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Explicit Solutions for Coupled Parallel Queues

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Abstract: We consider a system of two coupled parallel queues, queue 1 and queue 2, with infinite waiting rooms. The time setting is *discrete*. In either queue, the service of a customer requires exactly one discrete time slot. Arrivals of new customers occur independently from slot to slot, but the numbers of arrivals of both types within a slot may be mutually interdependent. Their joint probability generating function (pgf) is indicated as $A(z_1, z_2)$ and characterizes the whole model. It is well-known that, in general, determining the steady-state joint probability mass function (pmf) $u(m,n), m, n \geq 0$ or the corresponding joint pgf $U(z_1, z_2)$ of the system contents (i.e., numbers of customers present) in both queues is a formidable task. Only in a number of isolated cases, for very specific choices of the arrival pgf $A(z_1, z_2)$, explicit results are known in the literature. In this paper, we identify a multi-parameter, *generic class* of arrival pgfs $A(z_1, z_2)$, for which we can explicitly determine the system-content pgf $U(z_1, z_2)$. We find that, for arrival pgfs of this class, $U(z_1, z_2)$ has a denominator which is a product, say $r_1(z_1)r_2(z_2)$ of two univariate functions. This property allows a straightforward inversion of $U(z_1, z_2)$, resulting in a pmf $u(m,n)$ which can be expressed as a finite linear combination of bivariate geometric terms. We also observe that our generic model encompasses most of the previously known results as special cases.

Keywords: parallel queues; discrete-time; joint system-content distribution; explicit solutions

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1. Introduction

This paper fits into a greater scientific effort which aims at finding explicit analytic solutions for the joint probability distribution (or probability generating function) of the numbers of customers in a system of two coupled discrete-time queues. Various instances of such systems have been studied before, both differing in the *cause of the coupling* between the two queues or in the *scientific perspective* taken in the study.

With no claim on completeness, we mention a number of *possible causes* for the presence of coupling between queues. A first cause may be that the arrival streams into the queues are *mutually interdependent* or *state-dependent*, i.e., dependent on the *system contents*, i.e., the numbers of customers present in the queues. Mutual dependence between arrivals occurs, for instance, in the context of communications networks, where the nodes of the network contain switching systems which have to forward digital packets from many different origins to many different destinations. In such switches, each destination has (at least, conceptually) its own dedicated buffer to temporarily store arriving packets, and, since packets destined to one destination do not enter the output buffer associated to another destination, the arrivals within such output buffers are mutually correlated. Buffered slotted switches were studied, e.g., in [1–6]. *State-dependence* of arrivals occurs, for instance, in *join-the-shortest-queue* systems, where arriving customers adapt their behavior at the entrance of the system to the system contents upon arrival; see, e.g., [7–9]. More conceptual studies of queues with interdependent arrivals include [5,6,10,11].

Another (major) cause of coupling may be that the queues of the system have to *share the same service facilities*. This situation occurs, for instance, in *polling systems*, where one server periodically visits multiple customer queues to serve a number of customers and then go to the next queue; various variants of polling systems have been studied quite intensively in the past (see, e.g., [12–19]). Sharing of servers also occurs in so-called *alternating service* systems, where one server is allocated for alternating random durations of time to either of the two queues, regardless of the states of these queues, (see, e.g., [11,20–28]), or in *priority queues*, where one common service facility gives preferential service to one

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class of customers over the other class(es) of customers; a large body of research results is available on various types of *priority queueing* systems (see, e.g., [29–46]). Similar ideas are also implemented in so-called *(generalized) processor sharing (GPS)* systems, whereby the service facility is randomly allocated to multiple queues according to preset weights; as opposed to *randomly alternating service* systems, however, *GPS-systems* usually allow the server to deliver service to customers of a queue to which it is not allocated when the queue to which it is allocated is empty, thus making the system work-conserving. In fact, *GPS-systems* can also be viewed as systems with *alternating priorities*; some papers dealing with *GPS-systems* are [25,47–51]. We should also mention *serve-the-longest-queue* systems, where, upon a service completion, a server can autonomously decide to give preference to the queue that contains the largest number of customers; see, e.g, [52,53]. Recently, some authors have examined the combined *join-the-shortest-queue* and *serve-the-longest-queue* scenario; see [54,55].

