varphi(t) \leq \nu, t \in [t_0, t_0 + T] \tag{2}$$

$$\varphi(t + T) \leq \sigma\varphi(t) + \gamma e^{-\alpha t}, t \in [t_0, +\infty) \tag{3}$$

with $\alpha \in \left(0, -\frac{\ln\sigma}{T}\right)$, then

$$\varphi(t) \leq b_1 e^{-\beta t} + b_2 e^{-\alpha t}$$

where $\beta = -\frac{\ln\sigma}{T}, b_1 = \max\left\{\nu e^{\beta(t_0+T)}, \frac{\nu}{\sigma}e^{\beta t_0}\right\}$ and $b_2 = \frac{\gamma e^{\alpha T}}{1-\sigma e^{\alpha T}}$.

Proof. See Appendix A. □

3. Event-Triggered Consensus for General Linear Time-Varying Multi-Agent Systems

This section will give the overall framework for event-triggered control design methods. It may be applied to a collection of identical linear time-varying multi-agent systems, and the convergence of their consensus error systems will be analyzed.

3.1. Event-Triggered Consensus Control

Consider a team of linear time-varying multi-agent systems in the following form

$$\dot{\chi}_i = \Lambda(t)\chi_i + \Xi(t)u_i, t \in [0, \infty), i = 1, 2, \cdots, N \tag{4}$$

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where the control input is represented by $u_i \in R^m$ and the state of the ith agent is indicated by $\chi_i \in R^n$; $\Lambda : [0, \infty) \to R^{n\times n}$ and $\Xi : [0, +\infty) \to R^{n\times m}$ are two time-varying matrices.

To optimize the use of communication resources, the paper adopts an event-triggered communication scheme among agents. To particularize, we set a decentralized function, termed $H_i\left(t, \chi_i(t), t^i_{k_i}, \chi_i\left(t^i_{k_i}\right)\right)$, to act as the trigger function for agent $i, i = 1, 2, \cdots, N$, where $t^i_{k_i}$ is the k_i th communication instant of agent i. After $t^i_{k_i}$, agent i continuously monitors its own state $\chi_i(t)$ to see if the trigger condition

$$H_i\left(t, \chi_i(t), t^i_{k_i}, \chi_i\left(t^i_{k_i}\right)\right) > 0 \tag{5}$$

is satisfied. If satisfied, the current instant is denoted by $t^{i+1}_{k_{i+1}}$, and agent i transmits $t^{i+1}_{k_{i+1}}$ and $\chi_i\left(t^{i+1}_{k_{i+1}}\right)$ to its neighboring agents immediately. At this time instant, it is said that an event occurs for agent i. Note that $t^i_{k_i}, i = 1, \cdots, N$, are characterised by being independent with respect to all nodes, and does not need to be synchronized.

Definition 4. *The Zeno behavior is confirmed to be present in the system if an event happens in an infinite times over some finite time interval.*

We aim to design a distributed control $u_i = \nu\left(t, \cup_{j\in\{i\}\cup\mathcal{N}_i}\left(t^j_{k_j}, \chi_j\left(t^j_{k_j}\right)\right)\right)$ based on the event-triggered communication described above, where $t^j_{k_j}$ is the time instant of the most recent event occurring for agent i prior to the current time t, such that $\chi_i(t) - \chi_j(t) \to 0$ for all agents while avoiding the Zeno behavior.

Considering the information above, the event-triggered control law can be generally expressed as

$$\begin{aligned} u_i =& K(t) \sum_{i\in\mathcal{N}_i} a_{ij}\left[\psi\left(t^j_{k_j}\right)\chi_j\left(t^j_{k_j}\right) - \psi\left(t^i_{k_i}\right)\chi_i\left(t^i_{k_i}\right)\right] \\ t \in& \left[t^i_{k_i}, t^i_{k_i+1}\right) \end{aligned} \tag{6}$$

where a_{ij} refers to the elements of the adjacent matrix $\mathcal{A}$ within network topology; $K(t)$ and $\psi(t)$ are two bounded time-varying matrices which need to be designed. The trigger function is defined as

$$H_i\left(t, \chi_i(t), t^i_{k_i}, \chi_i\left(t^i_{k_i}\right)\right) = \|e_{\chi_i}(t)\|^2 - ce^{-\alpha t}, t > t^i_{k_i} \tag{7}$$

where $e_{\chi_i}(t) = \psi(t)\chi_i(t) - \psi\left(t^i_{k_i}\right)\chi_i\left(t^i_{k_i}\right)$, and $c, \alpha > 0$ are two positive design parameters.

Remark 1. *The consensus control law (6) is different from that in general linear time-invariant systems such as [55]. First, the control gain matrix is time-varying. Second, an extra term $\psi_i(t)$ appears in the consensus term. Thus, how to design $K(t)$ and $\psi(t)$ is a key issue. Moreover, the trigger function (7) is inspired by that in [32], where c and α are two design parameters. To achieve the objective, $K(t)$, $\psi(t)$ and α are designed by following the conditions of the Theorem 1 next.*

For convenience of denotations, let $\chi = [\chi_1^{\mathrm{T}}, \cdots, \chi_N^{\mathrm{T}}]$ and $e_\chi(t) = \left[e^{\mathrm{T}}_{\chi_1}(t), \cdots, e^{\mathrm{T}}_{\chi_N}(t)\right]^{\mathrm{T}}$. The closed-loop system is given by

$$\dot{\chi} = [I_N \otimes \Lambda(t) + \mathcal{L} \otimes (\Xi(t)K(t))]\chi + \mathcal{L} \otimes (\Xi(t)K(t))e_\chi(t). \tag{8}$$

The following consensus analysis will be based on this closed loop system.

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3.2. Consensus Analysis

In the consensus analysis, to tackle the consensus problem, it is usually transformed into a stabilization issue through a fitting nonsingular transformation. As a result, we establish the subsequent assumptions regrading the system (8).

