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Weighted Composition Operators between Bers-Type Spaces on Generalized Hua-Cartan-Hartogs Domains †

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Abstract: We address weighted composition operators between Bers-type spaces on generalized Hua-Cartan-Hartogs domains and provide necessary and sufficient conditions for their boundedness and compactness. We then apply our results to study the boundedness and the compactness of weighted composition operators between Bers-type spaces on five different domains: generalized Hua domains, Cartan-Hartogs domains, generalized Cartan-Hartogs domains, generalized Cartan-Hartogs domains over different Cartan domains and generalized ellipsoidal-type domains.

Keywords: the generalized Hua-Cartan-Hartogs domains; Bers-type space; weighted composition operators; boundedness and compactness

MSC: 32A27; 47B33

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1. Introduction

Throughout this paper, we denote by Ω a bounded domain of $\mathbb{C}^n$ and by $H(\Omega)$ the set of all holomorphic functions on Ω. We also denote by ϕ a holomorphic map of Ω on itself (self-map) and by ψ a generic element of $H(\Omega)$. A weighted composition operator ψC_ϕ is defined as

$$\psi C_\phi f(z) = \psi(z)(f \circ \phi(z)), \ \ z \in \Omega.$$

If $\psi \equiv 1$, ψC_ϕ reduces to the composition operator, usually denoted by C_ϕ. If $\phi(z) = z$, we have the multiplication operator, usually denoted by M_ψ.

In recent years, there has been great interest in the study of (weighted) composition operators between spaces of various domains. For example, on the unit disk, some properties of weighted composition operators between H^∞ and Bloch space and between Bloch space and weighted Banach space have been extensively discussed by Allen in [1,2], respectively. Pu characterised the sufficient and necessary conditions of bounded and compact composition operators from Bloch space to Bers-type space and small Bers-type space [3]. Zhong, Wang and Liu [4,5] also discussed sufficient and necessary conditions for the boundedness and compactness of weighted composition operator between Bers-type spaces, obtaining the same results.

For the unit ball, Jin and Tang [6] investigated the sufficient and necessary conditions for the boundedness and compactness of weighted composition operator between Bers-type spaces, which generalize the results obtained for the unit disk in [4,5]. Du and Li [7] studied the properties of weighted composition operators from H^∞ space to Bloch space. In [8], Dai described the characteristics of Lipschitz space on the unit ball, and gave the necessary and sufficient conditions for the weighted composition operators on Lipschitz space to be bounded and compact. Zhou et al. studied the properties of weighted composition operators from Bers-type space to Bloch space [9].

Concerning the polydisc, Li and Zhang [10] discussed the equivalence of compactness conditions for composition operators between Bloch type spaces on the polydisc, and gave the simplest representation of the compactness conditions. The boundedness and compactness of composition operators of general weight Bloch spaces were studied by Hu in [11]. Stević investigated the boundedness

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and compactness of composition operators between special weight Bloch space and H^∞ space in [12]. Later, in [13,14], together with Li , the conclusions were extended to the case of weighted composition operators.

In 1930, E. Cartan [15] fully characterized the irreducible bounded symmetric domains into six types: four types of Cartan domains and two exceptional domains of complex dimensions 16 and 27, respectively. The four types of Cartan domains are defined as follows:

$$\begin{aligned}
\Re_{\mathrm{I}}(m,n) &:= \{Z \in \mathbb{C}^{m\times n} : I - Z\overline{Z}' > 0\}.\\
\Re_{\mathrm{II}}(p) &:= \{Z \in \mathbb{C}^{p\times p} : I - Z\overline{Z} > 0, Z = Z'\}.\\
\Re_{\mathrm{III}}(q) &:= \{Z \in \mathbb{C}^{q\times q} : I + Z\overline{Z} > 0, Z = -Z'\}.\\
\Re_{\mathrm{IV}}(N) &:= \{z \in \mathbb{C}^{N} : 1 + |zz'|^2 - 2z\overline{z}' > 0, 1 - |zz'|^2 > 0\}.
\end{aligned}$$

where m, n, p, q, N are positive integers, $Z \in \mathbb{C}^{m\times n}$ means that Z is a $m \times n$ complex matrix, $\overline{Z}$ denotes the conjugate of Z and Z' denotes the transpose of Z.

For the sake of convenience, the four Cartan domains will be denoted by the shorthands $\Re_{\mathrm{I}}, \Re_{\mathrm{II}}, \Re_{\mathrm{III}}$ and $\Re_{\mathrm{IV}}$, respectively, throughout the papers. Su revisited the classical extremal problem on Cartan domains in [16]. Wang and Liu studied the Bloch constant on Cartan domain of the first kind in [17].

In 1998, building on the notion of Cartan domains, Yin was inspired by Roos to construct a new type of domain called the Cartan-Hartogs domains:

$$\begin{aligned}
\mathrm{Y}_{\mathrm{I}}(N;m,n;k) &:= \{\xi \in \mathbb{C}^N, Z \in \Re_{\mathrm{I}}(m,n) : |\xi|^{2k} < \det(I - Z\overline{Z}')\},\\
\mathrm{Y}_{\mathrm{II}}(N;p;k) &:= \{\xi \in \mathbb{C}^N, Z \in \Re_{\mathrm{II}}(p) : |\xi|^{2k} < \det(I - Z\overline{Z}')\},\\
\mathrm{Y}_{\mathrm{III}}(N;q;k) &:= \{\xi \in \mathbb{C}^N, Z \in \Re_{\mathrm{III}}(q) : |\xi|^{2k} < \det(I - Z\overline{Z}')\},\\
\mathrm{Y}_{\mathrm{IV}}(N;n;k) &:= \{\xi \in \mathbb{C}^N, z \in \Re_{\mathrm{IV}}(n) : |\xi|^{2k} < (1 + |zz'|^2 - 2|z|^2)\}.
\end{aligned}$$

where $\Re_{\mathrm{I}}(m,n), \Re_{\mathrm{II}}(p), \Re_{\mathrm{III}}(q)$ and $\Re_{\mathrm{IV}}(n)$ denote respectively the Cartan domains of the first type, second type, third type and fourth type, $\overline{Z}$ denotes the conjugate of Z and Z' denotes the transpose of Z, N, m, n, p, q are positive integers, k is positive real number.

In this framework, Bai [18] discussed the boundedness and compactness of weighted composition operators between Bers-type spaces on Cartan-Hartogs domain of the first type, and obtained necessary and sufficient conditions. In [19], Su and Zhang studied the boundedness and compactness of weighted composition operators from H^∞ to the special weight Bloch space on Cartan-Hartogs domain of the first type. The boundedness and compactness of composition operators between special weight Bloch spaces on Cartan-Hartogs domain of the fourth type were studied by Su and Zhang in [20].

For $m = k = 1$, the Cartan-Hartogs domain of the first type Y_{I} reduces to the unit ball.

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Yin then constructed four types of Cartan-Egg domains [21], later extending the Cartan-Egg domains to Hua domains [22]:

$$\begin{aligned}
&\mathrm{HE_I}(n_1, n_2, \cdots, n_r; m, n; p_1, p_2, \cdots, p_r) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{I}}(m,n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z}'), j = 1, 2, \cdots, r \right\} \\
&\mathrm{HE_{II}}(n_1, n_2, \cdots, n_r; p; p_1, p_2, \cdots, p_r) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{II}}(p) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z}), j = 1, 2, \cdots, r \right\} \\
&\mathrm{HE_{III}}(n_1, n_2, \cdots, n_r; q; p_1, p_2, \cdots, p_r) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{III}}(q) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I + Z\overline{Z}), j = 1, 2, \cdots, r \right\} \\
&\mathrm{HE_{IV}}(n_1, n_2, \cdots, n_r; n; p_1, p_2, \cdots, p_r) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, z \in \Re_{\mathrm{IV}}(n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < (1 + |zz'|^2 - 2z\overline{z}'), j = 1, 2, \cdots, r \right\}
\end{aligned}$$

where $\xi_j = (\xi_{j1}, \cdots, \xi_{jn_j})$, $|\xi_j|^2 = \sum_{i=1}^{n_j} |\xi_{ji}|^2 (j = 1, 2, \cdots, r)$. $n_1, \cdots, n_r, m, n, p, q$ are positive integers, $p_1, \cdots, p_r$ are positive real numbers.

The explicit formula of the Bergman kernel function on the four types of Hua domains have been obtained in [22]. Li, Su and Wang [23] discussed an extremal problem on Hua domains of the second type. In [24–26], Liu et al. studied the convexity of Hua domains of the first, second and third type respectively, and computed the Carathéodory metric and Kobayashi metric on these three domains. Su et al investigated the boundness and compactness of composition operators between u-Bloch space and v-Bloch space on Hua domains of the first type [27]. The necessary and sufficient conditions for the boundedness and compactness of weighted composite operators between Bers-type spaces on four types of Hua domains are characterized by Jiang and Li in [28].

In 2003, Yin once again extended the Hua domains to the generalized Hua domains [29]:

$$\begin{aligned}
&\mathrm{GHE_I}(n_1, n_2, \cdots, n_r; m, n; p_1, p_2, \cdots, p_r; k) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{I}}(m,n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z}')^k, j = 1, 2, \cdots, r \right\} \\
&\mathrm{GHE_{II}}(n_1, n_2, \cdots, n_r; p; p_1, p_2, \cdots, p_r; k) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{II}}(p) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z})^k, j = 1, 2, \cdots, r \right\} \\
&\mathrm{GHE_{III}}(n_1, n_2, \cdots, n_r; q; p_1, p_2, \cdots, p_r; k) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{III}}(q) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I + Z\overline{Z})^k, j = 1, 2, \cdots, r \right\} \\
&\mathrm{GHE_{IV}}(n_1, n_2, \cdots, n_r; n; p_1, p_2, \cdots, p_r; k) \\
&= \left\{ \xi_j \in \mathbb{C}^{n_j}, z \in \Re_{\mathrm{IV}}(n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < (1 + |zz'|^2 - 2z\overline{z}')^k, j = 1, 2, \cdots, r \right\}
\end{aligned}$$

where $\xi_j = (\xi_{j1}, \cdots, \xi_{jn_j})$; $|\xi_j|^2 = \sum_{i=1}^{n_j} |\xi_{ji}|^2 (j = 1, 2, \cdots, r)$. $n_1, \cdots, n_r, m, n, p, q$ are positive integers, $p_1, \cdots, p_r, k$ are positive real numbers. When $k = 1$, the generalized Hua domains are the Hua domains.

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The explicit formula of the Bergman kernel function on the four types of generalized Hua domains have been obtained in [29]. Su and Wang studied the boundedness and compactness of operators $\psi C_\phi : \mathcal{B}^\alpha \to \mathcal{A}_\beta$ between Bloch space and Bers space on generalized Hua domains of the first type, and obtained some sufficient conditions and necessary conditions [30].

In 2005, Yin extended the generalized Hua domains and proposed a type of domains named Hua Constructions [31]. Both Hua domains and generalized Hua domains are special cases of Hua Construction. Cartan-Hartogs domains, Cartan-Egg domains, Hua domains, generalized Hua domains and Hua Constructions are collectively referred to as Hua domains, see [33]. The Hua domains is generally not transitive except for the unit ball. It is thus very meaningful to study the problem on Hua domains.

Ahn-Park [35]introduced the generalized Cartan-Hartogs domain:

$$\widehat{\Omega}_m = \{\xi \in \mathbb{C}^m, Z_k \in \Omega_k : |\xi|^2 < N_{\Omega_1}(Z_1, Z_1)^{\mu_1} \cdots N_{\Omega_t}(Z_t, Z_t)^{\mu_t}, \mu_k > 0\}.$$

where Ω_k is one of the six bounded symmetric domains, and $N_{\Omega_k}(Z_k, Z_k)$ is the corresponding generic norm of Ω_k. $k = 1, \cdots, t$. m is positive integer. This type of domains generalizes the Cartan-Hartogs domain introduced by Yin and Roos.

Wang et al. proved vanishing theorem on generalized Cartan-Hartogs domains of the second type in [36,37], and [38] discussed the boundedness and compactness of composition operators between weighted Bloch spaces on generalized Cartan-Hartogs domain of the first type. A considerable attention has been devoted to Rawnsley's ε-functions and to the comparison theorem for the Einstein-Kahler and Kobayashi metrics on generalized Cartan-Hartogs domains, see e.g., [39,40]. Other conclusions on the generalized Cartan-Hartogs domains can be found in [41–43]. On the other hand, the properties of weighted composition operators between Bers-type spaces on generalized Cartan-Hartogs domains have not been studied.

Starting from these results, we introduce a novel type of domains, which we term the generalized Hua-Cartan-Hartogs domain:

$$\begin{aligned}&\mathscr{H}(n_1, \cdots, n_r; p_1, \cdots, p_r; A_1, \cdots, A_t)\\ &= \{\xi_j \in \mathbb{C}^{n_j}, Z_k \in \Re_{A_k} : \sum_{j=1}^{r} |\xi_j|^{2p_j} < N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}), j = 1, \cdots, r;\ k = 1, \cdots, t.\}.\end{aligned}$$

where $n_1, \cdots, n_r, t, r$ are positive integers, $p_1, \cdots, p_r$ are positive real numbers, and $\xi_j = (\xi_{j_1}, \xi_{j_2}, \cdots, \xi_{j_{n_j}})$, $\Re_{A_1}, \Re_{A_2}, \cdots, \Re_{A_t} \in \{\Re_{\mathrm{I}}, \Re_{\mathrm{II}}, \Re_{\mathrm{III}}, \Re_{\mathrm{IV}}\}$. The generic norm $N_k(Z_k, \overline{W_k})$ are holomorphic for Z_k and antl-holomorphic for W_k, where $Z_k, W_k \in \Re_{A_k}$, $k = 1, \cdots, t$. $N_k(Z_k, \overline{W_k})$ should meet the following two conditions:

$$(1) \qquad 0 < N_k(Z_k, \overline{Z_k}) \leq 1,\ k = 1, \cdots, t. \tag{1.1}$$

$$(2) \qquad 2|N_k(Z_k, \overline{W_k})| \geq N_k(Z_k, \overline{Z_k}) + N_k(W_k, \overline{W_k}),\ Z_k, W_k \in \Re_{A_k},\ k = 1, \cdots, t. \tag{1.2}$$

We use the shorthand $\mathscr{H}$ for the generalized Hua-Cartan-Hartogs domain and denote the points of $\mathscr{H}$ by $(Z_1, Z_2, \cdots, Z_t, \xi_1, \xi_2, \cdots, \xi_r) = (Z, \xi)$, where $(Z_1, Z_2, \cdots, Z_t) = Z$, $(\xi_1, \xi_2, \cdots, \xi_r) = \xi$.