A third possible cause of coupling in two-queue systems can be that (part of) the output stream of one queue constitutes (part of) the input stream into the other queue, such as in the context of *tandem queues* (see, e.g., [45,56,57]), or, more generally, in a network environment.

As far as the *scientific perspective* taken by various authors in the literature is concerned, we see a main difference between considering the involved (two-queue) queueing system as the basic concept of the study, where the determination of the joint (or, total) system-content distribution of both queues, the overflow probabilities, the customer delays, etc. is the main objective, versus a more fundamental, mathematically-oriented point of view, whereby the underlying *random walks* that model the system contents of both queues are the basic concepts of the study (see, e.g., [58–62]). Usually, the aim of such papers is to shed more light on the structural properties of these random walks required to admit elegant solutions. Moreover, the involved random walks are often of nearest-neighbor type, which is rather restrictive in a queueing context, and the structure of their transition probabilities may be quite arbitrary and not necessarily reflect the behavior of a queueing system.

The present paper does not take the mathematical study of the random walk that models the two-queue system explicitly as a major point of interest, but rather concentrates on the explicit determination of the joint pgf of the two system contents in the two queues of the system. Specifically, we consider a conceptually very simple system of two coupled *parallel* discrete-time queues. The queues are named queue 1 and queue 2, both have their own dedicated server and infinite storage capacity. Customers arriving to queue 1 and to queue 2 are referred to as type-1 and type-2 customers, respectively. The service times of the customers are deterministically equal to one time slot, regardless of the customer type. New customer arrivals of both types occur independently from slot to slot, but are possibly *type-interdependent* within a slot. This is the only source of coupling in this model. Earlier studies of various instances of this type of two-queue system include [1–6,10,11,63].

It is well-known that, in general, determining the steady-state joint pgf $U(z_1, z_2)$ of the system contents in a system of two coupled queues is a formidable task, because it requires the solution of a possibly complicated, nonlinear *functional equation* for $U(z_1, z_2)$, which contains the unknown boundary functions $U(z_1, 0)$ and/or $U(0, z_2)$. In many previous papers, the analysis consists of first determining these boundary functions through various complex-analysis techniques, upon which $U(z_1, z_2)$ can be computed from the functional equation by subsequent substitution of the expressions found for $U(z_1, 0)$ and/or $U(0, z_2)$. In this paper, we use a different, purely *algebraic*, technique, which can be best described as a two-step process: first, we make an *educated guess* of the solution for $U(z_1, z_2)$, and, next, we prove that the proposed $U(z_1, z_2)$ indeed satisfies the functional equation. Of course, in this approach, making an *educated guess* of the solution is crucial. In fact, this step is essentially a process of trial and error, based on the intuition gained from the preliminary study of a large number of simple special cases in earlier papers.

For the specific coupled-queues system considered in this paper, explicit results have been obtained thus far only in a number of isolated cases, for very specific choices of the arrival pgf $A(z_1, z_2)$ (see, e.g., [5]). Furthermore, these special cases are of a rather simple nature: either the arrivals of both types should be *mutually independent*, or the two queues should receive *identical numbers of arrivals* in

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each slot, or one of both queues should receive *no more than one single arrival per slot*, implying that in this queue no accumulation of customers occurs. Some initial indications to extend the class of "solvable" arrival pgfs $A(z_1,z_2)$ were also given in [5], but the extensions are limited.

In this paper, we introduce a multi-parameter, *generic class* of arrival pgfs $A(z_1,z_2)$, for which we succeed to explicitly determine the joint pgf $U(z_1,z_2)$, using the *algebraic* approach described above. By making specific choices for the many parameters of the model, we also define three interesting *subclasses* of arrival pgfs which lead to even more explicit solutions. We find that for arrival pgfs of the classes considered in this paper, the bivariate joint system-content pgf $U(z_1,z_2)$ has a denominator which is a product, say $r_1(z_1)r_2(z_2)$ of two univariate functions. This property allows a straightforward inversion of the pgf $U(z_1,z_2)$ by means of an inversion technique we developed in a previous paper [11], resulting in a pmf $u(m,n)$ which can be expressed as a *finite linear combination* of bivariate geometric terms. We observe that our generic model encompasses most of the previously known results as special cases. In fact, it was by studying these special cases that we developed the intuition needed to be able to identify the class of arrival pgfs introduced in this paper.