Assumption 1. *Under the nonsingular transformation $z_i(t) = \psi(t)\chi_i(t) + \zeta$, there are two time-varying matrices $\Upsilon(t)$ and $\Theta(t)$, a positive semi-definite matrix $\Psi(t)$, and a constant vector ζ that satisfy the described conditions:*

$$\dot{z}(t) = \Upsilon(t)\Psi(t)z(t) + \Theta(t)e_z(t) \tag{9}$$

$z = \left[z_1^{\mathrm{T}}, \cdots, z_N^{\mathrm{T}}\right]^{\mathrm{T}}$, *and* $e_z(t) = e_\chi(t)$ *with* $e_{zi}(t) = z_i(t) - z_i\left(t_{k_i}^i\right) = [\psi(t)\chi_i(t) + \zeta] - \left[\psi\left(t_{k_i}^i\right)\chi_i\left(t_{k_i}^i\right) + \zeta\right] = e_{\chi_i}(t)$. *Furthermore,* $\left\|\psi^{-1}(t)\right\|$ *is uniformly bounded.*

Assumption 2. *There exists a function $V(x)$ such that $\kappa_1 z^{\mathrm{T}} z \leq V(z) \leq \kappa_2 z^{\mathrm{T}} z$ and $\frac{dV(z)}{dt} \leq -z^{\mathrm{T}}\Omega(t)z + \varrho e^{-\alpha t}$ where $\Omega(t)$ is a positive semi-definite matrix, ϱ and κ are two constants.*

Assumption 3. *There exist constants $\varpi_\Upsilon, \varpi_\Psi, \varpi_\Theta$ and ϖ_Ω such that $\|\Upsilon(t)\| \leq \varpi_\Upsilon$, $\|\Psi(t)\| \leq \varpi_\Psi$, $\|\Theta(t)\| \leq \varpi_\Theta$, and $\|\Omega(t)\| \leq \varpi_\Omega$ for all $t \geq 0$. Moreover, assume that $\Omega(t) \geq \Psi(t)$ and $\Omega(t)$ is PE, implying the existence of two constants T and ε such that $\sigma_{\min}\left(\int_t^{t+T}\Omega(v)dv\right) \geq \varepsilon$ is valid.*

With the preliminaries out of the way, we can now introduce our foremost result.

Theorem 1. *Take into consideration the closed-loop system given by equation (8), which integrates the system (4), operates under the event-triggered control law presented in (6) and is governed by the triggered condition stated in (7). Under Assumptions 1-3, if α is designed such that $\alpha \in (0, \beta)$ with*

$$\beta = -\frac{1}{T}\ln\left(1 - \frac{\varepsilon^2}{(\varepsilon + \kappa_2)\left(\varepsilon + 2\kappa_2\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\right)}\right),$$

then we have:

(i) the existence of two positive constants ζ_1 and ζ_2, ensuring $\sum_{i=1}^N\left\|x_i(t) + \psi^{-1}(t)\zeta\right\|^2 \leq \zeta_1 e^{-\beta t} + \zeta_2 e^{-\alpha t}$;

(ii) Zeno behavior is absent.

Proof of Theorem 1: (i) In accordance with Assumption 2, we derive that

$$\frac{dV(t)}{dt} \leq -z^{\mathrm{T}}\Omega(t)z + \varrho e^{-\alpha t}. \tag{10}$$

Then, by performing integration on both sides of (10) from t to $t+T$, one gets

$$\begin{aligned} &V(t+T) - V(t) \\ &\leq -\int_t^{t+T}\left\|\Omega^{\frac{1}{2}}(v)z(v)\right\|^2 dv + \frac{\varrho}{\alpha}\left(e^{-\alpha t} - e^{-\alpha(t+T)}\right) \end{aligned} \tag{11}$$

where $\Omega^{\frac{1}{2}}$ is derived from the decomposition of the positive semi-definite matrix Ω, i.e., $\Omega = \Omega^{\frac{1}{2}}\Omega^{\frac{1}{2}}$, and based on (9), it is concluded that

$$z(v) = z(t) + \int_t^v (\Upsilon(s)\Psi(s)z(s) + \Theta(s)e_z(s))ds. \tag{12}$$

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Substituting (12) in (11) yields

$$\begin{aligned} & V(t+T) - V(t) \\ & \leq -\int_t^{t+T} \|\Omega^{\frac{1}{2}}(v)\Big(z(t) + \int_t^v (\Upsilon(s)\Psi(s)z(s) \\ & +\Theta(s)e_z(s))ds)\|^2 dv + \frac{\varrho}{\alpha}\Big(e^{-\alpha t} - e^{-\alpha(t+T)}\Big). \end{aligned} \tag{13}$$

According to Lemma 4 with $m = \Omega^{\frac{1}{2}}(v)z(t)$ and $n = \Omega^{\frac{1}{2}}(v)\int_t^v(\Upsilon(s)\Psi(s)z(s) + \Theta(s)e_z(s))ds$, we have

$$\begin{aligned} & V(t+T) - V(t) \\ & \leq -\frac{\mu}{1+\mu}\int_t^{t+T} \left\|\Omega^{\frac{1}{2}}(v)z(t)\right\|^2 dv \\ & +\mu\int_t^{t+T} \|\Omega^{\frac{1}{2}}(v)\int_t^v (\Upsilon(s)\Psi(s)z(s) \\ & +\Theta(s)e_z(s))ds\|^2 dv + \frac{\varrho}{\alpha}\Big(e^{-\alpha t} - e^{-\alpha(t+T)}\Big) \end{aligned} \tag{14}$$

where μ is an arbitrary positive constant. From Assumption 3, one gets

$$\begin{aligned} & V(t+T) - V(t) \\ & \leq -\frac{\mu\varepsilon}{1+\mu}\|z(t)\|^2 \\ & +\mu\varpi_\Omega\int_t^{t+T} \left\|\int_t^v \Upsilon(s)\Psi(s)z(s) + \Theta(s)e_z(s)ds\right\|^2 dv \\ & +\frac{\varrho}{\alpha}\Big(e^{-\alpha t} - e^{-\alpha(t+T)}\Big) \\ & \leq -\frac{\mu\varepsilon}{1+\mu}\|z(t)\|^2 \\ & +\mu\varpi_\Omega\int_t^{t+T}\bigg(\int_t^v \|\Upsilon(s)\Psi(s)z(s) \\ & +\Theta(s)e_z(s)\|^2 ds\bigg)^2 dv + \frac{\varrho}{\alpha}\Big(e^{-\alpha t} - e^{-\alpha(t+T)}\Big) \end{aligned} \tag{15}$$