In the fourth Section of this paper, we will further prove that generalized Hua domains, Cartan-Hartogs domains, generalized Cartan-Hartogs domains, generalized Cartan-Hartogs domains with different types of Cartan domains as bases, and generalized ellipsoidal-type domains are special generalized Hua-Cartan-Hartogs domain.

A Bers-type space on $\mathscr{H}$ is defined as follows:

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Definition 1.1. *Let $\alpha > 0$. A Bers-type space on $\mathscr{H}$, denoted by $\mathcal{A}_\alpha(\mathscr{H})$, consists of all holomorphic functions on $\mathscr{H}$ satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathscr{H})} = \sup_{(Z,\xi)\in\mathscr{H}} \left[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \sum_{j=1}^{r}|\xi_j|^{2p_j}\right]^\alpha |f(Z,\xi)| < +\infty.$$

It is easy to see that $\mathcal{A}_\alpha(\mathscr{H})$ is a Banach space.

In this paper, we study the boundedness and compactness of weighted composition operator between Bers-type spaces on the generalized Hua-Cartan-Hartogs domain, and obtain necessary and sufficient conditions, which are relevant generalizations of some previous conclusions.

As some applications, in the fourth section of this paper, we will prove sufficient and necessary conditions for the boundedness and compactness of weighted composition operators between Bers-type spaces on five domains: generalized Hua domains, Cartan-Hartogs domains, generalized Cartan-Hartogs domains, generalized Cartan-Hartogs domains with different types of Cartan domains as bases, and generalized ellipsoidal-type domains. Here, we briefly anticipate them:

(1) For $t = 1$, if $Z \in \Re_{\mathrm{A}}, \mathrm{A} = \mathrm{I}, \mathrm{II}$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = \det(I - Z\overline{Z}')^k$, if $z \in \Re_{\mathrm{IV}}$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = (1 + |zz'|^2 - 2|z|^2)^k$, where k is positive real number. In this case, the generalized Hua-Cartan-Hartogs domains are equivalent to generalized Hua domains.

(2) For $t = 1$, starting from the generalized Hua domains, let $p_1 = p_2 = \cdots p_r = 1$. If $Z \in \Re_{\mathrm{A}}, \mathrm{A} = \mathrm{I}, \mathrm{II}$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = \det(I - Z\overline{Z}')^k$, if $z \in \Re_{\mathrm{IV}}$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = (1 + |zz'|^2 - 2|z|^2)^k$, where k is positive real number. In this case, the generalized Hua-Cartan-Hartogs domains are Cartan-Hartogs domains.

(3) For $t = 2$, if $Z_k \in \Re_{\mathrm{A_k}}, \mathrm{A_k} = \mathrm{I}, \mathrm{II}$, then let $N_k(Z_k,\overline{Z_k}) = \det(I - Z_k\overline{Z_k}')^{s_k}$, if $z_k \in \Re_{\mathrm{IV}}$, then let $N_k(Z_k,\overline{Z_k}) = (1 + |z_k z_k'|^2 - 2|z_k|^2)^{s_k}$, where s_k are positive real numbers, $k = 1, 2$. In this case, the generalized Hua-Cartan-Hartogs domains are generalized Cartan-Hartogs domains. The boundedness and compactness of weighted composition operator between Bers-type spaces on generalized Cartan-Hartogs domains will be discussed by examples from generalized Cartan-Hartogs domains and generalized Cartan-Hartogs domains over different types of Cartan domains, respectively.

(4) For $t = 1$, if $Z \in \Re_{\mathrm{I}}(m,n)$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = 1 - \frac{|Z|^2}{m}$; if $Z \in \Re_{\mathrm{II}}(p)$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = 1 - \frac{|Z|^2}{p}$; if $Z \in \Re_{\mathrm{III}}(q)$, then let $N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = 1 - \frac{|Z|^2}{[\frac{q}{2}]}$, where $[\frac{q}{2}]$ denotes the integer part of $\frac{q}{2}$ and if $z \in \Re_{\mathrm{IV}}(N)$, let$N_1(Z_1,\overline{Z_1}) = N(Z,\overline{Z}) = 1 - |z|^2$. In this case, we refer to this generalized Hua-Cartan-Hartogs domains as generalized ellipsoidal-type domains.

Constants will be denoted by $C, C_1, C_2, \cdots$, they are positive and may differ in the different cases. Without loss of generality, we assume that $n_j = 1$, that is $\xi_j \in \mathbb{C}$, $j = 1, 2, \cdots, r$, $\xi = (\xi_1, \xi_2, \cdots, \xi_r)$ and $\|\xi\|^2 = \sum_{j=1}^{r}|\xi_j|^{2p_j}$.

2. Preliminaries

This section is devoted to present and prove few lemmas that will be used in the following theorems about the boundedness and compactness of weighted composition operators $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$.

Lemma 2.1 (see [34]). *Given the sequence $a_k, \in \mathbb{C}$, $k = 1, 2, \cdots, n$, then when $p \geq 1$*

$$\sum_{k=1}^{n}|a_k|^p \leq \left[\sum_{k=1}^{n}|a_k|\right]^p \leq n^{p-1}\sum_{k=1}^{n}|a_k|^p. \tag{2.1}$$

and when $0 < p < 1$

$$\sum_{k=1}^{n}|a_k|^p \geq \left[\sum_{k=1}^{n}|a_k|\right]^p \geq n^{p-1}\sum_{k=1}^{n}|a_k|^p. \tag{2.2}$$

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Lemma 2.2 (see [34]). *(The product-type Minkowski inequality) Let $a_k, b_k \geq 0$, $k = 1, 2, \cdots, n$, then*

$$\left[\prod_{k=1}^{n}(a_k + b_k)\right]^{\frac{1}{n}} \geq \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}}. \tag{2.3}$$

with the equality that holds iff $a_k = Cb_k, k = 1, 2, \cdots, n$.

Lemma 2.3 (see [32]). *Given the $m \times n$ matrix $(m \leq n)$*

$$Z = \begin{pmatrix} z_{11} & z_{12} & \dots & z_{1n} \\ z_{21} & z_{22} & \dots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \dots & z_{mn} \end{pmatrix},$$

then there exist an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that

$$Z = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_m & 0 & \dots & 0 \end{pmatrix} V \ (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0).$$

Lemma 2.4 (see [32]). *Given two diagonal $m \times m$ matrices Λ_1, Λ_2*

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix} \ (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$$

and

$$\Lambda_2 = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_m \end{pmatrix} \ (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0)$$

satisfying

$$\lambda_j \mu_k < 1 \ (j, k = 1, \cdots, m),$$

then there exists a permutation matrix P such that

$$\inf_{U\overline{U}' = I,\ V\overline{V}' = I} |\det(I - \Lambda_1 U \Lambda_2 \overline{U}' V)| = |\det(I - \Lambda_1 P \Lambda_2 P')|.$$

The infimum is achieved for $U = \Theta P, V = I$, where

$$\Theta = \begin{pmatrix} e^{i\theta_1} & 0 & \dots & 0 \\ 0 & e^{i\theta_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{i\theta_m} \end{pmatrix}.$$

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Lemma 2.5. *For any $\alpha > 0$, $(Z, \xi) \in \mathscr{H}$ and $f \in \mathcal{A}_\alpha(\mathscr{H})$, one has*

$$|f(Z,\xi)| \leq \frac{\|f\|_{\mathcal{A}_\alpha(\mathscr{H})}}{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\alpha}.$$

Proof. By $f \in \mathcal{A}_\alpha(\mathscr{H})$, we have

$$\begin{aligned}\|f\|_{\mathcal{A}_\alpha(\mathscr{H})} &= \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\alpha |f(Z,\xi)| \\ &\geq [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\alpha |f(Z,\xi)|,\end{aligned}$$

and so

$$|f(Z,\xi)| \leq \frac{\|f\|_{\mathcal{A}_\alpha(\mathscr{H})}}{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\alpha},$$

and the lemma is proved. □

Lemma 2.6. *Given $\alpha, \beta > 0$, the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is compact if and only if ψC_ϕ is bounded in $\mathcal{A}_\beta(\mathscr{H})$ and for any bounded sequence $\{f_k\}_{k\geq 1}$ in $\mathcal{A}_\alpha(\mathscr{H})$ converging to 0 uniformly on every compact subsets of $\mathscr{H}$ one has that*

$$\lim_{k\to\infty} \|\psi C_\phi f_k\|_{\mathcal{A}_\beta(\mathscr{H})} = 0.$$

Proof. Assume that $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is compact, then it must be a bounded operator. Let $\{f_k\}_{k\geq 1}$ be a bounded sequence in $\mathcal{A}_\alpha(\mathscr{H})$ such that $f_k \rightrightarrows 0$ on every compact subsets of $\mathscr{H}$ as $k \to \infty$.

If $\|\psi C_\phi f_k\|_{\mathcal{A}_\beta(\mathscr{H})} \nrightarrow 0$ as $k \to \infty$, then there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ such that

$$\inf_{j\in N} \|\psi C_\phi f_{k_j}\|_{\mathcal{A}_\beta(\mathscr{H})} > 0.$$

Since ψC_ϕ is compact, then there exists a function $g \in \mathcal{A}_\beta(\mathscr{H})$ and a subsequence of $\{f_{k_j}\}$ (still written by $\{f_{k_j}\}$ without loss of generality), such that

$$\lim_{j\to\infty} \|\psi C_\phi f_{k_j} - g\|_{\mathcal{A}_\beta(\mathscr{H})} = 0.$$

Let K be a compact subset of $\mathscr{H}$, for $\forall (Z,\xi) \in K \subset \mathscr{H}$ and $\psi C_\phi f_{k_j} - g \in \mathcal{A}_\beta(\mathscr{H})$, it follows from Lemma 2.5 that

$$|\psi C_\phi f_{k_j}(Z,\xi) - g(Z,\xi)| \leq \frac{\|\psi C_\phi f_{k_j} - g\|_{\mathcal{A}_\beta(\mathscr{H})}}{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta}.$$

This means that $\psi C_\phi f_{k_j} - g \rightrightarrows 0$ on compact subset K as $j \to \infty$, and that for $\forall \varepsilon > 0$, $\exists J_1 > 0$, such that for $j > J_1$, we have

$$|\psi C_\phi f_{k_j}(Z,\xi) - g(Z,\xi)| < \varepsilon, (Z,\xi) \in K.$$

Since $f_{k_j} \rightrightarrows 0$ on every compact subsets of $\mathscr{H}$ as $j \to \infty$, then for the above ε, $\exists J_2 > 0$, such that for $j > J_2$ we have

$$|f_{k_j}(\phi(Z,\xi))| < \varepsilon, (Z,\xi) \in K.$$

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In this case, let $J = \max\{J_1, J_2\}$, $M = \max_{(Z,\xi)\in K} |\psi(Z,\xi)|$, that is $|\psi(Z,\xi)| \leq M$ on K, when $j > J$, for $(Z,\xi) \in K$, we obtain

$$\begin{aligned} |g(Z,\xi)| < |\psi C_\phi f_{k_j}(Z,\xi)| + \varepsilon &= |\psi(Z,\xi)||f_{k_j}(\phi(Z,\xi))| + \varepsilon \\ &\leq M\varepsilon + \varepsilon \\ &= (M+1)\varepsilon. \end{aligned}$$

From the arbitrariness of ε, we have $g(Z,\xi) \equiv 0$ on K. Then, by the uniqueness theorem of analytic functions, we have $g \equiv 0$ on $\mathscr{H}$. It follows that

$$\lim_{j\to\infty} \|\psi C_\phi f_{k_j}\|_{\mathcal{A}_\beta(\mathscr{H})} = 0.$$

which contradicts with $\inf_{j\in N} \|\psi C_\phi f_{k_j}\|_{\mathcal{A}_\beta(\mathscr{H})} > 0$. Therefore, $\lim_{k\to\infty} \|\psi C_\phi f_k\|_{\mathcal{A}_\beta(\mathscr{H})} = 0$.

Conversely, assume that $\{f_k\}_{k\geq 1}$ is a bounded sequence in $\mathcal{A}_\alpha(\mathscr{H})$, and that $\|f_k\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq C_1$,$k = 1, 2, \cdots$. Then $\{f_k\}_{k\geq 1}$ is locally uniformly bounded on $\mathscr{H}$. According to Montel's theorem, there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ such that $f_{k_j} \rightrightarrows f$ on every compact subset of $\mathscr{H}$ as $j \to \infty$. This means that $f_{k_j} - f \rightrightarrows 0$ on every compact subset of $\mathscr{H}$ as $j \to \infty$.