The rest of this paper is organized as follows. In section 2 we introduce the detailed mathematical model of the system under study and establish a *functional equation* for the joint pgf $U(z_1,z_2)$. The solution of this functional equation is, in fact, the main purpose of the paper. Section 3 defines the generic class of arrival pgfs $A(z_1,z_2)$ that will be studied in this paper. In section 4, we present and prove the main result of the paper in the form of Theorem 1 which gives an explicit expression for the joint system-content pgf $U(z_1,z_2)$ associated to the joint arrival pgf $A(z_1,z_2)$ defined in section 3. Section 5 defines three interesting subclasses, named A, B and C, of the generic class of arrival pgfs $A(z_1,z_2)$ defined in section 3, and establishes even more explicit formulas for the associated system-content pgfs $U(z_1,z_2)$ in these cases, in the form of three corrolaries, also named A, B and C, of Theorem 1. In section 6, we consider several instances of subclasses A, B and C, whereby specific choices are made for the various parameters and functions appearing in the formulations of Corrolaries A, B and C. Section 7 discusses a fundamental method to invert the system-content pgf $U(z_1,z_2)$, i.e., to determine the pmf $u(m,n)$ from the pgf $U(z_1,z_2)$, and illustrates this techniques by means of specific examples within subclasses A, B and C. Finally, we state some concluding remarks in section 8.

2. Mathematical Model and Queueing Analysis

We define the random variables $a_{1,k}$ and $a_{2,k}$ as the numbers of type-1 and type-2 arrivals, respectively, during slot k. Their joint probability mass function (pmf) $a(i,j)$ and probability generating function (pgf)$A(z_1,z_2)$ are indicated as

$$a(i,j) \triangleq \text{Prob}[a_{1,k}=i \text{ and } a_{2,k}=j] \ , \quad i,j \geq 0 \ ,$$

$$A(z_1,z_2) \triangleq E\Big[z_1^{a_{1,k}} z_2^{a_{2,k}}\Big] \triangleq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a(i,j)\, z_1^i z_2^j \ , \tag{1}$$

which are independent of k. The (marginal) pgfs of $a_{1,k}$ and $a_{2,k}$ are given by

$$A_1(z_1) \triangleq E\Big[z_1^{a_{1,k}}\Big] = A(z_1,1) \quad , \quad A_2(z_2) \triangleq E\Big[z_2^{a_{2,k}}\Big] = A(1,z_2) \ , \tag{2}$$

respectively. The mean number of arrivals of type i per slot is denoted as $\lambda_i \triangleq A_i'(1)$. A graphical representation of the system under study is shown in Figure 1.

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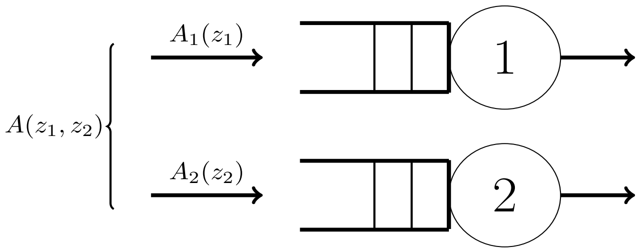

Figure 1. System of two coupled parallel queues

Let $u_{1,k}$ and $u_{2,k}$ indicate the *system contents*, i.e., the *total* numbers of customers present, in queue 1 and queue 2, respectively, *including* the customer(s) in service, if any, at the beginning of slot k. We indicate their joint pgf as

$$U_k(z_1, z_2) \triangleq E\left[z_1^{u_{1,k}} z_2^{u_{2,k}}\right] \ . \tag{3}$$

Furthermore, let $q_{1,k}$ and $q_{2,k}$ indicate the *queue contents*, i.e., the numbers of *waiting* customers, in queue 1 and queue 2, respectively, *excluding* the customer(s) in service, if any, at the beginning of slot k. We indicate their joint pgf as

$$Q_k(z_1, z_2) \triangleq E\left[z_1^{q_{1,k}} z_2^{q_{2,k}}\right] \ . \tag{4}$$

It is not difficult to see that the following relationships exist between the *system contents* and the *queue contents*:

$$q_{1,k} = (u_{1,k} - 1)^+ \quad , \quad q_{2,k} = (u_{2,k} - 1)^+ \ , \tag{5}$$

where we have introduced the notation $(x)^+$ to indicate the quantity $\max(0, x)$.