Using Lemma 5 with $f(s) = 1$ and $g(s) = \|\Upsilon(s)\Psi(s)z(s) + \Theta(s)e_z(s)\|$, combining with $v - t \leq T$, we conclude that

$$\begin{aligned} & V(t+T) - V(t) \\ & \leq -\frac{\mu\varepsilon}{1+\mu}\|z(t)\|^2 + \mu\varpi_\Omega T\int_t^{t+T}\int_t^v \|\Upsilon(s)\Psi(s)z(s) \\ & +\Theta(s)e_z(s)\|^2 ds dv + \frac{\varrho}{\alpha}\Big(e^{-\alpha t} - e^{-\alpha(t+T)}\Big) \end{aligned} \tag{16}$$

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After altering the integration order for the second term in inequality (16), in conjunction with $t+T-s \leq T$, we derive

$$\begin{aligned}
&V(t+T)-V(t)\\
\leq &-\frac{\mu\varepsilon}{1+\mu}\|z(t)\|^2+\mu\varpi_\Omega T^2\int_t^{t+T}\|\Upsilon(v)\Psi(v)z(v)\\
&+\Theta(v)e_z(v)\|^2dv+\frac{\varrho}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right)\\
\leq &-\frac{\mu\varepsilon}{1+\mu}\|z(t)\|^2+2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\int_t^{t+T}z(v)^{\mathrm{T}}\\
&\cdot\Psi(v)z(v)dv+2\mu\varpi_\Omega T^2\varpi_\Theta^2\int_t^{t+T}e_z(v)^{\mathrm{T}}e_z(v)dv\\
&+\frac{\varrho}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right)
\end{aligned} \tag{17}$$

where the known inequality $\|m+n\|^2 \leq 2\|m\|^2+2\|n\|^2$ to vectors m and n is applied to generate the final inequality. From $e_z(t)^{\mathrm{T}}e_z(t) \leq Nce^{-\alpha t}$, it implies that

$$\begin{aligned}
&V(t+T)-V(t)\\
\leq &-\frac{\mu\varepsilon}{1+\mu}\|z(t)\|^2+2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\int_t^{t+T}z(v)^{\mathrm{T}}\\
&\cdot\Psi(v)z(v)dv+\frac{2\mu\varpi_\Omega T^2\varpi_\Theta^2 Nc}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right)\\
&+\frac{\varrho}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right).
\end{aligned} \tag{18}$$

According to Assumptions 2 and 3, we have $-\|z(t)\|^2 \leq -\kappa_2^{-1}V(t)$ and $\Psi(t) \leq \Omega(t)$. Then,

$$\begin{aligned}
&V(t+T)-V(t)\\
\leq &-\frac{\mu\varepsilon}{(1+\mu)\kappa_2}V(t)+2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\int_t^{t+T}z(v)^{\mathrm{T}}\\
&\cdot\Omega(v)z(v)dv+\frac{2\mu\varpi_\Omega T^2\varpi_\Theta^2 Nc}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right)\\
&+\frac{\varrho}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right).
\end{aligned} \tag{19}$$

By substituting (11) into (19) leads to

$$\begin{aligned}
&V(t+T)-V(t)\\
\leq &-\frac{\mu\varepsilon}{(1+\mu)\kappa_2}V(t)+2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi(V(t)\\
&-V(t+T))+\frac{2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\varrho}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right)\\
&+\frac{2\mu\varpi_\Omega T^2\varpi_\Theta^2 Nc}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right)\\
&+\frac{\varrho}{\alpha}\left(e^{-\alpha t}-e^{-\alpha(t+T)}\right).
\end{aligned} \tag{20}$$

After that, one receives

$$V(t+T) \leq \kappa_3 V(t)+\kappa_4 e^{-\alpha t} \tag{21}$$

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where

$$\kappa_3 := 1 - \frac{\mu\varepsilon}{\kappa_2(1+\mu)\left(1+2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\right)},$$
$$\kappa_4 := \frac{2\mu\varpi_\Omega T^2\left(\varpi_\Upsilon^2\varpi_\Psi\varrho+\varpi_\Theta^2 Nc\right)+\varrho}{\alpha\left(1+2\mu\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\right)}.$$

Let μ be $\frac{\kappa_2}{\varepsilon}$, and then we have

$$\kappa_3 = 1 - \frac{\varepsilon^2}{(\varepsilon+\kappa_2)\left(\varepsilon+2\kappa_2\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\right)} \in (0,1),$$
$$\kappa_4 = \frac{2\kappa_2\varpi_\Omega T^2\left(\varpi_\Upsilon^2\varpi_\Psi\varrho+\varpi_\Theta^2 Nc\right)+\varrho\varepsilon}{\alpha\left(\varepsilon+2\kappa_2\varpi_\Omega T^2\varpi_\Upsilon^2\varpi_\Psi\right)}.$$

Based on equation (10), this implies that $V(t) \leq \varrho T + V(0), t \in [0,T)$. Thus, utilizing Lemma 6, we can conclude

$$V(t) \leq \eta_1 e^{-\beta t} + \eta_2 e^{-\alpha t} \tag{22}$$

if $\alpha \in (0,\beta)$, where $\beta = -\frac{\ln\kappa_3}{T}, \eta_1 = \max\{(\varrho T + V(0))e^{\beta T}, \frac{\varrho T+V(0)}{\kappa_3}\}$, and $\eta_2 = \frac{\kappa_4 e^{\alpha T}}{1-\kappa_3 e^{\alpha T}}$.

(i) From $z_i(t) = \psi(t)\chi_i(t) + \zeta$ and $\kappa_1 z^{\mathrm{T}} z \leq V(t)$, it is simple to ascertain

$$\sum_{i=1}^{N}\left\|x_i(t)+\psi^{-1}(t)\zeta\right\|^2 \leq \frac{V(t)}{\psi_0^2\kappa_1} \tag{23}$$

where ψ_0 as the upper bound of $\psi^{-1}(t)$. Substituting (22) in (23), it results in

$$\sum_{i=1}^{N}\left\|\chi_i(t)+\psi^{-1}(t)\zeta\right\|^2 \leq \zeta_1 e^{-\beta t} + \zeta_2 e^{-\alpha t} \tag{24}$$

where $\zeta_1 = \frac{\eta_1}{\psi_0^2\kappa_1}$ and $\zeta_2 = \frac{\eta_2}{\psi_0^2\kappa_1}$.