For $\forall (Z_0, \xi_0) \in \mathscr{H}$, there exists a compact subset $K_{(Z_0,\xi_0)}$ such that $(Z_0, \xi_0) \in K_{(Z_0,\xi_0)}$. Since $f_{k_j} \rightrightarrows f$, $j \to \infty$, $(Z,\xi) \in K_{(Z_0,\xi_0)}$, then $\exists J_0 > 0$, such that for $j > J_0$, we have

$$|f_{k_j}(Z,\xi) - f(Z,\xi)| < 1, (Z,\xi) \in K_{(Z_0,\xi_0)},$$

and we also know that

$$|f(Z_0,\xi_0)| \leq |f(Z_0,\xi_0) - f_{k_j}(Z_0,\xi_0)| + |f_{k_j}(Z_0,\xi_0)|.$$

Hence,

$$\begin{aligned} [\prod_{k=1}^{t} N_k(Z_{k,0}, \overline{Z_{k,0}}) - \|\xi_0\|^2]^\alpha |f(Z_0,\xi_0)| &\leq [\prod_{k=1}^{t} N_k(Z_{k,0}, \overline{Z_{k,0}}) - \|\xi_0\|^2]^\alpha |f(Z_0,\xi_0) - f_{k_j}(Z_0,\xi_0)| \\ &\quad + [\prod_{k=1}^{t} N_k(Z_{k,0}, \overline{Z_{k,0}}) - \|\xi_0\|^2]^\alpha |f_{k_j}(Z_0,\xi_0)| \\ &\leq 1 + C_1. \end{aligned}$$

From the arbitrariness of (Z_0, ξ_0),

$$[\prod_{k=1}^{t} N_k(Z_k, \overline{Z_k}) - \|\xi\|^2]^\alpha |f(Z,\xi)| \leq 1 + C_1, (Z,\xi) \in \mathscr{H}$$

i.e.,

$$\|f\|_{\mathcal{A}_\alpha(\mathscr{H})} = \sup_{(Z,\xi)\in\mathscr{H}} [\prod_{k=1}^{t} N_k(Z_k, \overline{Z_k}) - \|\xi\|^2]^\alpha |f(Z,\xi)| \leq 1 + C_1.$$

Then, for the sequence of functions $\{f_{k_j} - f\}_{j\geq 1}$, we have

$$\|f_{k_j} - f\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq \|f_{k_j}\|_{\mathcal{A}_\alpha(\mathscr{H})} + \|f\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq C_1 + 1 + C_1 = 1 + 2C_1,$$

that is, $\{f_{k_j} - f\}_{j\geq 1}$ is bounded in $\mathcal{A}_\alpha(\mathscr{H})$. Due to the hypothesis,

$$\lim_{k\to\infty} \|\psi C_\phi (f_{k_j} - f)\|_{\mathcal{A}_\beta(\mathscr{H})} = \lim_{k\to\infty} \|\psi C_\phi f_{k_j} - \psi C_\phi f\|_{\mathcal{A}_\beta(\mathscr{H})} = 0,$$

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which shows that $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is compact, and the lemma is proved. □

Lemma 2.7 (Hua-type inequality). *If $Z, S \in \Re_{A_1} \times \Re_{A_2} \times \cdots \times \Re_{A_t}$, then*

$$N_k(Z_k, \overline{Z_k}) \cdot N_k(S_k, \overline{S_k}) \leq |N_k(Z_k, \overline{S_k})|^2, \ \ k = 1, 2, \cdots, t. \tag{2.4}$$

Proof. If $Z, S \in \Re_{A_1} \times \Re_{A_2} \times \cdots \times \Re_{A_t}$, combine (1.2) and $a^2 + b^2 \geq 2ab$, we have

$$\begin{aligned} 2|N_k(Z_k, \overline{S_k})| &\geq N_k(Z_k, \overline{Z_k}) + N_k(S_k, \overline{S_k}) \\ &\geq 2[N_k(Z_k, \overline{Z_k})]^{\frac{1}{2}} \cdot [N_k(S_k, \overline{S_k})]^{\frac{1}{2}}. \end{aligned}$$

Then, we have,

$$N_k(Z_k, \overline{Z_k}) \cdot N_k(S_k, \overline{S_k}) \leq |N_k(Z_k, \overline{S_k})|^2, k = 1, 2, \cdots, t.$$

and the lemma is proved. □

Lemma 2.8. *If $Z, S \in \Re_{A_1} \times \Re_{A_2} \times \cdots \times \Re_{A_t}$, then*

$$N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) \cdot N_1(S_1, \overline{S_1}) \cdots N_t(S_t, \overline{S_t}) \leq |N_1(Z_1, \overline{S_1}) \cdots N_t(Z_t, \overline{S_t})|^2. \tag{2.5}$$

Proof. By (2.4), we get directly

$$\begin{aligned} &N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) \cdot N_1(S_1, \overline{S_1}) \cdots N_t(S_t, \overline{S_t}) \\ &= N_1(Z_1, \overline{Z_1}) N_1(S_1, \overline{S_1}) \cdots N_t(Z_t, \overline{Z_t}) N_t(S_t, \overline{S_t}) \\ &\leq |N_1(Z_1, \overline{S_1})|^2 \cdots |N_t(Z_t, \overline{S_t})|^2 \\ &= |N_1(Z_1, \overline{S_1}) \cdots N_t(Z_t, \overline{S_t})|^2, \end{aligned}$$

and the lemma is proved. □

Lemma 2.9. *If $(Z, \xi), (S, \varsigma) \in \mathscr{H}$, then*

$$|\sum_{j=1}^{r} \xi_j^{p_j} \overline{\varsigma}_j^{p_j}| \leq \|\xi\| \|\varsigma\| \tag{2.6}$$

and

$$\|\xi\| \|\varsigma\| < |N_1(Z_1, \overline{S_1}) \cdots N_t(Z_t, \overline{S_t})|.$$

Proof.

$$\begin{aligned} |\sum_{j=1}^{r} \xi_j^{p_j} \overline{\varsigma}_j^{p_j}| \leq \sum_{j=1}^{r} |\xi_j|^{p_j} \cdot |\overline{\varsigma}_j|^{p_j} &\leq (\sum_{j=1}^{r} |\xi_j|^{2p_j} \cdot \sum_{j=1}^{r} |\overline{\varsigma}_j|^{2p_j})^{\frac{1}{2}} \\ &= (\|\xi\|^2 \|\varsigma\|^2)^{\frac{1}{2}} \\ &= \|\xi\| \|\varsigma\| . \end{aligned}$$

For $(Z, \xi), (S, \varsigma) \in \mathscr{H}$, we have

$$\|\xi\|^2 < N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t})$$

$$\|\varsigma\|^2 < N_1(S_1, \overline{S_1}) \cdots N_t(S_t, \overline{S_t}).$$

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From (2.5), it follows that

$$\begin{aligned}\|\xi\|\|\varsigma\| &< [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})\cdot N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})]^{\frac{1}{2}}\\ &\leq |N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})|,\end{aligned}$$

and the lemma is proved. □

Lemma 2.10. *If $(Z,\xi),(S,\varsigma)\in\mathscr{H}$, then*

$$\begin{aligned}&[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2][N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]\\ &\leq [|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})|-\|\xi\|\|\varsigma\|]^2.\end{aligned}\tag{2.7}$$

Proof. Assume that a,b,c,d are nonnegative real numbers with $b\leq a,d\leq c$, then we have $(a^2-b^2)(c^2-d^2)\leq(ac-bd)^2$. From this inequality and (2.5), we obtain

$$\begin{aligned}&[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2][N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]\\ &\leq \{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})]^{\frac{1}{2}}[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})]^{\frac{1}{2}}-\|\xi\|\|\varsigma\|\}^2\\ &\leq [|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})|-\|\xi\|\|\varsigma\|]^2,\end{aligned}$$

and the lemma is proved. □

Lemma 2.11. *Let $\alpha>0$, if $(S,\varsigma)\in\mathscr{H}$, then $f_{(S,\varsigma)}(Z,\xi)\in\mathcal{A}_\alpha(\mathscr{H})$, where the function $f_{(S,\varsigma)}(Z,\xi)$ is defined as*

$$f_{(S,\varsigma)}(Z,\xi)=\frac{[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]^\alpha}{[N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})-\sum_{j=1}^r\xi_j^{p_j}\overline{\varsigma}_j^{p_j}]^{2\alpha}}$$

with $\|f_{(S,\varsigma)}\|_{\mathcal{A}_\alpha(\mathscr{H})}\leq 1$.

Proof. From (2.6) and (2.7), we have

$$\begin{aligned}&[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\alpha|f_{(S,\varsigma)}(Z,\xi)|\\ &=[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\alpha\frac{[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]^\alpha}{|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})-\sum_{j=1}^r\xi_j^{p_j}\overline{\varsigma}_j^{p_j}|^{2\alpha}}\\ &\leq[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\alpha\frac{[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]^\alpha}{[|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})|-|\sum_{j=1}^r\xi_j^{p_j}\overline{\varsigma}_j^{p_j}|]^{2\alpha}}\\ &\leq[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\alpha\frac{[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]^\alpha}{[|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})|-\|\xi\|\|\varsigma\|]^{2\alpha}}\\ &\leq 1,\end{aligned}$$

which shows that

$$\|f_{(S,\varsigma)}\|_{\mathcal{A}_\alpha(\mathscr{H})}=\sup_{(Z,\xi)\in\mathscr{H}}[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\alpha|f_{(S,\varsigma)}(Z,\xi)|\leq 1,$$

and $f_{(S,\varsigma)}\in\mathcal{A}_\alpha(\mathscr{H})$. The lemma is thus proved. □

Lemma 2.12. *If $Z,S\in\Re_{A_1}\times\Re_{A_2}\times\cdots\times\Re_{A_t}$, then*

$$2^t|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})|\geq N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})+N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t}).\tag{2.8}$$

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Proof. By (1.2)

$$2|N_k(Z_k,\overline{S_k})| \geq N_k(Z_k,\overline{Z_k}) + N_k(S_k,\overline{S_k}), k = 1,2,\cdots,t.$$

So

$$\begin{aligned}
&2^t|N_1(Z_1,\overline{S_1})\cdots N_t(Z_t,\overline{S_t})| \\
&= 2|N_1(Z_1,\overline{S_1})|\cdot 2|N_2(Z_2,\overline{S_2})|\cdots 2|N_t(Z_t,\overline{S_t})| \\
&\geq [N_1(Z_1,\overline{Z_1}) + N_1(S_1,\overline{S_1})][N_2(Z_2,\overline{Z_2}) + N_2(S_2,\overline{S_2})]\cdots[N_t(Z_t,\overline{Z_t}) + N_t(S_t,\overline{S_t})] \\
&\geq N_1(Z_1,\overline{Z_1})N_2(Z_2,\overline{Z_2})\cdots N_t(Z_t,\overline{Z_t}) + N_1(S_1,\overline{S_1})N_2(S_2,\overline{S_2})\cdots N_t(S_t,\overline{S_t}),
\end{aligned}$$

and the lemma is proved. □

Lemma 2.13. *If* $(Z,\xi),(S,\varsigma) \in \mathscr{H}$, *then*

$$2^t\Big|\prod_{k=1}^{t} N_k(Z_k,\overline{S_k}) - \|\xi\|\|\varsigma\|\Big| \geq \prod_{k=1}^{t} N_k(Z_k,\overline{Z_k}) - \|\xi\|^2 + \prod_{k=1}^{t} N_k(S_k,\overline{S_k}) - \|\varsigma\|^2. \tag{2.9}$$

Proof. Since $(Z,\xi),(S,\varsigma) \in \mathscr{H}$, we have

$$\|\xi\|^2 < \prod_{k=1}^{t} N_k(Z_k,\overline{Z_k}),$$

$$\|\varsigma\|^2 < \prod_{k=1}^{t} N_k(S_k,\overline{S_k}).$$

Given a permutation $i_1, i_2, \cdots, i_t$ of $1,2,\cdots,t$, we write

$$M_{i_1,\cdots,i_k;i_{k+1},\cdots,i_t} = \prod_{l=1}^{k} N_{i_l}(Z_{i_l},\overline{Z_{i_l}}) \prod_{l=k+1}^{t} N_{i_l}(S_{i_l},\overline{S_{i_l}}), k = 1,2,\cdots,t-1.$$

then

$$\begin{aligned}
&M_{i_1,\cdots,i_k;i_{k+1},\cdots,i_t} + M_{i_{k+1},\cdots,i_t;i_1,\cdots,i_k} - 2\|\xi\|\|\varsigma\| \\
&= \prod_{l=1}^{k} N_{i_l}(Z_{i_l},\overline{Z_{i_l}}) \prod_{l=k+1}^{t} N_{i_l}(S_{i_l},\overline{S_{i_l}}) + \prod_{l=1}^{k} N_{i_l}(S_{i_l},\overline{S_{i_l}}) \prod_{l=k+1}^{t} N_{i_l}(Z_{i_l},\overline{Z_{i_l}}) - 2\|\xi\|\|\varsigma\| \\
&\geq 2\prod_{l=1}^{k} N_{i_l}(Z_{i_l},\overline{Z_{i_l}})^{\frac{1}{2}} \prod_{l=k+1}^{t} N_{i_l}(S_{i_l},\overline{S_{i_l}})^{\frac{1}{2}} \prod_{l=1}^{k} N_{i_l}(S_{i_l},\overline{S_{i_l}})^{\frac{1}{2}} \prod_{l=k+1}^{t} N_{i_l}(Z_{i_l},\overline{Z_{i_l}})^{\frac{1}{2}} - 2\|\xi\|\|\varsigma\| \\
&= 2\prod_{l=1}^{t} N_{i_l}(Z_{i_l},\overline{Z_{i_l}})^{\frac{1}{2}} \prod_{l=1}^{t} N_{i_l}(S_{i_l},\overline{S_{i_l}})^{\frac{1}{2}} - 2\|\xi\|\|\varsigma\| \\
&> 2\|\xi\|\|\varsigma\| - 2\|\xi\|\|\varsigma\| \\
&= 0.
\end{aligned}$$

Hence,

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$$
\begin{aligned}
&2^t|\prod_{k=1}^{t} N_k(Z_k,\overline{S_k}) - \|\xi\|\|\varsigma\||\\
&\geq 2^t|\prod_{k=1}^{t} N_k(Z_k,\overline{S_k})| - 2^t\|\xi\|\|\varsigma\|\\
&\geq \prod_{k=1}^{t}[N_k(Z_k,\overline{Z_k}) + N_k(S_k,\overline{S_k})] - 2\|\xi\|\|\varsigma\| - (2^t-2)\|\xi\|\|\varsigma\|\\
&= \prod_{k=1}^{t} N_k(Z_k,\overline{Z_k}) + \prod_{k=1}^{t} N_k(S_k,\overline{S_k}) - 2\|\xi\|\|\varsigma\| + \mathcal{M} - (2^t-2)\|\xi\|\|\varsigma\|\\
&\geq \prod_{k=1}^{t} N_k(Z_k,\overline{Z_k}) + \prod_{k=1}^{t} N_k(S_k,\overline{S_k}) - \|\xi\|^2 - \|\varsigma\|^2\\
&= \prod_{k=1}^{t} N_k(Z_k,\overline{Z_k}) - \|\xi\|^2 + \prod_{k=1}^{t} N_k(S_k,\overline{S_k}) - \|\varsigma\|^2,
\end{aligned}
$$

where $\mathcal{M}$ denotes the sum of all terms in $\prod_{k=1}^{t}[N_k(Z_k,\overline{Z_k}) + N_k(S_k,\overline{S_k})]$ except $\prod_{k=1}^{t} N_k(Z_k,\overline{Z_k})$ and $\prod_{k=1}^{t} N_k(S_k,\overline{S_k})$, and $\mathcal{M} - (2^t-2)\|\xi\|\|\varsigma\| \geq 0$. The lemma is thus proved. □

3. Main Results

In this Section, we present few charaterization theorems about the boundedness and compactness of weighted composition operators $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$.