The main purpose of the paper is to analyze the steady-state behavior of the queueing system under study, i.e., we are interested in determining the steady-state joint pgfs of the two *system contents* and/or *queue contents*, provided that such a steady state exists. Specifically, we wish to study the following limit functions:

$$U(z_1, z_2) \triangleq \lim_{k\to\infty} U_k(z_1, z_2) \quad , \quad Q(z_1, z_2) \triangleq \lim_{k\to\infty} Q_k(z_1, z_2) \ , \tag{6}$$

if they exist. A steady state exists if and only if both queues are stable, i.e., receive, on average, less customers per slot than they can serve, i.e., if and only if the following stability conditions are fulfilled:

$$\lambda_1 < 1 \ , \ \lambda_2 < 1 \ . \tag{7}$$

As mentioned in e.g., [5,6], the evolution of the *system contents* is described by the following *system equations*:

$$u_{1,k+1} = a_{1,k} + (u_{1,k} - 1)^+ \quad , \quad u_{2,k+1} = a_{2,k} + (u_{2,k} - 1)^+ \ . \tag{8}$$

Using standard z-tranform techniques, the equations (8) can be translated into one corresponding transform equation between the joint pgfs $U_k(z_1, z_2)$ and $U_{k+1}(z_1, z_2)$, by using definition (3). Assuming the system reaches a steady state, i.e., assuming the stability conditions (7) are met, letting the time parameter k go to infinity, and using the definitions (3) and (6), the latter transform equation translates into the following *functional equation* for the steady-state *system-content* pgf $U(z_1, z_2)$:

$$K(z_1, z_2)U(z_1, z_2) = A(z_1, z_2)L(z_1, z_2) \ , \tag{9}$$

where the unknown function $L(z_1, z_2)$ is defined as

$$L(z_1, z_2) \triangleq (z_2 - 1)U(z_1, 0) + (z_1 - 1)U(0, z_2) + (z_1 - 1)(z_2 - 1)U(0, 0) \tag{10}$$

and the *kernel* $K(z_1, z_2)$ is given by

$$K(z_1, z_2) \triangleq z_1 z_2 - A(z_1, z_2) \ . \tag{11}$$

Although, in general, the functional Equation (9) is hard to solve for $U(z_1, z_2)$, it is fairly easy to derive explicit expressions for the marginal pgfs $U(z_1, 1)$ and $U(1, z_2)$ of the individual system contents in queues 1 and 2, by choosing either $z_2 = 1$ or $z_1 = 1$ in (9), because such choices greatly simplify the L-function. As a result, we then obtain

$$U(z_1, 1) = \frac{U(0,1)(z_1 - 1)A_1(z_1)}{z_1 - A_1(z_1)} \ , \ U(1, z_2) = \frac{U(1,0)(z_2 - 1)A_2(z_2)}{z_2 - A_2(z_2)} \ . \tag{12}$$

Invoking the normalization condition $U(1,1) = 1$ yields $U(0,1) = 1 - \lambda_1$ and $U(1,0) = 1 - \lambda_2$. The expressions in (12) are very well-known in the context of discrete-time queueing theory; see, for instance [64]; they will be very useful further in this paper.