(ii) Starting with $e_{\chi_i}(t) = e_{zi}(t) = z_i(t) - z_i\left(t_{k_i}^i\right)$, it is discovered that $\dot{e}_{zi}(t) = \dot{z}_i(t)$ for $t \in \left[t_{k_i}^i, t_{k_{i+1}}^i\right)$, and $e_{zi}\left(t_{k_i}^i\right) = 0$. Thus, based on (9), we conclude that

$$\begin{aligned}\|e_{\chi_i}(t)\| &\leq \int_{t_{k_i}^i}^{t}\|\dot{z}_i(v)\|dv \\ &\leq \int_{t_{k_i}^i}^{t}\|\Upsilon(v)\Psi(v)z(v)+\Theta(v)e_z(v)\|dv.\end{aligned} \tag{25}$$

Based on $\kappa_1 z^{\mathrm{T}} z \leq V(t)$ and (22), we have

$$\|z(t)\| \leq \sqrt{\frac{\eta_1}{\kappa_1}}e^{-\frac{\beta}{2}t_{k_i}^i} + \sqrt{\frac{\eta_2}{\kappa_1}}e^{\frac{-\alpha}{2}t_{k_i}^i}. \tag{26}$$

From (5) and (7), we obtain that

$$\|e(t)\| \leq \sqrt{N}ce^{-\frac{\alpha}{2}t_{k_i}^i}. \tag{27}$$

Substituting (26) and (27) back into (25), together with Assumption 3, leads to

$$\|e_{\chi_i}(t)\| \leq \left(\mu_1 e^{-\beta t_{k_i}^i} + \mu_2 e^{-\alpha t_{k_i}^i}\right)\left(t - t_{k_i}^i\right) \tag{28}$$

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where $\mu_1 = \varpi_\Upsilon \varpi_\Psi \sqrt{\frac{\eta_1}{\kappa_1}}$ and $\mu_2 = \varpi_\Upsilon \varpi_\Psi \sqrt{\frac{\eta_2}{\kappa_1}} + \varpi_\Theta \sqrt{Nc}$. The subsequent event will occur once $\|e_i(t)\|^2 = ce^{-\alpha t}$. Therefore, the value of $v = t - t^i_{k_i}$ that satisfies equation

$$\left(\mu_1 e^{\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)t^i_{k_i}} + \mu_2\right) v = \sqrt{Nc} e^{-\frac{\alpha}{2}v} \tag{29}$$

serves as a minimum limit for the time intervals between events. Because of $0 < \alpha < \beta$, the inequality $\mu_2 \leq \mu_1 e^{\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)t^i_{k_i}} + \mu_2 \leq \mu_1 + \mu_2$ holds. For all $t^i_{k_i} \geq 0$, the solution $v\left(t^i_{k_i}\right)$ is the greater or equal to v given by $(\mu_1 + \mu_2)v = \sqrt{Nc} e^{-\frac{\alpha_i}{2}v}$, which is strictly positive. Consequently, it is clear that the system is free from Zeno behavior. □

Remark 2. *Assumptions 1-3 are crucial elements in establishing the proof of Theorem 1. In applications, how to choose the transformation and the Lyapunov function is key for the event-triggered consensus analysis. In the next section, we will illustrate how to identify a proper transformation and Lyapunov function for a specific linear time-varying multi-agent system within a network that includes a spanning tree.*

4. Applications to Event-Triggered Consensus of Linear Time-Varying Multi-Agent Systems Having a Spanning Tree

This section utilizes the findings from Section III to engineer an event-triggered consensus control strategy for a collective of identical linear time-varying multi-agent systems over a network having a spanning tree.

4.1. Event-Triggered Control Design

Take into account a collective of identical multi-agent systems

$$\dot{x}_i = A(t)x_i + B(t)u_i, t \in [0, \infty), i = 1, 2, \cdots, N \tag{30}$$

that are linear and change over time, where the control input is represented by $u_i \in R^m$ and the state of the ith agent is indicated by $x_i \in R^n$; $A : [0, \infty) \to R^{n\times n}$ and $B : [0, +\infty) \to R^{n\times m}$ are two time-varying matrices.

Based on the control law (6), we design $K(t) = B(t)^{\mathrm{T}} \Phi_A(0, t)^{\mathrm{T}}$ and $\psi\left(t^j_{k_j}\right) = \Phi_A\left(0, t^j_{k_j}\right)$. Then, the detailed control law for agent i is expressed by

$$\begin{aligned} u_i =& B(t)^{\mathrm{T}} \Phi_A(0, t)^{\mathrm{T}} \sum_{j\in\mathcal{N}_i} \gamma_{ij} \left(\Phi_A\left(0, t^j_{k_j}\right) x_j\left(t^j_{k_j}\right)\right. \\ & \left. - \Phi_A\left(0, t^i_{k_i}\right) x_i\left(t^i_{k_i}\right)\right) \end{aligned} \tag{31}$$

where $\gamma_{i,j}$ are the entries of the adjacency matrix Γ corresponding to the graph $\mathcal{G}$, and $\Phi_A(\cdot, \cdot)$ is the state transformation matrix for $A(t)$. The design of trigger function is as

$$H_i\left(t, x_i(t), t^i_{k_i}, x_i\left(t^i_{k_i}\right)\right) = e^{\mathrm{T}}_{xi}(t) e_{xi}(t) - ce^{-\alpha t}, t > t^i_{k_i} \tag{32}$$

where $e_{xi}(t) = \Phi_A(0, t)x_i(t) - \Phi_A\left(0, t^i_{k_i}\right) x_i\left(t^i_{k_i}\right)$, and $c, \alpha > 0$ are two positive design parameters.

To provide context of our main result, we must first detail the subsequent assumptions.

Assumption 4. *There exist $\bar{m} \geq 1$ and $\bar{n} \geq 1$ such that $\|\Phi_A(t_1, t_2)\| \leq \bar{m}$ and $\|B(t)\| \leq \bar{n}$ for all $t_1, t_2, t \geq 0$.*

Assumption 5. *The pair $(A(t), B(t))$ is uniformly controllable meaning a pair of positive numbers (ε, T) can be found to ensure $\sigma_{\min}(W_c(t, t+T)) \geq \varepsilon$ for all $t \geq 0$.*

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Now that the foundation has been laid, we are prepared to introduce our primary result.