Theorem 3.1. *Consider two positive numbers $\alpha, \beta > 0$, a holomorphic self-map ϕ of $\mathscr{H}$ and a function $\psi \in H(\mathscr{H})$. Then, the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is bounded if and only if*

$$
\widetilde{M} = \sup_{(Z,\xi)\in\mathscr{H}} |\psi(Z,\xi)| \frac{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t}) - \|\eta\|^2]^\alpha} < +\infty,
$$

where $(W,\eta) = \phi(Z,\xi)$.

Proof. Assume that $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is bounded. Then, for each $f \in \mathcal{A}_\alpha(\mathscr{H})$, there exists a positive constant C such that $\|\psi C_\phi f\|_{\mathcal{A}_\beta(\mathscr{H})} \leq C\|f\|_{\mathcal{A}_\alpha(\mathscr{H})}$. For the fixed point $(S,\varsigma) \in \mathscr{H}$, consider the function

$$
f_{(S,\varsigma)}(Z,\xi) = \frac{[N_1(A_1,\overline{A_1})\cdots N_t(A_t,\overline{A_t}) - \|\zeta\|^2]^\alpha}{[N_1(Z_1,\overline{A_1})\cdots N_t(Z_t,\overline{A_t}) - \sum_{j=1}^{r}\xi_j^{p_j}\overline{\zeta}_j^{p_j}]^{2\alpha}}, (Z,\xi) \in \mathscr{H}
$$

where $(A,\zeta) = \phi(S,\varsigma)$. From Lemma 2.11, it follows that $f_{(S,\varsigma)} \in \mathcal{A}_\alpha(\mathscr{H})$ and $\|f_{(S,\varsigma)}\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq 1$. By direct computation, we have

$$
\begin{aligned}
&\|\psi C_\phi f_{(S,\varsigma)}\|_{\mathcal{A}_\beta(\mathscr{H})}\\
&= \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi C_\phi f_{(S,\varsigma)}(Z,\xi)|\\
&\geq [N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t}) - \|\varsigma\|^2]^\beta |\psi C_\phi f_{(S,\varsigma)}(S,\varsigma)|\\
&= [N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t}) - \|\varsigma\|^2]^\beta |\psi(S,\varsigma) f_{(S,\varsigma)}(A,\zeta)|\\
&= [N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t}) - \|\varsigma\|^2]^\beta |\psi(S,\varsigma)| \frac{[N_1(A_1,\overline{A_1})\cdots N_t(A_t,\overline{A_t}) - \|\zeta\|^2]^\alpha}{[N_1(A_1,\overline{A_1})\cdots N_t(A_t,\overline{A_t}) - \sum_{j=1}^{r}\zeta_j^{p_j}\overline{\zeta}_j^{p_j}]^{2\alpha}}\\
&= |\psi(S,\varsigma)| \frac{[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t}) - \|\varsigma\|^2]^\beta}{[N_1(A_1,\overline{A_1})\cdots N_t(A_t,\overline{A_t}) - \|\zeta\|^2]^\alpha}.
\end{aligned}
$$

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We also know that $\|\psi C_\phi f_{(S,\varsigma)}\|_{\mathcal{A}_\beta(\mathscr{H})} \leq C\|f_{(S,\varsigma)}\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq C < +\infty$, which shows that,

$$|\psi(S,\varsigma)|\frac{[N_1(S_1,\overline{S_1})\cdots N_t(S_t,\overline{S_t})-\|\varsigma\|^2]^\beta}{[N_1(A_1,\overline{A_1})\cdots N_t(A_t,\overline{A_t})-\|\zeta\|^2]^\alpha} \leq C, (A,\zeta)=\phi(S,\varsigma),$$

that is,

$$\widetilde{M} = \sup_{(Z,\xi)\in\mathscr{H}} |\psi(Z,\xi)|\frac{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t})-\|\eta\|^2]^\alpha} < +\infty, (W,\eta)=\phi(Z,\xi).$$

Conversely, if

$$\widetilde{M} = \sup_{(Z,\xi)\in\mathscr{H}} |\psi(Z,\xi)|\frac{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t})-\|\eta\|^2]^\alpha} < +\infty,$$

then, for all $f \in \mathcal{A}_\alpha(\mathscr{H})$, we obtain from Lemma 2.5

$$\begin{aligned}
&\|\psi C_\phi f\|_{\mathcal{A}_\beta(\mathscr{H})}\\
&= \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta|\psi C_\phi f(Z,\xi)|\\
&= \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta|\psi(Z,\xi)f(W,\eta)|\\
&\leq \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta|\psi(Z,\xi)|\frac{\|f\|_{\mathcal{A}_\alpha(\mathscr{H})}}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t})-\|\eta\|^2]^\alpha}\\
&= \widetilde{M}\|f\|_{\mathcal{A}_\alpha(\mathscr{H})},
\end{aligned}$$

which implies that $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is bounded. This proves the desired result. □

Theorem 3.2. *Consider two positive numbers $\alpha, \beta > 0$, a holomorphic self-map ϕ of $\mathscr{H}$, and a function $\psi \in H(\mathscr{H})$. Then, the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is compact if and only if $\psi \in \mathcal{A}_\beta(\mathscr{H})$ and*

$$\lim_{\phi(Z,\xi)\to\partial\mathscr{H}} |\psi(Z,\xi)|\frac{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t})-\|\eta\|^2]^\alpha} = 0, \tag{3.1}$$

where $(W,\eta)=\phi(Z,\xi)$.

Proof. Assume that the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is compact, then ψC_ϕ is bounded. By taking $f \equiv 1$, we have

$$\begin{aligned}
&[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta|\psi(Z,\xi)|\\
&= [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t})-\|\xi\|^2]^\beta|\psi C_\phi f(Z,\xi)| < +\infty.
\end{aligned}$$

This shows that $\psi \in \mathcal{A}_\beta(\mathscr{H})$. Let us now consider a sequence $(S^i,\varsigma^i)(i=1,2,\cdots.)$ in $\mathscr{H}$ such that $\phi(S^i,\varsigma^i) \to \partial\mathscr{H}$ as $i \to \infty$. If such a sequence does not exist, then (3.1) holds. If such a sequence exists, let $(A^i,\zeta^i)=\phi(S^i,\varsigma^i), i=1,2,\cdots$, and define the sequence of functions $f_i(Z,\xi)=f_{(S^i,\varsigma^i)}(Z,\xi), i=1,2,\cdots$.

$$f_i(Z,\xi)=f_{(S^i,\varsigma^i)}(Z,\xi)=\frac{[N_1(A_1^i,\overline{A_1^i})\cdots N_t(A_t^i,\overline{A_t^i})-\|\zeta^i\|^2]^\alpha}{[N_1(Z_1,\overline{A_1^i})\cdots N_t(Z_t,\overline{A_t^i})-\sum_{j=1}^r \xi_j^{p_j}\overline{\zeta_j^i}^{p_j}]^{2\alpha}}.$$

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From Lemma 2.11, it follows that $f_{(S^i,\varsigma^i)} \in \mathcal{A}_\alpha(\mathscr{H})$ and $\|f_{(S^i,\varsigma^i)}\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq 1$, that is $\{f_i\}$ is bounded in $\mathcal{A}_\alpha(\mathscr{H})$. Taking $i \to \infty$, it follows that $(A^i, \zeta^i) \to \partial\mathscr{H}$, and therefore $[N_1(A_1^i, \overline{A_1^i}) \cdots N_t(A_t^i, \overline{A_t^i}) - \|\zeta^i\|^2] \to 0$, as $i \to \infty$.

According to the two inequalities (2.6) and (2.9),

$$\begin{aligned}
|N_1(Z_1, \overline{A_1^i}) \cdots N_t(Z_t, \overline{A_t^i}) - \sum_{j=1}^r \xi_j^{p_j} \overline{\zeta_j^i}^{p_j}| &\geq |N_1(Z_1, \overline{A_1^i}) \cdots N_t(Z_t, \overline{A_t^i})| - |\sum_{j=1}^r \xi_j^{p_j} \overline{\zeta_j^i}^{p_j}| \\
&\geq |N_1(Z_1, \overline{A_1^i}) \cdots N_t(Z_t, \overline{A_t^i})| - \|\xi\|\|\zeta^i\| \\
&\geq 2^{-t}[N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) - \|\xi\|^2] \\
&+ 2^{-t}[N_1(A_1^i, \overline{A_1^i}) \cdots N_t(A_t^i, \overline{A_t^i}) - \|\zeta^i\|^2] \\
&\geq 2^{-t}[N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) - \|\xi\|^2].
\end{aligned}$$

$[N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) - \|\xi\|^2]$ has thus a positive lower bound in the compact subset. Hence, $f_i \rightrightarrows 0$ on every compact subsets of $\mathscr{H}$ as $i \to \infty$. Then, by making use of Lemma 2.6,

$$\lim_{i\to\infty} \|\psi C_\phi f_i\|_{\mathcal{A}_\beta(\mathscr{H})} = 0.$$

$$\begin{aligned}
&\|\psi C_\phi f_i\|_{\mathcal{A}_\beta(\mathscr{H})} \\
&= \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) - \|\xi\|^2]^\beta |\psi C_\phi f_i(Z,\xi)| \\
&\geq [N_1(S_1^i, \overline{S_1^i}) \cdots N_t(S_t^i, \overline{S_t^i}) - \|\varsigma^i\|^2]^\beta |\psi C_\phi f_i(S^i,\varsigma^i)| \\
&= [N_1(S_1^i, \overline{S_1^i}) \cdots N_t(S_t^i, \overline{S_t^i}) - \|\varsigma^i\|^2]^\beta |\psi(S^i,\varsigma^i) f_i(A^i,\zeta^i)| \\
&= [N_1(S_1^i, \overline{S_1^i}) \cdots N_t(S_t^i, \overline{S_t^i}) - \|\varsigma^i\|^2]^\beta |\psi(S^i,\varsigma^i)| \frac{[N_1(A_1^i, \overline{A_1^i}) \cdots N_t(A_t^i, \overline{A_t^i}) - \|\zeta^i\|^2]^\alpha}{[N_1(A_1^i, \overline{A_1^i}) \cdots N_t(A_t^i, \overline{A_t^i}) - \sum_{j=1}^r \zeta_j^{i\,p_j} \overline{\zeta_j^i}^{p_j}]^{2\alpha}} \\
&= |\psi(S^i,\varsigma^i)| \frac{[N_1(S_1^i, \overline{S_1^i}) \cdots N_t(S_t^i, \overline{S_t^i}) - \|\varsigma^i\|^2]^\beta}{[N_1(A_1^i, \overline{A_1^i}) \cdots N_t(A_t^i, \overline{A_t^i}) - \|\zeta^i\|^2]^\alpha}.
\end{aligned}$$

This leads to

$$\lim_{i\to\infty} |\psi(S^i,\varsigma^i)| \frac{[N_1(S_1^i, \overline{S_1^i}) \cdots N_t(S_t^i, \overline{S_t^i}) - \|\varsigma^i\|^2]^\beta}{[N_1(A_1^i, \overline{A_1^i}) \cdots N_t(A_t^i, \overline{A_t^i}) - \|\zeta^i\|^2]^\alpha} = 0,$$

where $(A^i, \zeta^i) = \phi(S^i, \varsigma^i)$, and then

$$\lim_{\phi(Z,\xi)\to\partial\mathscr{H}} |\psi(Z,\xi)| \frac{[N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) - \|\xi\|^2]^\beta}{[N_1(W_1, \overline{W_1}) \cdots N_t(W_t, \overline{W_t}) - \|\eta\|^2]^\alpha} = 0,$$

where $(W, \eta) = \phi(Z, \xi)$.

Conversely, suppose that (3.1) holds, then

$$\sup_{(Z,\xi)\in\mathscr{H}} |\psi(Z,\xi)| \frac{[N_1(Z_1, \overline{Z_1}) \cdots N_t(Z_t, \overline{Z_t}) - \|\xi\|^2]^\beta}{[N_1(W_1, \overline{W_1}) \cdots N_t(W_t, \overline{W_t}) - \|\eta\|^2]^\alpha} < +\infty,$$

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that is, ψC_ϕ is bounded. Let $\{f_i\}_{i\geq 1}$ be a bounded sequence of functions in $\mathcal{A}_\alpha(\mathscr{H})$ which converges to 0 uniformly on every compact subsets of $\mathscr{H}$. Upon denoting by $\|f_i\|_{\mathcal{A}_\alpha(\mathscr{H})} \leq C_2, i = 1, 2, \cdots$, we have, by (3.1), that $\forall \varepsilon > 0$, $\exists \sigma > 0$, such that $\forall (Z, \xi) \in E = \{(Z, \xi) \in \mathscr{H} : \mathrm{dist}(\phi(Z, \xi), \partial\mathscr{H}) < \sigma\}$, we have

$$|\psi(Z,\xi)| \frac{[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t}) - \|\eta\|^2]^\alpha} < \varepsilon. \tag{3.2}$$

By (3.2) and Lemma 2.5, we obtain

$$\begin{aligned}
&\sup_{(Z,\xi)\in E} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi C_\phi f_i(Z,\xi)| \\
&= \sup_{(Z,\xi)\in E} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi(Z,\xi)||f_i(W,\eta)| \\
&\leq \sup_{(Z,\xi)\in E} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi(Z,\xi)| \frac{\|f_i\|_{\mathcal{A}_\alpha(\mathscr{H})}}{[N_1(W_1,\overline{W_1})\cdots N_t(W_t,\overline{W_t}) - \|\eta\|^2]^\alpha} \\
&\leq C_2\varepsilon.
\end{aligned}$$

On the other hand, if we set

$$E_\sigma = \{(Z,\xi) \in \mathscr{H} : \mathrm{dist}(\phi(Z,\xi), \partial\mathscr{H}) \geq \sigma\},$$

it is clear that E_σ is a compact subset of $\mathscr{H}$. By hypothesis, we know that $\{f_i\}$ converges to 0 uniformly on every compact subsets of $\mathscr{H}$. Since $\psi \in \mathcal{A}_\beta(\mathscr{H})$, we can assume that $\|\psi\|_{\mathcal{A}_\beta(\mathscr{H})} \leq C_3$. Then, for such $\varepsilon > 0$,

$$\begin{aligned}
&\sup_{(Z,\xi)\in E_\sigma} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi C_\phi f_i(Z,\xi)| \\
&= \sup_{(Z,\xi)\in E_\sigma} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi(Z,\xi)||f_i(\phi(Z,\xi))| \\
&\leq \sup_{(Z,\xi)\in E_\sigma} |\psi(Z,\xi)|[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta \cdot \sup_{(Z,\xi)\in E_\sigma} |f_i(\phi(Z,\xi))| \\
&\leq C_3\varepsilon.
\end{aligned}$$

By combining the above two cases, for $i \to \infty$, we have

$$\begin{aligned}
\|\psi C_\phi f_i\|_{\mathcal{A}_\beta(\mathscr{H})} &= \sup_{(Z,\xi)\in\mathscr{H}} [N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi C_\phi f_i(Z,\xi)| \\
&\leq (\sup_{(Z,\xi)\in E} + \sup_{(Z,\xi)\in E_\sigma})[N_1(Z_1,\overline{Z_1})\cdots N_t(Z_t,\overline{Z_t}) - \|\xi\|^2]^\beta |\psi C_\phi f_i(Z,\xi)| \\
&\leq (C_2 + C_3)\varepsilon.
\end{aligned}$$

That is, $\lim_{i\to\infty} \|\psi C_\phi f_i\|_{\mathcal{A}_\beta(\mathscr{H})} = 0$. From Lemma 2.6, we obtain that the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathscr{H}) \to \mathcal{A}_\beta(\mathscr{H})$ is compact. This completes the proof of the theorem. □

4. Main applications

In this Section, we discuss several special cases of generalized Hua-Cartan-Hartogs domains, and obtain five specific domains in which the necessary and sufficient condition for the boundness and compactness of weighted composition operators between Bers-type spaces may be specifically characterized.