Theorem 2. *Given that Assumptions 4 and 5 are met and the graph under consideration features a spanning tree. If the constant α satisfies $0 < \alpha < \beta$ with*

$$\beta = -\frac{1}{T}\ln\left(1 - \frac{\varepsilon^2}{(\varepsilon + \sigma_{\max}(P))(\varepsilon + 2\sigma_{\max}(P)\bar{m}^4\bar{n}^4 l^2 T^2)}\right)$$

where l is $\sup\{\|\mathcal{L} + \mathbf{1}_N r^{\mathrm{T}}\|\}$. Then,

(i) there exist two positive constants ζ_1 and ζ_2 such that

$$\|x - \bar{x}\| \leq \zeta_1 e^{\frac{-\beta}{2}t} + \zeta_2 e^{-\frac{\alpha}{2}t} \tag{33}$$

i.e., all agents reach consensus;

(ii) Zeno behavior is absent in the linear time-varying system, which implies that there is a fixed positive constant that severs as a lower bound for all time intervals $t_{k_i}^{i+1} - t_{k_i}^{i}, i = 1, \cdots, N, k_i = 1, 2, \cdots$. The complete proof is scheduled to be detailed in the next section.

4.2. Proof of Theorem 2

The problem of consensus is altered into a stabilization problem through the introduction of a new variable $z = \left[z_1(t)^{\mathrm{T}}, \cdots, z_N(t)^{\mathrm{T}}\right]^{\mathrm{T}}$ with $z_i = \Phi_A(0,t)x_i(t) - \sum_{i=1}^{N} r_i x_i(0)$, where $r_i > 0$ is defined as in subsection 2.1. We also let $e_x(t) = [e_{x1}(t), \cdots, e_{xN}(t)]^{\mathrm{T}}$. Then, we have the following lemma which gives the dynamic of $z(t)$.

Lemma 7. *The dynamic of the variable $z(t)$ is governed by*

$$\dot{z}(t) = \left(\left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right) \otimes G(t)\right) z(t) - (\mathcal{L} \otimes G(t)) e_x(t) \tag{34}$$

where $G(t) = \Phi_A(0,t)B(t)B(t)^{\mathrm{T}}\Phi_A(0,t)^{\mathrm{T}}$, and $\mathcal{L}$ is the Lapalacian matrix of graph $\mathcal{G}$.

Proof. See Appendix B. □

Proof of Theorem 2: To employ Theorem 1, we need to check Assumptions 1-3.

1) Let $\Upsilon(t) = \left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right) \otimes I, \Psi(t) = I \otimes G(t), \Theta(t) = -(\mathcal{L} \otimes G(t))$, and $\zeta = -\sum_{i=1}^{N} r_i x_i(0)$. Obviously, $\Psi(t)$ is a positive semi-definite matrix and under a nonsingular transition $z_i(t) = \psi(t)x_i(t) + \zeta = \Phi_A(0,t)x_i(t) - \sum_{i=1}^{N} r_i x_i(0)$, the dynamic described in (34) suggests that Assumption 1 is satisfied.

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2) As the network includes a spanning tree, a positive matrix P that meets the Lemma 1 can be determined. With the Lyapunov function defined as $V = z^{\mathrm{T}}(P \otimes I_n)z$,

$$\begin{aligned}\frac{dV}{dt} =& \dot{z}(t)^{\mathrm{T}}(P \otimes I_n)z(t) + z(t)^{\mathrm{T}}(P \otimes I_n)\dot{z}(t) \\ =& z^{\mathrm{T}}\left(\left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right) \otimes \Psi(t)\right)^{\mathrm{T}}(P \otimes I_n)z \\ &+ z^{\mathrm{T}}(P \otimes I_n)\left(\left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right) \otimes \Psi(t)\right)z \\ &+ 2z^{\mathrm{T}}(P \otimes I_n)((-\mathcal{L}) \otimes \Psi(t))e_x(t) \\ =& -2z^{\mathrm{T}}(I_n \otimes \Psi(t))z \\ &- 2e_x(t)^{\mathrm{T}}\left(P\mathcal{L} \otimes \Psi^{\frac{1}{2}}(t)\right)\left(I_n \otimes \Psi^{\frac{1}{2}}(t)\right)z \\ \leq& -z^{\mathrm{T}}(I_n \otimes \Psi(t))z \\ &+ e_x^{\mathrm{T}}(t)\left(P\mathcal{L}(P\mathcal{L})^{\mathrm{T}} \otimes \Psi(t)\right)e_x(t) \\ \leq& -z^{\mathrm{T}}(I_n \otimes \Psi(t))z + \rho N c e^{-\alpha t}\end{aligned} \tag{35}$$

where $\rho = \sigma_{\max}^2(P\mathcal{L})\bar{m}^2\bar{n}^2$. Let $\kappa_1 = \sigma_{\min}(P), \kappa_2 = \sigma_{\max}(P), \varrho = \rho N c$ and $\Omega(t) = I_n \otimes G(t)$. Obviously, Assumption 2 holds.

3) Let $\varpi_{\Upsilon} = l, \varpi_{\Psi} = \bar{m}^2\bar{n}^2, \varpi_{\Theta} = \|\mathcal{L}\|\bar{m}^2\bar{n}^2$ and $\varpi_{\Omega} = \bar{m}^2\bar{n}^2$. It can be seen from Assumptions 4 and 5 that Assumption 3 holds.

The proof has been completed according to Theorem 1.

4.3. Simulation Example

In this section, We consider an example to demonstrate the efficacy of our consensus control method. Consider the subsequent agent model (30) with $i = 1, 2, 3, 4$, where

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

The transmition matrix for $A(t)$ is

$$\Phi(0,t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \tag{36}$$

It is easily shown that Assumptions 4 and 5 hold, where $\bar{m} = 1, \bar{n} = 1, T = 2\pi$ and $\varepsilon = 1.2$. Moreover, the network configuration is illustrated in Figure 1.

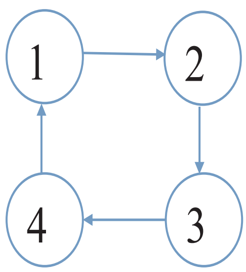

Figure 1. The network topology

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It is evident that a spanning tree exists, and the adjacency matrix is given by

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

In the simulation, we use the control law (31) along with the trigger function (32), and set the design parameter $c = 1$ and $\alpha = 0.5 \in (0, 0.68)$, and the initial states are given by $x_1 = [1.4, 1]$, $x_2 = [0.3, 0]$, $x_3 = [-1.2, -0.5]$, $x_4 = [-0.5, -1]$. Figures 2–4 display the outcomes of the simulation. The states of all agents presented in Figure 2 confirm the achievement of state consensus. The curves of the supervised errors and the threshold function are shown in Figure 3, and the time vs event interval is shown in Figure 4. It is evident that there is no occurrence of Zeno behavior in the closed-loop system. This also accords with the results that presented in Theorem 2.

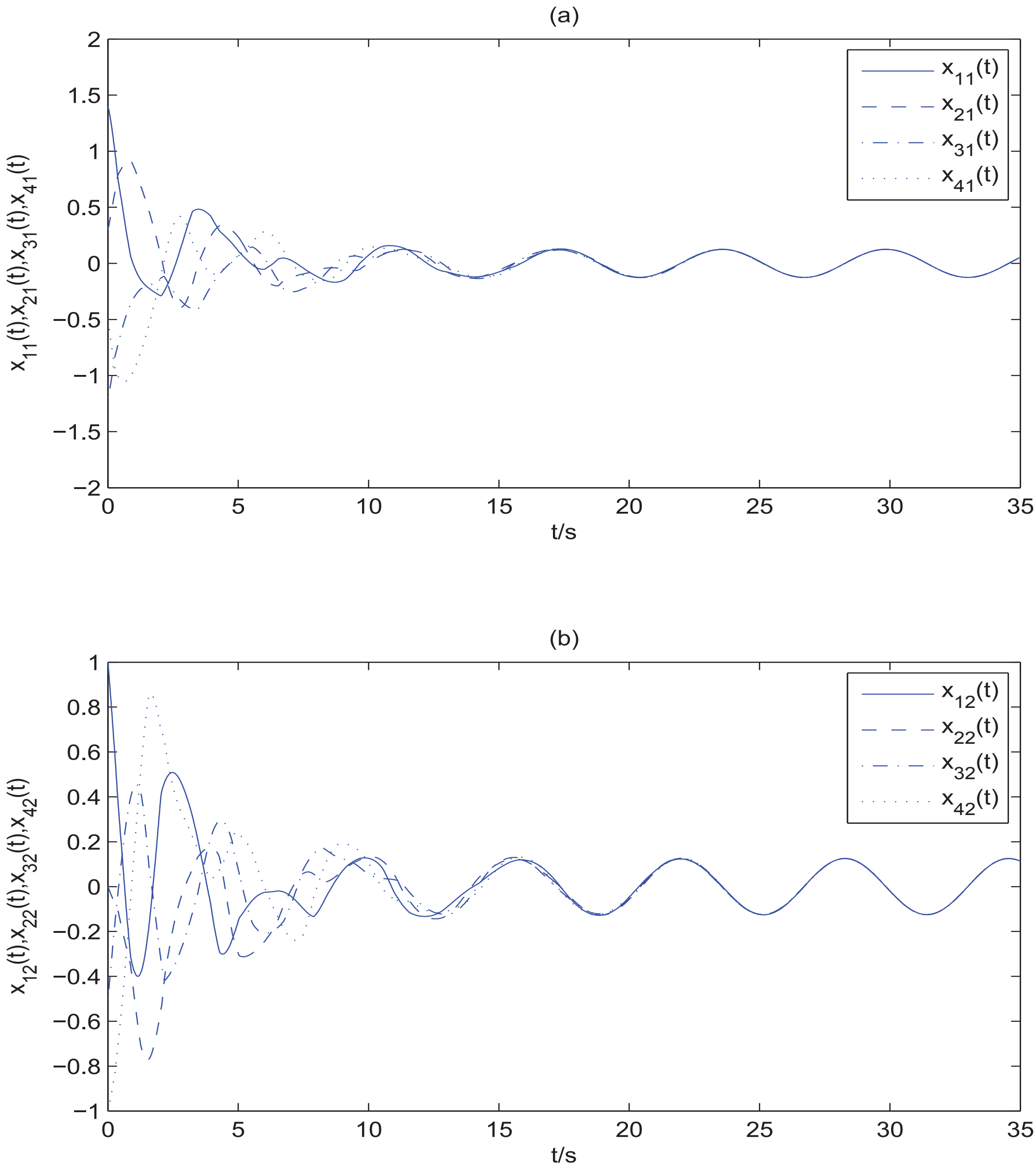

Figure 2. (a) Curves of states $x_{11}(t)$; $x_{21}(t)$; $x_{31}(t)$ and $x_{41}(t)$;(b) Curves of states $x_{12}(t)$; $x_{22}(t)$; $x_{32}(t)$ and $x_{42}(t)$.

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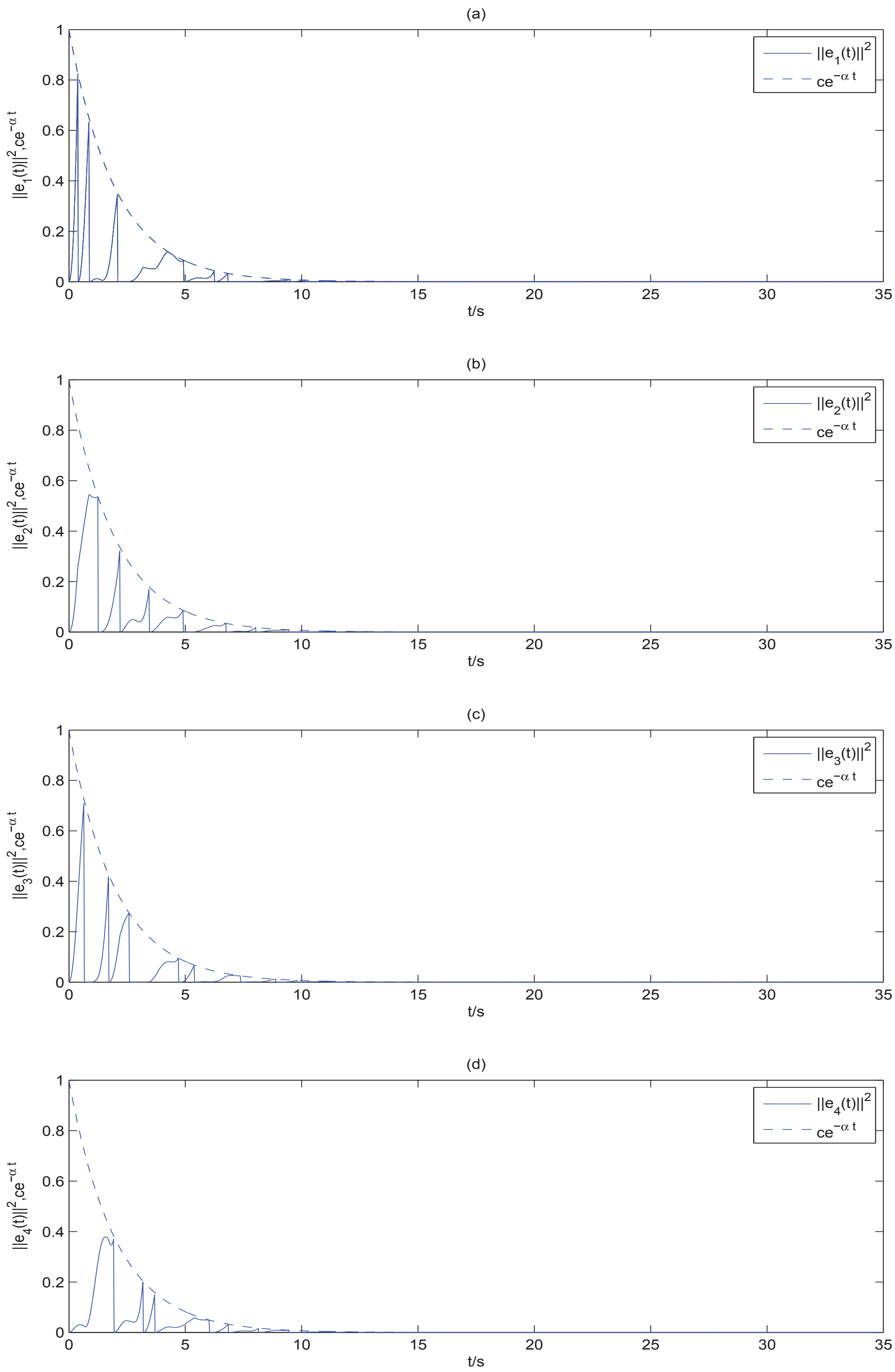