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4.1. The generalized Hua domains

For $t = 1$, consider the quantity

$$N_1(Z_1, \overline{Z_1}) = N(Z, \overline{Z}) = \begin{cases} \det(I - Z\overline{Z}')^k, & Z \in \Re_{\mathrm{I}}(m,n) \\ \det(I - Z\overline{Z}')^k, & Z \in \Re_{\mathrm{II}}(p) \\ (1 + |zz'|^2 - 2|z|^2)^k, & z \in \Re_{\mathrm{IV}}(N) \end{cases} \tag{4.1}$$

where k is positive real number. In this case, the generalized Hua-Cartan-Hartogs domains are the generalized Hua domains:

$$\mathrm{GHE_I} := \{\xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{I}}(m,n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z}')^k, j = 1, \cdots, r\},$$

$$\mathrm{GHE_{II}} := \{\xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{II}}(p) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z\overline{Z}')^k, j = 1, \cdots, r\},$$

$$\mathrm{GHE_{IV}} := \{\xi_j \in \mathbb{C}^{n_j}, z \in \Re_{\mathrm{IV}}(N) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < (1 + |zz'|^2 - 2|z|^2)^k, j = 1, \cdots, r\}.$$

$\mathrm{GHE_I}, \mathrm{GHE_{II}}, \mathrm{GHE_{IV}}$ denote respectively the generalized Hua domains of the first, second, and the fourth kind. The Bers-type spaces on $\mathrm{GHE_I}, \mathrm{GHE_{II}}, \mathrm{GHE_{IV}}$ may be defined as follows

Definition 4.1. *Let $\alpha > 0$.*

(i) *The Bers-type space $\mathcal{A}_\alpha(\mathrm{GHE_I})$ consists of all $f \in H(\mathrm{GHE_I})$ satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{GHE_I})} = \sup_{(Z,\xi)\in \mathrm{GHE_I}} [\det(I - Z\overline{Z}')^k - \sum_{j=1}^{r} |\xi_j|^{2p_j}]^\alpha |f(Z,\xi)| < +\infty.$$

(ii) *The Bers-type space $\mathcal{A}_\alpha(\mathrm{GHE_{II}})$ consists of all $f \in H(\mathrm{GHE_{II}})$ satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{GHE_{II}})} = \sup_{(Z,\xi)\in \mathrm{GHE_{II}}} [\det(I - Z\overline{Z}')^k - \sum_{j=1}^{r} |\xi_j|^{2p_j}]^\alpha |f(Z,\xi)| < +\infty.$$

(iii) *The Bers-type space $\mathcal{A}_\alpha(\mathrm{GHE_{IV}})$ consists of all $f \in H(\mathrm{GHE_{IV}})$ satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{GHE_{IV}})} = \sup_{(z,\xi)\in \mathrm{GHE_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - \sum_{j=1}^{r} |\xi_j|^{2p_j}]^\alpha |f(z,\xi)| < +\infty.$$

To prove that the above generalized Hua domains are special cases of the generalized Hua-Cartan-Hartogs domains, we need to verify that $\det(I - Z\overline{W}')^k$, $Z, W \in \Re_{\mathrm{I}}, \Re_{\mathrm{II}}$ and $(1 + zz'\overline{ww}' - 2z\overline{w}')^k$, $z, w \in \Re_{\mathrm{IV}}$ satisfy conditions (1.1) and (1.2). Indeed, we have

Lemma 4.1. *The following statements hold.*

(i) *If $Z, W \in \Re_{\mathrm{I}}(m,n)$ and $0 < km \leq 1$, then*

$$2|\det(I - Z\overline{W}')^k| \geq \det(I - Z\overline{Z}')^k + \det(I - W\overline{W}')^k. \tag{4.2}$$

(ii) *If $Z, W \in \Re_{\mathrm{II}}(p)$ and $0 < kp \leq 1$, then*

$$2|\det(I - Z\overline{W}')^k| \geq \det(I - Z\overline{Z}')^k + \det(I - W\overline{W}')^k. \tag{4.3}$$

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Proof. (1) Assume that $Z, W \in \Re_1(m,n)$, and that $m \leq n$.
For $m = n$, applying Lemma 2.3, there exist $m \times m$ unitary matrixes U_1, U_2, V_1, V_2 such that

$$Z = U_1 \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix} V_1 = U_1 \Lambda_1 V_1 \quad (1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0), \tag{4.4}$$

and

$$W = U_2 \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_m \end{pmatrix} V_2 = U_2 \Lambda_2 V_2 \quad (1 > \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0). \tag{4.5}$$

Then, it turns out that

$$\begin{aligned} \det(I - Z\overline{W}') &= \det(I - U_1\Lambda_1 V_1 \overline{V_2}' \overline{\Lambda_2}' \overline{U_2}') \\ &= \det(U_1 \overline{U_1}' - U_1 \Lambda_1 V_1 \overline{V_2}' \overline{\Lambda_2}' \overline{U_2}') \\ &= \det U_1 \det(\overline{U_1}' - \Lambda_1 V_1 \overline{V_2}' \overline{\Lambda_2}' \overline{U_2}') \\ &= \det(I - \Lambda_1 V_1 \overline{V_2}' \overline{\Lambda_2}' V_2 \overline{V_1}' V_1 \overline{V_2}' \overline{U_2}' U_1), \end{aligned}$$

and according to Lemma 2.4, there exists a square matrix P, such that

$$|\det(I - Z\overline{W}')| \geq |\det(I - \Lambda_1 P \Lambda_2 P')| = \prod_{i=1}^{m} (1 - \lambda_i \mu_{k_i}),$$

where $k_1, k_2, \cdots, k_m$ is a permutation of $1, 2, \cdots, m$. Using (2.2) and (2.3), we have

$$\begin{aligned} 2|\det(I - Z\overline{W}')^k| &= 2^{1-mk} 2^{mk} |\det(I - Z\overline{W}')^k| \\ &= 2^{1-mk} \Big[2^m |\det(I - Z\overline{W}')| \Big]^k \\ &\geq 2^{1-mk} \Big[2^m \prod_{i=1}^{m} (1 - \lambda_i \mu_{k_i}) \Big]^k \\ &= 2^{1-mk} \Big[\prod_{i=1}^{m} (2 - 2\lambda_i \mu_{k_i}) \Big]^k \\ &\geq 2^{1-mk} \Big[\prod_{i=1}^{m} (2 - \lambda_i^2 - \mu_{k_i}^2) \Big]^k \\ &= 2^{1-mk} \Big\{ \Big\{ \prod_{i=1}^{m} [(1 - \lambda_i^2) + (1 - \mu_{k_i}^2)] \Big\}^{\frac{1}{m}} \Big\}^{mk} \\ &\geq 2^{1-mk} \Big\{ \Big[\prod_{i=1}^{m} (1 - \lambda_i^2) \Big]^{\frac{1}{m}} + \Big[\prod_{i=1}^{m} (1 - \mu_{k_i}^2) \Big]^{\frac{1}{m}} \Big\}^{mk} \\ &\geq 2^{1-mk} 2^{mk-1} \Big\{ \Big[\prod_{i=1}^{m} (1 - \lambda_i^2) \Big]^{\frac{1}{m} \times mk} + \Big[\prod_{i=1}^{m} (1 - \mu_{k_i}^2) \Big]^{\frac{1}{m} \times mk} \Big\} \\ &= \Big[\prod_{i=1}^{m} (1 - \lambda_i^2) \Big]^k + \Big[\prod_{i=1}^{m} (1 - \mu_{k_i}^2) \Big]^k \\ &= \det(I - Z\overline{Z}')^k + \det(I - W\overline{W}')^k. \end{aligned} \tag{4.6}$$

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For $m < n$, there exists a unitary matrix $U^{(n)}$ such that

$$Z = (Z_1^{(m)}, 0)U, \ \ W = (W_1^{(m)}, W_2)U.$$

By (4.6), we obtain

$$\begin{aligned} 2|\det(I - Z\overline{W}')^k| &= 2|\det(I - Z_1\overline{W_1}')^k| \\ &\geq \det(I - Z_1\overline{Z_1}')^k + \det(I - W_1\overline{W_1}')^k \\ &\geq \det(I - Z_1\overline{Z_1}')^k + \det(I - W_1\overline{W_1}' - W_2\overline{W_2}')^k \\ &= \det(I - Z\overline{Z}')^k + \det(I - W\overline{W}')^k. \end{aligned}$$

This completes the proof of (4.2).

(2) Assume $Z, W \in \Re_{\mathrm{II}}(p)$, then the polar decompositions of Z, W are similar to (4.4) and (4.5), thus (4.3) can be proved in the same way. □

Lemma 4.2 (see [32]). *The linear transformation:*

$$w_1 = z_1 + iz_2, w_2 = z_1 - iz_2$$

$$w_3 = iz_3 - z_4, w_4 = iz_3 + z_4$$

maps the domain $\Re_{\mathrm{IV}}(4)$:

$$1 + |z_1^2 + z_2^2 + z_3^2 + z_4^2|^2 - 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) > 0$$

$$1 - |z_1^2 + z_2^2 + z_3^2 + z_4^2|^2 > 0$$

onto the domain $\Re_{\mathrm{I}}(2,2)$:

$$I - W\overline{W}' > 0, W = \begin{pmatrix} w_1 & w_3 \\ w_4 & w_2 \end{pmatrix}.$$

Lemma 4.3 (see [32]). *If Z, W are $m \times n$ matrices which satisfy $I - Z\overline{Z}' \geq 0, I - W\overline{W}' > 0$, then $I - \overline{W}'Z(\overline{\overline{W}'Z})' > 0$ and $I - \overline{W}'Z$ is nonsingular.*

Lemma 4.4. *Let $z, w \in \Re_{\mathrm{IV}}(N)$, then there exist two second-order square matrices Z, W such that*

$$1 + zz'\overline{w}\overline{w}' - 2z\overline{w}' = \det(I - \overline{W}'Z) \neq 0,$$

$$1 + |zz'|^2 - 2|z|^2 = \det(I - Z\overline{Z}'),$$

$$1 + |ww'|^2 - 2|w|^2 = \det(I - W\overline{W}').$$

Proof. If $z, w \in \Re_{\mathrm{IV}}(N)$, then there exists a $N \times N$ real orthogonal square matrix Γ such that the two N-dimensional vectors z and w can be written as

$$z = (z_1^*, z_2^*, z_3^*, z_4^*, 0, \cdots, 0)\Gamma,$$

$$w = (w_1^*, w_2^*, w_3^*, w_4^*, 0, \cdots, 0)\Gamma.$$

According to Lemma 4.2, we have

$$I - Z\overline{Z}' > 0, I - W\overline{W}' > 0,$$

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where

$$Z = \begin{pmatrix} z_1^* + iz_2^* & iz_3^* - z_4^* \\ iz_3^* + z_4^* & z_1^* - iz_2^* \end{pmatrix} \in \Re_{\mathrm{I}}(2,2), \tag{4.7}$$

$$W = \begin{pmatrix} w_1^* + iw_2^* & iw_3^* - w_4^* \\ iw_3^* + w_4^* & w_1^* - iw_2^* \end{pmatrix} \in \Re_{\mathrm{I}}(2,2). \tag{4.8}$$

By Lemma 4.3, we get $\det(I - \overline{W}'Z) \neq 0$. Hence,

$$\begin{aligned} 1 + zz'\overline{ww}' - 2z\overline{w}' &= 1 + [(z_1^*)^2 + (z_2^*)^2 + (z_3^*)^2 + (z_4^*)^2]\overline{[(w_1^*)^2 + (w_2^*)^2 + (w_3^*)^2 + (w_4^*)^2]} \\ &\quad - 2(z_1^*\overline{w_1^*} + z_2^*\overline{w_2^*} + z_3^*\overline{w_3^*} + z_4^*\overline{w_4^*}) \\ &= \det(I - \overline{W}'Z) \neq 0. \end{aligned}$$

and

$$\begin{aligned} 1 + |zz'|^2 - 2|z|^2 &= 1 + |(z_1^*)^2 + (z_2^*)^2 + (z_3^*)^2 + (z_4^*)^2|^2 - 2(|z_1^*|^2 + |z_2^*|^2 + |z_3^*|^2 + |z_4^*|^2) \\ &= \det(I - Z\overline{Z}'), \end{aligned}$$

$$\begin{aligned} 1 + |ww'|^2 - 2|w|^2 &= 1 + |(w_1^*)^2 + (w_2^*)^2 + (w_3^*)^2 + (w_4^*)^2|^2 - 2(|w_1^*|^2 + |w_2^*|^2 + |w_3^*|^2 + |w_4^*|^2) \\ &= \det(I - W\overline{W}'). \end{aligned}$$

This completes the proof of the Lemma. □

Lemma 4.5. *If $z, w \in \Re_{\mathrm{IV}}(N)$, and $0 < k \leq \frac{1}{2}$, then*

$$(1 + |zz'|^2 - 2|z|^2)^k + (1 + |ww'|^2 - 2|w|^2)^k \leq 2|(1 + zz'\overline{ww}' - 2z\overline{w}')^k|. \tag{4.9}$$