Figure 3. Supervised error $||e_1(t)||^2$, $||e_2(t)||^2$, $||e_3(t)||^2$, $||e_4(t)||^2$ and threshold $ce^{\alpha t}$.

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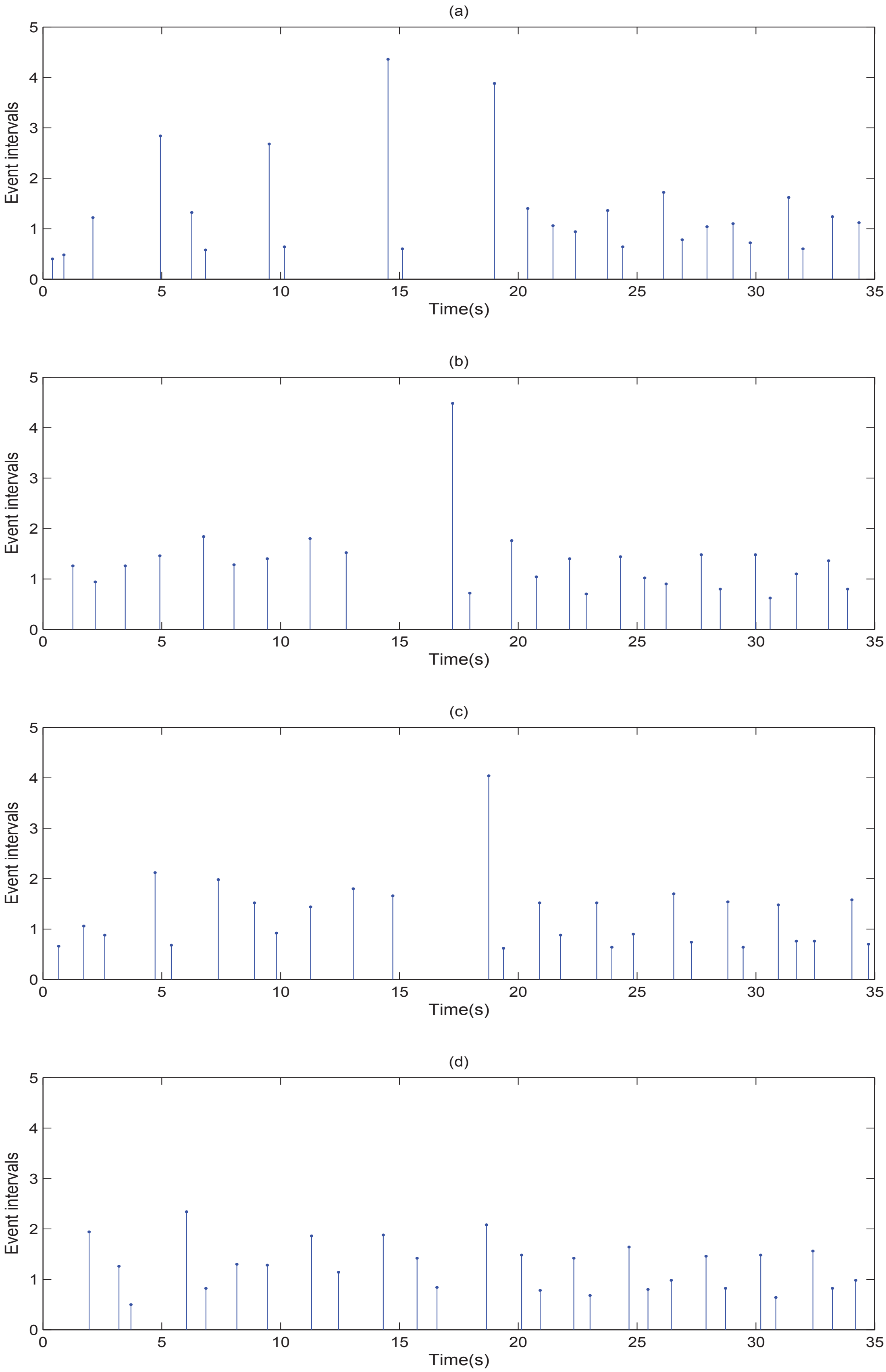

Figure 4. Time vs event interval.

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5. Conclusions

Our goal in this study is to establish a general framework for the design of consensus control in a set of linear time-varying multi-agent systems using an event-triggered communication approach. Based on certain assumptions, we proved the exponential convergence of all states of systems, and the occurrence of Zeno behavior is also precluded. The event-triggered scheme is used in communication process. Thus, our scheme can save the communication resource. Moreover, we implement the proposed event-triggered scheme to address the issue of event-triggered consensus within specific linear time-varying multi-agent systems.

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Appendix A. PROOF OF LEMMA 6

For all $t \geq t_0$, consider N to be the positive number that satisfies $t_0 + (N-1)T \leq t \leq t_0 + NT$. Then,

Case $(i): t \geq t_0 + T$. From (3), it can be inferred that

$$\begin{aligned}
\varphi(t) \leq & \sigma\varphi(t-T) + \gamma e^{-\alpha(t-T)} \\
\leq & \sigma^2\varphi(t-2T) + \sigma\gamma e^{-\alpha(t-2T)} + \gamma e^{-\alpha(t-T)} \\
& \vdots \\
\leq & \sigma^{N-1}\varphi(t-(N-1)T) + \sigma^{N-2}\gamma e^{-\alpha(t-(N-1)T)} \\
& + \cdots + \gamma e^{-\alpha(t-T)} \\
\leq & \sigma^{N-1}\varphi(t-(N-1)T) + \Big[\sigma^{N-2}e^{(N-1)\alpha T} \\
& + \cdots + \sigma e^{2\alpha T} + e^{\alpha T}\Big]\gamma e^{-\alpha t}.
\end{aligned} \tag{A1}$$

It follows from $t_0 + NT \geq t$ that $N \geq \frac{t-t_0}{T}$ and $t - (N-1)T \in [t_0, t_0 + T)$. Together with (2), we have

$$\begin{aligned}
\sigma^{N-1}\varphi(t-(N-1)T) \leq & \frac{\nu}{\sigma}\sigma^{\frac{t-t_0}{T}} \\
= & \frac{\nu}{\sigma}e^{-(-\frac{\ln\sigma}{T})(t-t_0)} \\
= & \frac{\nu}{\sigma}e^{-\beta(t-t_0)}
\end{aligned} \tag{A2}$$

where $\beta = -\frac{\ln\sigma}{T}$. For $0 < \alpha < -\frac{\ln\sigma}{T}$, the inequality $\sigma e^{\alpha T} < 1$ holds. Therefore,

$$\begin{aligned}
& \Big[\sigma^{N-2}e^{(N-1)\alpha T} + \cdots + \sigma e^{2\alpha T} + e^{\alpha T}\Big]\gamma e^{-\alpha t} \\
\leq & \gamma e^{-\alpha t}\frac{e^{\alpha T}\left(1-\left(\sigma e^{\alpha T}\right)^{N-1}\right)}{1-\sigma e^{\alpha T}} \\
\leq & \frac{\gamma e^{\alpha T}}{1-\sigma e^{\alpha T}}e^{-\alpha t}.
\end{aligned} \tag{A3}$$

Substituting (A1) and (A2) into (A3) yields

$$\varphi(t) \leq \frac{\nu}{\sigma} e^{\beta t_0} e^{-\beta t} + \frac{\gamma e^{\alpha T}}{1 - \sigma e^{\alpha T}} e^{-\alpha t} \tag{A4}$$

Case (ii): $t_0 \leq t \leq t_0 + T$. It is easy to get

$$\varphi(t) \leq \nu \leq \frac{\nu e^{\beta(t_0+T)}}{e^{\beta t}} \tag{A5}$$

Definite $b_1 := \max\left\{\nu e^{\beta(t_0+T)}, \frac{\nu}{\sigma} e^{\beta t_0}\right\}$ and $b_2 := \frac{\gamma_1 e^{\alpha T}}{1-\sigma e^{\alpha T}}$. Then, from (A4) and (A5), one gets

$$\varphi(t) \leq b_1 e^{-\beta t} + b_2 e^{-\alpha t}, t \in [t_0, +\infty). \tag{A6}$$

This complete this proof.

Appendix B. PROOF OF LEMMA 7

Differentiating both sides of $\Phi_A(0,t)\Phi_A(t,0) = I_n$ results into

$$\begin{aligned} \dot{\Phi}_A(0,t) &= -\Phi_A(0,t)\dot{\Phi}_A(t,0)\Phi_A(0,t) \\ &= -\Phi_A(0,t)A(t). \end{aligned} \tag{A7}$$

Then, one gets

$$\begin{aligned} \dot{z}_i(t) =& \dot{\Phi}_A(0,t)x_i(t) + \Phi_A(0,t)\dot{x}_i(t) \\ =& \Phi_A(0,t)B(t)B(t)^{\mathrm{T}}\Phi_A(0,t)^{\mathrm{T}} \\ & \times \sum_{j\in\mathcal{N}_i} r_{ij}\left(\Phi_A\left(0,t^j_{k_j}\right)x_j\left(t^j_{k_j}\right)\right. \\ & \left. -\Phi_A\left(0,t^i_{k_i}\right)x_i\left(t^i_{k_i}\right)\right). \end{aligned} \tag{A8}$$

Considering $e_{xi} = \Phi_A\left(0,t^i_{k_i}\right)x_i\left(t^i_{k_i}\right) - \Phi_A(0,t)x_i(t)$ and $z_i(t) = \Phi_A(0,t)x_i(t) - \sum_{i=1}^{N} r_i x_i(0)$, we further have

$$\begin{aligned} \dot{z}_i(t) =& \Phi_A(0,t)B(t)B(t)^{\mathrm{T}}\Phi_A(0,t)^{\mathrm{T}} \\ & \times \sum_{j\in\mathcal{N}_i} r_{ij}(z_j(t) - z_i(t)) \\ & + \Phi_A(0,t)B(t)B(t)^{\mathrm{T}}\Phi_A(0,t)^{\mathrm{T}} \\ & \times \sum_{j\in\mathcal{N}_i} r_{ij}(e_{xj}(t) - e_{xi}(t)) \end{aligned} \tag{A9}$$

$$\dot{z}(t) = ((-\mathcal{L}) \otimes G(t))(z(t) + e_x(t)). \tag{A10}$$

Observe that $(\mathbf{1}_N r^{\mathrm{T}} \otimes I_n)\dot{z}(t) = 0$ and $(\mathbf{1}_N r^{\mathrm{T}} \otimes I_n)z(0) = 0$. From this, we deduce that

$$\dot{z}(t) = \left(\left(-\mathcal{L} - \mathbf{1}_N r^{\mathrm{T}}\right) \otimes G(t)\right)z(t) - (\mathcal{L} \otimes G(t))e_x(t).$$

The proof comes to an end.

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