Proof. For $z, w \in \Re_{\mathrm{IV}}(N)$, Lemma 4.4 ensures that there exist two second-order square matrices $Z, W \in \Re_{\mathrm{I}}(2,2)$ such that

$$\begin{aligned} (1 + |zz'|^2 - 2|z|^2)^k &= \det(I - Z\overline{Z}')^k, \\ (1 + |ww'|^2 - 2|w|^2)^k &= \det(I - W\overline{W}')^k, \\ (1 + zz'\overline{ww}' - 2z\overline{w}')^k &= \det(I - Z\overline{W}')^k, \end{aligned}$$

where Z, W are identical to (4.7) and (4.8), respectively. By (4.2), we have

$$(1 + |zz'|^2 - 2|z|^2)^k + (1 + |ww'|^2 - 2|w|^2)^k \leq 2|(1 + zz'\overline{ww}' - 2z\overline{w}')^k|.$$

This completes the proof of the Lemma. □

Lemma 4.6. *If $z \in \Re_{\mathrm{IV}}(N)$, then $0 < 1 + |zz'|^2 - 2|z|^2 \leq 1$.*

Proof. Since $z \in \Re_{\mathrm{IV}}(N)$, there exists a real orthogonal square matrix Γ such that

$$z = e^{i\theta}(\lambda_1, i\lambda_2, 0, \cdots, 0)\Gamma \quad (\lambda_1 \geq \lambda_2 > 0,\ \ 1 > \lambda_1 + \lambda_2),$$

where $e^{i\theta} = \cos\theta + i\sin\theta$. Therefore,

$$zz' = e^{i\theta}(\lambda_1, i\lambda_2, 0, \cdots, 0)\Gamma\Gamma'(\lambda_1, i\lambda_2, 0, \cdots, 0)'e^{i\theta} = e^{2i\theta}(\lambda_1^2 - \lambda_2^2),$$

which implies that

$$0 < |zz'| = |e^{2i\theta}(\lambda_1^2 - \lambda_2^2)| = \lambda_1^2 - \lambda_2^2 < 1.$$

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At the same time,

$$\begin{aligned}2|z|^2 &= 2z\overline{z}' = 2e^{i\theta}(\lambda_1, i\lambda_2, 0, \cdots, 0)\Gamma\overline{\Gamma}'(\lambda_1, -i\lambda_2, 0, \cdots, 0)'\overline{e^{i\theta}}' \\ &= 2|e^{i\theta}|^2(\lambda_1^2+\lambda_2^2) = 2(\lambda_1^2+\lambda_2^2) \geq \lambda_1^2-\lambda_2^2 \geq (\lambda_1^2-\lambda_2^2)^2 = |zz'|^2,\end{aligned}$$

that is $|zz'|^2 - 2|z|^2 \leq 0$, then $1+|zz'|^2-2|z|^2 \leq 1$. Furthermore, when $z \in \Re_{\mathrm{IV}}(N)$, we have $1+|zz'|^2-2|z|^2 > 0$. Hence, $0 < 1+|zz'|^2-2|z|^2 \leq 1$. This completes the proof of the Lemma. □

By the definition of the Cartan domains of the first kind and second kind, $\det(I-Z\overline{Z}')^k$, $Z \in \Re_{\mathrm{I}}, \Re_{\mathrm{II}}$ clearly meet (1.1). From Lemma 4.6, $(1+|zz'|^2-2|z|^2)^k$, $z \in \Re_{\mathrm{IV}}$ meets (1.1). Meanwhile, upon using Lemmas 4.1 and 4.5, we have that when k meets certain conditions, $\det(I-Z\overline{W}')^k$, $Z, W \in \Re_{\mathrm{I}}, \Re_{\mathrm{II}}$ and $(1+zz'\overline{w}\overline{w}'-2z\overline{w}')^k$, $z, w \in \Re_{\mathrm{IV}}$ meet conditions (1.2). Therefore, those generalized Hua domains are special cases of generalized Hua-Cartan-Hartogs domains. Using Theorems 3.1 and 3.2, we obtain the conditions for the boundedness and compactness of weighted composition operators between Bers-type spaces on generalized Hua domains.

Let us now consider a holomorphic self-map ϕ of GHE $\in \{\mathrm{GHE_I}, \mathrm{GHE_{II}}, \mathrm{GHE_{IV}}\}$. Let us write $(W,\eta) = \phi(Z,\xi)$ for $(Z,\xi) \in \mathrm{GHE_I}, \mathrm{GHE_{II}}$ and $(w,\eta) = \phi(z,\xi)$ for $(z,\xi) \in \mathrm{GHE_{IV}}$.

Corollary 4.1. *If $\alpha, \beta > 0$ are positive numbers, ϕ is a holomorphic self-map of* GHE *and $\psi \in H(\mathrm{GHE})$, then the following statements hold.*

(i) *If $0 < km \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is bounded iff*

$$\widetilde{M}_I = \sup_{(Z,\xi)\in \mathrm{GHE_I}} |\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|^2]^\beta}{[\det(I-W\overline{W}')^k - \|\eta\|^2]^\alpha} < +\infty.$$

(ii) *If $0 < kp \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GHE_{II}}) \to \mathcal{A}_\beta(\mathrm{GHE_{II}})$ is bounded iff*

$$\widetilde{M}_{II} = \sup_{(Z,\xi)\in \mathrm{GHE_{II}}} |\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|^2]^\beta}{[\det(I-W\overline{W}')^k - \|\eta\|^2]^\alpha} < +\infty.$$

(iii) *If $0 < k \leq \frac{1}{2}$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GHE_{IV}}) \to \mathcal{A}_\beta(\mathrm{GHE_{IV}})$ is bounded iff*

$$\widetilde{M}_{IV} = \sup_{(z,\xi)\in \mathrm{GHE_{IV}}} |\psi(z,\xi)| \frac{[(1+|zz'|^2-2|z|^2)^k - \|\xi\|^2]^\beta}{[(1+|ww'|^2-2|w|^2)^k - \|\eta\|^2]^\alpha} < +\infty.$$

Corollary 4.2. *If $\alpha, \beta > 0$ are positive numbers, ϕ is a holomorphic self-map of* GHE *and $\psi \in H(\mathrm{GHE})$, then the following statements hold.*

(i) *If $0 < km \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GHE_I}) \to \mathcal{A}_\beta(\mathrm{GHE_I})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{GHE_I})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{GHE_I}} |\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|^2]^\beta}{[\det(I-W\overline{W}')^k - \|\eta\|^2]^\alpha} = 0.$$

(ii) *If $0 < kp \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GHE_{II}}) \to \mathcal{A}_\beta(\mathrm{GHE_{II}})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{GHE_{II}})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{GHE_{II}}} |\psi(Z,\xi)| \frac{[\det(I-Z\overline{Z}')^k - \|\xi\|^2]^\beta}{[\det(I-W\overline{W}')^k - \|\eta\|^2]^\alpha} = 0.$$

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(iii) *If* $0 < k \leq \frac{1}{2}$, *then the weighted composition operator* $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GHE_{IV}}) \to \mathcal{A}_\beta(\mathrm{GHE_{IV}})$ *is compact iff* $\psi \in \mathcal{A}_\beta(\mathrm{GHE_{IV}})$ *and*

$$\lim_{\phi(z,\xi)\to\partial\mathrm{GHE_{IV}}} |\psi(z,\xi)| \frac{[(1+|zz'|^2-2|z|^2)^k - \|\xi\|^2]^\beta}{[(1+|ww'|^2-2|w|^2)^k - \|\eta\|^2]^\alpha} = 0.$$

Remark 4.1. *Corollaries* 4.1 *and* 4.2 *express the sufficient and necessary conditions for boundedness and compactness of weighted composition operators between Bers-type spaces on generalized Hua domains of the first, second, and the fourth kind. This is new result, not obtained before. In order to prove analogue results for the generalized Hua domains of the third type, we should prove that " if* $Z, W \in \Re_{\mathrm{III}}(q)$, *then* $2|\det(I - Z\overline{W}')^k| \geq \det(I - Z\overline{Z}')^k + \det(I - W\overline{W}')^k$ *". We have tried to prove this inequality by the polar decomposition of* $\Re_{III}$, *without successs so far. This will be the subject of future analysis.*

4.2. The Cartan-Hartogs domains

For $t = 1$ and $p_1 = p_2 = \cdots = p_r = 1$, $N_1(Z_1, \overline{Z_1})$ is equivalent to (4.1). In this case, the generalized Hua-Cartan-Hartogs domains are the following Cartan-Hartogs domains:

$$\begin{aligned}
\mathrm{Y_I} :&= \{\xi \in \mathbb{C}^r, Z \in \Re_{\mathrm{I}}(m,n) : |\xi|^{2l} < \det(I - Z\overline{Z}')\} \\
&= \{\xi \in \mathbb{C}^r, Z \in \Re_{\mathrm{I}}(m,n) : |\xi|^{2} < \det(I - Z\overline{Z}')^k\}, \\
\mathrm{Y_{II}} :&= \{\xi \in \mathbb{C}^r, Z \in \Re_{\mathrm{II}}(p) : |\xi|^{2l} < \det(I - Z\overline{Z}')\} \\
&= \{\xi \in \mathbb{C}^r, Z \in \Re_{\mathrm{II}}(P) : |\xi|^{2} < \det(I - Z\overline{Z}')^k\}, \\
\mathrm{Y_{IV}} :&= \{\xi \in \mathbb{C}^r, z \in \Re_{\mathrm{IV}}(N) : |\xi|^{2l} < 1 + |zz'|^2 - 2|z|^2\} \\
&= \{\xi \in \mathbb{C}^r, z \in \Re_{\mathrm{IV}}(N) : |\xi|^{2} < (1 + |zz'|^2 - 2|z|^2)^k\},
\end{aligned}$$

where $l = \frac{1}{k} > 0$. $\mathrm{Y_I}, \mathrm{Y_{II}}, \mathrm{Y_{IV}}$ denote the Cartan-Hartogs domains of the first, second, and the fourth kind, respectively. The Bers-type spaces on $\mathrm{Y_I}, \mathrm{Y_{II}}, \mathrm{Y_{IV}}$ are defined by:

Definition 4.2. *Let* $\alpha > 0$.

(i) *The Bers-type space* $\mathcal{A}_\alpha(\mathrm{Y_I})$ *consists of all* $f \in H(\mathrm{Y_I})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{Y_I})} = \sup_{(Z,\xi)\in\mathrm{Y_I}} [\det(I - Z\overline{Z}')^k - |\xi|^2]^\alpha |f(Z,\xi)| < +\infty.$$

(ii) *The Bers-type space* $\mathcal{A}_\alpha(\mathrm{Y_{II}})$ *consists of all* $f \in H(\mathrm{Y_{II}})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{Y_{II}})} = \sup_{(Z,\xi)\in\mathrm{Y_{II}}} [\det(I - Z\overline{Z}')^k - |\xi|^2]^\alpha |f(Z,\xi)| < +\infty.$$

(iii) *The Bers-type space* $\mathcal{A}_\alpha(\mathrm{Y_{IV}})$ *consists of all* $f \in H(\mathrm{Y_{IV}})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{Y_{IV}})} = \sup_{(z,\xi)\in\mathrm{Y_{IV}}} [(1 + |zz'|^2 - 2|z|^2)^k - |\xi|^2]^\alpha |f(z,\xi)| < +\infty.$$

By the definition of the Cartan domains of the first kind and second kind, $\det(I - Z\overline{Z}')^k$, $Z \in \Re_{\mathrm{I}}, \Re_{\mathrm{II}}$ clearly meet (1.1). From Lemma 4.6, $(1 + |zz'|^2 - 2|z|^2)^k$, $z \in \Re_{\mathrm{IV}}$ meets (1.1). Meanwhile, using Lemmas 4.1 and 4.5, we have that when k meets certain conditions, $\det(I - Z\overline{W}')^k$, $Z, W \in \Re_{\mathrm{I}}, \Re_{\mathrm{II}}$ and $(1 + zz'\overline{w}\overline{w}' - 2z\overline{w}')^k$, $z, w \in \Re_{\mathrm{IV}}$ meet conditions (1.2). Therefore, those Cartan-Hartogs domains are special cases of generalized Hua-Cartan-Hartogs domains. According to Theorems 3.1 and 3.2, we obtain the conditions for boundedness and compactness of weighted composition operators between Bers-type spaces on the Cartan-Hartogs domains .

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Assume that ϕ is a holomorphic self-map of $\mathrm{Y} \in \{\mathrm{Y_I}, \mathrm{Y_{II}}, \mathrm{Y_{IV}}\}$, and let us write $(W, \eta) = \phi(Z, \xi)$ for $(Z, \xi) \in \mathrm{Y_I}, \mathrm{Y_{II}}$ and $(w, \eta) = \phi(z, \xi)$ for $(z, \xi) \in \mathrm{Y_{IV}}$. We have

Corollary 4.3. *If $\alpha, \beta > 0$ are positive numbers, ϕ is a holomorphic self-map of Y, and $\psi \in H(\mathrm{Y})$, then the following statements hold.*

(i) *If $0 < km \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{Y_I}) \to \mathcal{A}_\beta(\mathrm{Y_I})$ is bounded iff*

$$\widetilde{N}_I = \sup_{(Z,\xi)\in \mathrm{Y_I}} |\psi(Z,\xi)| \frac{[\det(I - Z\overline{Z}')^k - |\xi|^2]^\beta}{[\det(I - W\overline{W}')^k - |\eta|^2]^\alpha} < +\infty.$$

(ii) *If $0 < kp \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{Y_{II}}) \to \mathcal{A}_\beta(\mathrm{Y_{II}})$ is bounded iff*

$$\widetilde{N}_{II} = \sup_{(Z,\xi)\in \mathrm{Y_{II}}} |\psi(Z,\xi)| \frac{[\det(I - Z\overline{Z}')^k - |\xi|^2]^\beta}{[\det(I - W\overline{W}')^k - |\eta|^2]^\alpha} < +\infty.$$

(iii) *If $0 < k \leq \frac{1}{2}$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{Y_{IV}}) \to \mathcal{A}_\beta(\mathrm{Y_{IV}})$ is bounded iff*

$$\widetilde{N}_{IV} = \sup_{(Z,\xi)\in \mathrm{Y_{IV}}} |\psi(Z,\xi)| \frac{[(1 + |zz'|^2 - 2|z|^2)^k - |\xi|^2]^\beta}{[(1 + |ww'|^2 - 2|w|^2)^k - |\eta|^2]^\alpha} < +\infty.$$

Corollary 4.4. *If $\alpha, \beta > 0$ are positive numbers, ϕ is a holomorphic self-map of Y and $\psi \in H(\mathrm{Y})$, then the following statements hold.*

(i) *If $0 < km \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{Y_I}) \to \mathcal{A}_\beta(\mathrm{Y_I})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{Y_I})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{Y_I}} |\psi(Z,\xi)| \frac{[\det(I - Z\overline{Z}')^k - |\xi|^2]^\beta}{[\det(I - W\overline{W}')^k - |\eta|^2]^\alpha} = 0.$$

(ii) *If $0 < kp \leq 1$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{Y_{II}}) \to \mathcal{A}_\beta(\mathrm{Y_{II}})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{Y_{II}})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{Y_{II}}} |\psi(Z,\xi)| \frac{[\det(I - Z\overline{Z}')^k - |\xi|^2]^\beta}{[\det(I - W\overline{W}')^k - |\eta|^2]^\alpha} = 0.$$

(iii) *If $0 < k \leq \frac{1}{2}$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{Y_{IV}}) \to \mathcal{A}_\beta(\mathrm{Y_{IV}})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{Y_{IV}})$ and*

$$\lim_{\phi(z,\xi)\to\partial \mathrm{Y_{IV}}} |\psi(z,\xi)| \frac{[(1 + |zz'|^2 - 2|z|^2)^k - |\xi|^2]^\beta}{[(1 + |ww'|^2 - 2|w|^2)^k - |\eta|^2]^\alpha} = 0.$$

For $m = 1, k = 1$, the Cartan-Hartogs domain of the first kind $\mathrm{Y_I}$ is the simple unit ball $\mathrm{B_n}$. The sufficient and necessary conditions for the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{B_n}) \to \mathcal{A}_\beta(\mathrm{B_n})$ to be bounded and compact are summarized as follows

Corollary 4.5. *Given $\alpha, \beta > 0$, a holomorphic self-map ϕ of $\mathrm{B_n}$, and $\psi \in H(\mathrm{B_n})$, then the weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{B_n}) \to \mathcal{A}_\beta(\mathrm{B_n})$*

(i) *is bounded if and only if*

$$\sup_{z\in \mathrm{B_n}} |\psi(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\phi(z)|^2)^\alpha} < +\infty.$$

(ii) *is compact if and only if $\psi \in \mathcal{A}_\beta(\mathrm{B_n})$ and*

$$\lim_{\phi(z)\to 1} |\psi(z)| \frac{(1 - |z|^2)^\beta}{(1 - |\phi(z)|^2)^\alpha} = 0.$$

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This result is consistent with the conclusion of Jin and Tang in [11].

From Corollary 4.5, by setting $n = 1$, we may obtain the boundedness and compactness conditions for weighted composition operators between Bers-type space on the unit disk, which are consistent with the results of [9, 10].

Corollary 4.6. *Given $\alpha, \beta > 0$, a holomorphic self-map ϕ of D and $\psi \in H(\mathrm{D})$, then the weighted composition $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{D}) \to \mathcal{A}_\beta(\mathrm{D})$*

(i) *is bounded if and only if*

$$\sup_{z\in \mathrm{D}} |\psi(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^\alpha} < +\infty.$$

(ii) *is compact if and only if $\psi \in \mathcal{A}_\beta(\mathrm{D})$ and*

$$\lim_{\phi(z)\to 1} |\psi(z)| \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^\alpha} = 0.$$

4.3. The generalized Cartan-Hartogs domains

For $t = 2$, let us introduce

$$N_k(Z_k, \overline{Z_k}) = \begin{cases} \det(I - Z_k\overline{Z_k}')^{s_k}, & Z_k \in \Re_{\mathrm{I}}(m,n) \\ \det(I - Z_k\overline{Z_k}')^{s_k}, & Z_k \in \Re_{\mathrm{II}}(p) \\ (1 + |z_k z_k'|^2 - 2|z_k|^2)^{s_k}, & z_k \in \Re_{\mathrm{IV}}(N) \end{cases} \tag{4.10}$$

where s_k are positive real numbers, $k = 1, 2$. In this case, those generalized Hua-Cartan-Hartogs domains are generalized Cartan-Hartogs domains. We consider the case in which Z_1, Z_2 simultaneously belong to $\Re_{\mathrm{I}}, \Re_{\mathrm{II}}, \Re_{\mathrm{IV}}$, and assume $Z_1 \in \Re_{\mathrm{I}}(m,n), Z_2 \in \Re_{\mathrm{I}}(g,l)$, then let $N_1(Z_1, \overline{Z_1})N_2(Z_2, \overline{Z_2}) = \det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2}$. We refer to this domain as $\mathrm{GW}_{\mathrm{I,I}}$, with formal definition

$$\begin{aligned} \mathrm{GW}_{\mathrm{I,I}} := \{ & \xi_j \in \mathbb{C}^{n_j}, Z_1 \in \Re_{\mathrm{I}}(m,n), Z_2 \in \Re_{\mathrm{I}}(g,l) : \\ & \sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2}, j = 1, \cdots, r \}. \end{aligned}$$

Definition 4.3. *Let $\alpha > 0$, the Bers-type space $\mathcal{A}_\alpha(\mathrm{GW}_{\mathrm{I,I}})$ consists of all $f \in H(\mathrm{GW}_{\mathrm{I,I}})$ satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{GW}_{\mathrm{I,I}})} = \sup_{(Z,\xi)\in \mathrm{GW}_{\mathrm{I,I}}} [\det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2} - \sum_{j=1}^{r} |\xi_j|^{2p_j}]^\alpha |f(Z,\xi)| < +\infty.$$

For $Z_1 \in \Re_{\mathrm{I}}(m,n)$, $Z_2 \in \Re_{\mathrm{I}}(g,l)$, by the definition of the Cartan domain of the first kind, $\det(I - Z_k\overline{Z_k}')^{s_k}, k = 1, 2$ clearly meet (1.1), if s_k $(k = 1, 2.)$ satisfy $0 < s_1 m \le 1, 0 < s_2 g \le 1$, then Lemma 4.1 implies that $\det(I - Z_k\overline{W_k}')^{s_k}, k = 1, 2$ satisfy conditions (1.2). Therefore, the generalized Cartan-Hartogs domain $\mathrm{GW}_{\mathrm{I,I}}$ is a generalized Hua-Cartan-Hartogs domain, and we may easily obtain conditions for the boundedness and compactness of weighted composition operator between Bers-type spaces on $\mathrm{GW}_{\mathrm{I,I}}$.

Assume that ϕ is a holomorphic self-map of $\mathrm{GW}_{\mathrm{I,I}}$, and consider $(W, \eta) = \phi(Z, \xi)$ for $(Z, \xi) \in \mathrm{GW}_{\mathrm{I,I}}$.

Corollary 4.7. *For $\alpha, \beta > 0$, $0 < s_1 m \le 1$, $0 < s_2 g \le 1$, a holomorphic self-map ϕ of $\mathrm{GW}_{\mathrm{I,I}}$ and $\psi \in H(\mathrm{GW}_{\mathrm{I,I}})$, the following statements hold.*

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(i) *The weighted composition operator* $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GW_{I,I}}) \to \mathcal{A}_\beta(\mathrm{GW_{I,I}})$ *is bounded iff*

$$\widetilde{N}_{I,I} = \sup_{(Z,\xi)\in \mathrm{GW_{I,I}}} |\psi(Z,\xi)| \frac{[\det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2} - \|\xi\|^2]^\beta}{[\det(I - W_1\overline{W_1}')^{s_1} \det(I - W_2\overline{W_2}')^{s_2} - \|\eta\|^2]^\alpha} < +\infty.$$

(ii) *The weighted composition operator* $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GW_{I,I}}) \to \mathcal{A}_\beta(\mathrm{GW_{I,I}})$ *is compact iff* $\psi \in \mathcal{A}_\beta(\mathrm{GW_{I,I}})$ *and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{GW_{I,I}}} |\psi(Z,\xi)| \frac{[\det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2} - \|\xi\|^2]^\beta}{[\det(I - W_1\overline{W_1}')^{s_1} \det(I - W_2\overline{W_2}')^{s_2} - \|\eta\|^2]^\alpha} = 0.$$

In particular, for $r = 1$ $\mathrm{GW_{I,I}}$ is the ordinary generalized Cartan-Hartogs domain $\mathrm{W_{I,I}}$.

$$\mathrm{W_{I,I}} := \{(Z_1, Z_2, \xi) \in \Re_{\mathrm{I}}(m,n) \times \Re_{\mathrm{I}}(g,l) \times \mathbb{C}^n : |\xi|^2 < \det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2}\}.$$

If we set $r = 1$ in Corollary 4.7, we obtain sufficient and necessary conditions for the boundedness and compactness of weighted composition operators between Bers-type spaces on the generalized Cartan-Hartogs domains $\mathrm{W_{I,I}}$.

4.4. The generalized Cartan-Hartogs domains over different Cartan domains

For $t = 2$, we introduce $N_k(Z_k, \overline{Z_k})$ as in (4.10), $k = 1, 2$. In this case, those generalized Hua-Cartan-Hartogs domains are generalized Cartan-Hartogs domains. Here, we consider the case in which Z_1, Z_2 belong to different Cartan domains. Taking $Z_1 \in \Re_{\mathrm{I}}(m,n), Z_2 \in \Re_{\mathrm{II}}(p)$ as an example, let $N_1(Z_1, \overline{Z_1})N_2(Z_2, \overline{Z_2}) = \det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2}$ and refer to this domain as $\mathrm{GW_{I,II}}$ by setting

$$\begin{aligned}\mathrm{GW_{I,II}} := \{&\xi_j \in \mathbb{C}^{n_j}, Z_1 \in \Re_{\mathrm{I}}(m,n), Z_2 \in \Re_{\mathrm{II}}(p) : \\ &\sum_{j=1}^{r} |\xi_j|^{2p_j} < \det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2}, j = 1, 2, \cdots, r\}.\end{aligned}$$

Definition 4.4. *For* $\alpha > 0$, *the Bers-type space* $\mathcal{A}_\alpha(\mathrm{GW_{I,II}})$ *consists of all* $f \in H(\mathrm{GW_{I,II}})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{GW_{I,II}})} = \sup_{(Z,\xi)\in \mathrm{GW_{I,II}}} [\det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2} - \sum_{j=1}^{r} |\xi_j|^{2p_j}]^\alpha |f(Z,\xi)| < +\infty$$

For $Z_1 \in \Re_{\mathrm{I}}(m,n), Z_2 \in \Re_{\mathrm{II}}(p)$, by the definition of the Cartan domains of the first kind and second kind, $\det(I - Z_k\overline{Z_k}')^{s_k}, k = 1, 2$ clearly meet (1.1), if s_k $(k = 1, 2.)$ satisfy $0 < s_1 m \le 1$, $0 < s_2 p \le 1$, then by Lemma 4.1 we know that $\det(I - Z_k\overline{W_k}')^{s_k}, k = 1, 2$ satisfy conditions (1.2). Therefore, the generalized Cartan-Hartogs domain $\mathrm{GW_{I,II}}$ is a special case of generalized Hua-Cartan-Hartogs domains. Taking Theorems 3.1 and 3.2 into account, we obtain the conditions for the boundedness and compactness of weighted composition operators between Bers-type spaces on $\mathrm{GW_{I,II}}$.

Assume that ϕ is a holomorphic self-map of $\mathrm{GW_{I,II}}$ and write $(W, \eta) = \phi(Z, \xi)$ for $(Z, \xi) \in \mathrm{GW_{I,II}}$. We have

Corollary 4.8. *Given* $\alpha, \beta > 0$, $0 < s_1 m \le 1$, $0 < s_2 p \le 1$, *a holomorphic self-map* ϕ *of* $\mathrm{GW_{I,II}}$ *and* $\psi \in H(\mathrm{GW_{I,II}})$, *then the following statements hold.*

(i) *The weighted composition operator* $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GW_{I,II}}) \to \mathcal{A}_\beta(\mathrm{GW_{I,II}})$ *is bounded iff*

$$\widetilde{N}_{I,II} = \sup_{(Z,\xi)\in \mathrm{GW_{I,II}}} |\psi(Z,\xi)| \frac{[\det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2} - \|\xi\|^2]^\beta}{[\det(I - W_1\overline{W_1}')^{s_1} \det(I - W_2\overline{W_2}')^{s_2} - \|\eta\|^2]^\alpha} < +\infty.$$

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(ii) *The weighted composition operator* $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{GW_{I,II}}) \to \mathcal{A}_\beta(\mathrm{GW_{I,II}})$ *is compact iff* $\psi \in \mathcal{A}_\beta(\mathrm{GW_{I,II}})$ *and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{GW_{I,II}}} |\psi(Z,\xi)| \frac{[\det(I - Z_1\overline{Z_1}')^{s_1} \det(I - Z_2\overline{Z_2}')^{s_2} - \|\xi\|^2]^\beta}{[\det(I - W_1\overline{W_1}')^{s_1} \det(I - W_2\overline{W_2}')^{s_2} - \|\eta\|^2]^\alpha} = 0.$$

4.5. The generalized ellipsoidal-type domains

For $t = 1$, let us introduce

$$N_1(Z_1, \overline{Z_1}) = N(Z, \overline{Z}) = \begin{cases} 1 - \dfrac{|Z|^2}{m}, & Z \in \Re_{\mathrm{I}}(m,n) \\ 1 - \dfrac{|Z|^2}{p}, & Z \in \Re_{\mathrm{II}}(p) \\ 1 - \dfrac{|Z|^2}{[\frac{q}{2}]}, & Z \in \Re_{\mathrm{III}}(q) \\ 1 - |z|^2, & z \in \Re_{\mathrm{IV}}(N) \end{cases}$$

where $[\frac{q}{2}]$ denotes the integer part of $\frac{q}{2}$. In this case, we refer to those generalized Hua-Cartan-Hartogs domains as the generalized ellipsoidal-type domains.

$$\mathrm{L_I} := \{\xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{I}}(m,n) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < 1 - \frac{|Z|^2}{m},\ j = 1,2,\cdots,r\},$$

$$\mathrm{L_{II}} := \{\xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{II}}(p) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < 1 - \frac{|Z|^2}{p},\ j = 1,2,\cdots,r\},$$

$$\mathrm{L_{III}} := \{\xi_j \in \mathbb{C}^{n_j}, Z \in \Re_{\mathrm{III}}(q) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < 1 - \frac{|Z|^2}{[\frac{q}{2}]},\ j = 1,2,\cdots,r\},$$

$$\mathrm{L_{IV}} := \{\xi_j \in \mathbb{C}^{n_j}, z \in \Re_{\mathrm{IV}}(N) : \sum_{j=1}^{r} |\xi_j|^{2p_j} < 1 - |z|^2,\ j = 1,2,\cdots,r\}.$$

$\mathrm{L_I}$,$\mathrm{L_{II}}$,$\mathrm{L_{III}}$ and $\mathrm{L_{IV}}$ denote generalized ellipsoidal-type domains of the first, second, third, and fourth kind, respectively. The Bers-type spaces on $\mathrm{L_I}$,$\mathrm{L_{II}}$,$\mathrm{L_{III}}$ and $\mathrm{L_{IV}}$ may be defined as follows

Definition 4.5. *Let* $\alpha > 0$.

(i) *The Bers-type space* $\mathcal{A}_\alpha(\mathrm{L_I})$ *consists of all* $f \in H(\mathrm{L_I})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{L_I})} = \sup_{(Z,\xi)\in \mathrm{L_I}} \Big(1 - \frac{|Z|^2}{m} - \sum_{j=1}^{r} |\xi_j|^{2p_j}\Big)^\alpha |f(Z,\xi)| < +\infty.$$

(ii) *The Bers-type space* $\mathcal{A}_\alpha(\mathrm{L_{II}})$ *consists of all* $f \in H(\mathrm{L_{II}})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{L_{II}})} = \sup_{(Z,\xi)\in \mathrm{L_{II}}} \Big(1 - \frac{|Z|^2}{p} - \sum_{j=1}^{r} |\xi_j|^{2p_j}\Big)^\alpha |f(Z,\xi)| < +\infty.$$

(iii) *The Bers-type space* $f \in \mathcal{A}_\alpha(\mathrm{L_{III}})$ *consists of all* $f \in H(\mathrm{L_{III}})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{L_{III}})} = \sup_{(Z,\xi)\in \mathrm{L_{III}}} \Big(1 - \frac{|Z|^2}{[\frac{q}{2}]} - \sum_{j=1}^{r} |\xi_j|^{2p_j}\Big)^\alpha |f(Z,\xi)| < +\infty.$$

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(iv) *The Bers-type space* $\mathcal{A}_\alpha(\mathrm{L_{IV}})$ *consists of all* $f \in H(\mathrm{L_{IV}})$ *satisfying*

$$\|f\|_{\mathcal{A}_\alpha(\mathrm{L_{IV}})} = \sup_{(z,\xi)\in \mathrm{L_{IV}}} \big(1-|z|^2-\sum_{j=1}^{r}|\xi_j|^{2p_j}\big)^\alpha |f(z,\xi)| < +\infty.$$

In order to prove that the above generalized ellipsoidal-type domains are special cases of generalized Hua-Cartan-Hartogs domains, we need to prove the following Lemma first.

Lemma 4.7. *The following statements hold.*

(i) *If* $Z \in \Re_{\mathrm{I}}(m,n)$, *then* $0 < 1-\frac{|Z|^2}{m} \le 1$ *and for all* $Z, W \in \Re_{\mathrm{I}}(m,n)$ *we have*

$$\big(1-\frac{\mathrm{tr}(Z\overline{Z}')}{m}\big)+\big(1-\frac{\mathrm{tr}(W\overline{W}')}{m}\big) \le 2\big|\big(1-\frac{\mathrm{tr}(Z\overline{W}')}{m}\big)\big|.$$

(ii) *If* $Z \in \Re_{\mathrm{II}}(p)$, *then* $0 < 1-\frac{|Z|^2}{p} \le 1$, *and for all* $Z, W \in \Re_{\mathrm{II}}(p)$ *we have*

$$\big(1-\frac{\mathrm{tr}(Z\overline{Z}')}{p}\big)+\big(1-\frac{\mathrm{tr}(W\overline{W}')}{p}\big) \le 2\big|\big(1-\frac{\mathrm{tr}(Z\overline{W}')}{p}\big)\big|.$$

(iii) *If* $Z \in \Re_{\mathrm{III}}(q)$, *then* $0 < 1-\frac{|Z|^2}{[\frac{q}{2}]} \le 1$, *and for all* $Z, W \in \Re_{\mathrm{III}}(q)$ *we have*

$$\big(1-\frac{\mathrm{tr}(Z\overline{Z}')}{2[\frac{q}{2}]}\big)+\big(1-\frac{\mathrm{tr}(W\overline{W}')}{2[\frac{q}{2}]}\big) \le 2\big|\big(1-\frac{\mathrm{tr}(Z\overline{W}')}{2[\frac{q}{2}]}\big)\big|.$$

(iv) *If* $z \in \Re_{\mathrm{IV}}(N)$, *then* $0 < 1-|z|^2 \le 1$, *and for all* $z, w \in \Re_{\mathrm{IV}}(N)$ *we have*

$$(1-|z|^2)+(1-|w|^2) \le 2|1-z\overline{w}'|.$$

Proof. (i) For $Z \in \Re_{\mathrm{I}}(m,n)$, we have $0 \le |Z|^2 = \mathrm{tr}(Z\overline{Z}') < m$, that is $0 < 1-\frac{|Z|^2}{m} \le 1$. According to the two inequalities $|\mathrm{tr}(Z\overline{W}')|^2 \le \mathrm{tr}(Z\overline{Z}')\mathrm{tr}(W\overline{W}')$ and $a^2+b^2 \ge 2ab$, we obtain

$$\begin{aligned}
2\big|\big(1-\frac{\mathrm{tr}(Z\overline{W}')}{m}\big)\big| &\ge 2-2\frac{|\mathrm{tr}(Z\overline{W}')|}{m} \ge 2-2\frac{\sqrt{\mathrm{tr}(Z\overline{Z}')}\sqrt{\mathrm{tr}(W\overline{W}')}}{m} \\
&= 2-2\sqrt{\frac{\mathrm{tr}(Z\overline{Z}')}{m}}\sqrt{\frac{\mathrm{tr}(W\overline{W}')}{m}} \ge 2-\Big(\frac{\mathrm{tr}(Z\overline{Z}')}{m}+\frac{\mathrm{tr}(W\overline{W}')}{m}\Big) \\
&= 1-\frac{\mathrm{tr}(Z\overline{Z}')}{m}+1-\frac{\mathrm{tr}(W\overline{W}')}{m}.
\end{aligned}$$

(ii) For $Z \in \Re_{\mathrm{II}}(p)$, we may write

$$Z = \begin{pmatrix} z_{11} & \frac{1}{\sqrt{2}}z_{12} & \cdots & \frac{1}{\sqrt{2}}z_{1p} \\ \frac{1}{\sqrt{2}}z_{21} & z_{22} & \cdots & \frac{1}{\sqrt{2}}z_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}}z_{p1} & \frac{1}{\sqrt{2}}z_{p2} & \cdots & z_{pp} \end{pmatrix},$$

then $0 \le |Z|^2 = \mathrm{tr}(Z\overline{Z}') < p$, and easily obtain the proof.

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(iii) For $Z \in \Re_{\mathrm{III}}(q)$, we have $0 \le |Z|^2 = \frac{1}{2}\mathrm{tr}(Z\overline{Z}') < [\frac{q}{2}]$, that is $0 < 1 - \frac{|Z|^2}{[\frac{q}{2}]} \le 1$. According to the two inequalities $|\mathrm{tr}(Z\overline{W}')|^2 \le \mathrm{tr}(Z\overline{Z}')\mathrm{tr}(W\overline{W}')$ and $a^2 + b^2 \ge 2ab$, we obtain

$$\begin{aligned}
2\big|(1 - \frac{\mathrm{tr}(Z\overline{W}')}{2[\frac{q}{2}]})\big| &\ge 2 - 2\frac{|\mathrm{tr}(Z\overline{W}')|}{2[\frac{q}{2}]} \ge 2 - 2\frac{\sqrt{\mathrm{tr}(Z\overline{Z}')}\sqrt{\mathrm{tr}(W\overline{W}')}}{2[\frac{q}{2}]} \\
&= 2 - 2\sqrt{\frac{\mathrm{tr}(Z\overline{Z}')}{2[\frac{q}{2}]}}\sqrt{\frac{\mathrm{tr}(W\overline{W}')}{2[\frac{q}{2}]}} \ge 2 - \Big(\frac{\mathrm{tr}(Z\overline{Z}')}{2[\frac{q}{2}]} + \frac{\mathrm{tr}(W\overline{W}')}{2[\frac{q}{2}]}\Big) \\
&= 1 - \frac{\mathrm{tr}(Z\overline{Z}')}{2[\frac{q}{2}]} + 1 - \frac{\mathrm{tr}(W\overline{W}')}{2[\frac{q}{2}]}.
\end{aligned}$$

(iv) For $z \in \Re_{\mathrm{IV}}(N)$, we have $0 \le |z|^2 < 1$, that is $0 < 1 - |z|^2 \le 1$. According to the two inequalities $|z\overline{w}'| \le |z||w|$ and $a^2 + b^2 \ge 2ab$, we obtain

$$2|1 - z\overline{w}'| \ge 2 - 2|z\overline{w}'| \ge 2 - 2|z||w| \ge 2 - (|z|^2 + |w|^2) = 1 - |z|^2 + 1 - |w|^2.$$

This completes the proof of the Lemma. □

The above Lemma shows that $1 - \frac{\mathrm{tr}(Z\overline{W}')}{m}, Z, W \in \Re_{\mathrm{I}}(m,n); 1 - \frac{\mathrm{tr}(Z\overline{W}')}{p}, Z, W \in \Re_{\mathrm{II}}(p); 1 - \frac{\mathrm{tr}(Z\overline{W}')}{2[\frac{q}{2}]}, Z, W \in \Re_{\mathrm{III}}(q)$ and $(1 - z\overline{w}'), z, w \in \Re_{\mathrm{IV}}(N)$ satisfy conditions (1.1) and (1.2). Therefore, the generalized ellipsoidal-type domains are special case of the generalized Hua-Cartan-Hartogs domains, and we may obtain the boundedness and compactness conditions for weighted composition operators between Bers-type spaces on these domains.

Assume that ϕ is a holomorphic self-map of $\mathrm{L} \in \{\mathrm{L_I}, \mathrm{L_{II}}, \mathrm{L_{III}}, \mathrm{L_{IV}}\}$. Let us write $(W, \eta) = \phi(Z, \xi)$ for $(Z, \xi) \in \mathrm{L_I}, \mathrm{L_{II}}, \mathrm{L_{III}}$, and write $(w, \eta) = \phi(z, \xi)$ for $(z, \xi) \in \mathrm{L_{IV}}$.

Corollary 4.9. *Given $\alpha, \beta > 0$, a holomorphic self-map ϕ of L, and $\psi \in H(\mathrm{L})$, then the following statements hold.*

(i) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_I}) \to \mathcal{A}_\beta(\mathrm{L_I})$ is bounded iff*

$$\widetilde{Q}_I = \sup_{(Z,\xi)\in \mathrm{L_I}} |\psi(Z,\xi)| \frac{(1 - \frac{|Z|^2}{m} - \|\xi\|^2)^\beta}{(1 - \frac{|W|^2}{m} - \|\eta\|^2)^\alpha} < +\infty.$$

(ii) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_{II}}) \to \mathcal{A}_\beta(\mathrm{L_{II}})$ is bounded iff*

$$\widetilde{Q}_{II} = \sup_{(Z,\xi)\in \mathrm{L_{II}}} |\psi(Z,\xi)| \frac{(1 - \frac{|Z|^2}{p} - \|\xi\|^2)^\beta}{(1 - \frac{|W|^2}{p} - \|\eta\|^2)^\alpha} < +\infty.$$

(iii) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_{III}}) \to \mathcal{A}_\beta(\mathrm{L_{III}})$ is bounded iff*

$$\widetilde{Q}_{III} = \sup_{(Z,\xi)\in \mathrm{L_{III}}} |\psi(Z,\xi)| \frac{(1 - \frac{|Z|^2}{[\frac{q}{2}]} - \|\xi\|^2)^\beta}{(1 - \frac{|W|^2}{[\frac{q}{2}]} - \|\eta\|^2)^\alpha} < +\infty.$$

(iv) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_{IV}}) \to \mathcal{A}_\beta(\mathrm{L_{IV}})$ is bounded iff*

$$\widetilde{Q}_{IV} = \sup_{(z,\xi)\in \mathrm{L_{IV}}} |\psi(z,\xi)| \frac{(1 - |z|^2 - \|\xi\|^2)^\beta}{(1 - |w|^2 - \|\eta\|^2)^\alpha} < +\infty.$$

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Corollary 4.10. *Given $\alpha, \beta > 0$, a holomorphic self-map ϕ of L, and $\psi \in H(\mathrm{L})$, then the following statements hold.*

(i) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_I}) \to \mathcal{A}_\beta(\mathrm{L_I})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{L_I})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{L_I}} |\psi(Z,\xi)| \frac{(1-\frac{|Z|^2}{m}-\|\xi\|^2)^\beta}{(1-\frac{|W|^2}{m}-\|\eta\|^2)^\alpha} = 0.$$

(ii) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_{II}}) \to \mathcal{A}_\beta(\mathrm{L_{II}})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{L_{II}})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{L_{II}}} |\psi(Z,\xi)| \frac{(1-\frac{|Z|^2}{p}-\|\xi\|^2)^\beta}{(1-\frac{|W|^2}{p}-\|\eta\|^2)^\alpha} = 0.$$

(iii) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_{III}}) \to \mathcal{A}_\beta(\mathrm{L_{III}})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{L_{III}})$ and*

$$\lim_{\phi(Z,\xi)\to\partial \mathrm{L_{III}}} |\psi(Z,\xi)| \frac{(1-\frac{|Z|^2}{[\frac{q}{2}]}-\|\xi\|^2)^\beta}{(1-\frac{|W|^2}{[\frac{q}{2}]}-\|\eta\|^2)^\alpha} = 0.$$

(iv) *The weighted composition operator $\psi C_\phi : \mathcal{A}_\alpha(\mathrm{L_{IV}}) \to \mathcal{A}_\beta(\mathrm{L_{IV}})$ is compact iff $\psi \in \mathcal{A}_\beta(\mathrm{L_{IV}})$ and*

$$\lim_{\phi(z,\xi)\to\partial \mathrm{L_{IV}}} |\psi(z,\xi)| \frac{(1-|z|^2-\|\xi\|^2)^\beta}{(1-|w|^2-\|\eta\|^2)^\alpha} = 0.$$

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