We now turn our attention to the *queue contents*. Using (5) in (8), we readily get

$$u_{1,k+1} = a_{1,k} + q_{1,k} \quad , \quad u_{2,k+1} = a_{2,k} + q_{2,k} \ .$$

Transforming these relationships to pgfs, we obtain, on account of the definitions (3) and (4),

$$U_{k+1}(z_1, z_2) = A(z_1, z_2) Q_k(z_1, z_2) \ .$$

In view of (6), this implies that

$$U(z_1, z_2) = A(z_1, z_2) Q(z_1, z_2) \quad , \quad Q(z_1, z_2) = \frac{U(z_1, z_2)}{A(z_1, z_2)} \ . \tag{13}$$

Equation (13) makes clear that $Q(z_1, z_2)$ is known as soon as $U(z_1, z_2)$ is known, and vice versa. In the remainder of this paper, we mainly concentrate on the determination of $U(z_1, z_2)$.

3. Defining a Class of Arrival pgfs $A(z_1, z_2)$

Let $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$, $g_2(z_2)$, $h_1(z_1)$, $h_2(z_2)$ denote one-dimensional probability generating functions. Furthermore, let n_{11}, n_{12}, n_{21}, n_{22} denote a set of normalized probabilities, i.e.,

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \quad , \quad n_{11} + n_{12} + n_{21} + n_{22} = 1 \ ,$$

and d_1 and d_2 two nonnegative real parameters. We then use all the above quantities to define a whole class of bivariate functions $A(z_1, z_2)$ as follows:

$$A(z_1, z_2) = \frac{n_{11} z_1 z_2 + n_{12} z_1 f_2(z_2) + n_{21} z_2 f_1(z_1) + n_{22} h_1(z_1) h_2(z_2)}{1 + d_1 + d_2 - d_1 g_1(z_1) - d_2 g_2(z_2)} \ . \tag{14}$$

We now show that the above function is a genuine joint pgf, i.e., it can be developed as a two-dimensional power series in z_1 and z_2 with nonnegative coefficients which add up to 1.

Let us denote the numerator and the denominator of (14) as $n(z_1, z_2)$ and $d(z_1, z_2)$, respectively, i.e.,

$$A(z_1, z_2) = \frac{n(z_1, z_2)}{d(z_1, z_2)} = n(z_1, z_2)\left(\frac{1}{d(z_1, z_2)}\right) \ , \tag{15}$$

with

$$n(z_1, z_2) \triangleq n_{11} z_1 z_2 + n_{12} z_1 f_2(z_2) + n_{21} z_2 f_1(z_1) + n_{22} h_1(z_1) h_2(z_2) \ ,$$
$$d(z_1, z_2) \triangleq 1 + d_1 + d_2 - d_1 g_1(z_1) - d_2 g_2(z_2) \ . \tag{16}$$

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It is clear that both $n(z_1, z_2)$ and $d(z_1, z_2)$ are normalized, i.e., $n(1, 1) = 1$ and $d(1, 1) = 1$, which implies that $A(z_1, z_2)$ is also normalized, i.e., $A(1, 1) = 1$. The numerator $n(z_1, z_2)$ is a probabilistic mixture of four valid pgfs, and, therefore, is a valid pgf too. The function $1/d(z_1, z_2)$ is also a genuine pgf since it can be developed as a two-dimensional power series in z_1 and z_2 with nonnegative coefficients as follows:

$$\begin{aligned}\frac{1}{d(z_1,z_2)} &= \frac{1}{(1+d_1+d_2) - d_1 g_1(z_1) - d_2 g_2(z_2)} \\ &= \frac{1}{(1+d_1+d_2)(1-\pi_1 g_1(z_1) - \pi_2 g_2(z_2))} \\ &= \left(\frac{1}{1+d_1+d_2}\right)\sum_{i=0}^{\infty}\left(\pi_1 g_1(z_1) + \pi_2 g_2(z_2)\right)^i ,\end{aligned}$$

where the probabilities π_1 and π_2 have been defined as

$$\pi_1 = \frac{d_1}{1+d_1+d_2} \quad , \quad \pi_2 = \frac{d_2}{1+d_1+d_2} \quad .$$

Equation (15) thus shows that $A(z_1, z_2)$ can be expressed as the product of two valid joint pgfs, and, therefore, is a valid joint pgf too.

In this paper, we will examine a *parallel-queues* system with joint arrival pgf $A(z_1, z_2)$ defined in (14). The corresponding marginal arrival pgfs $A_1(z_1)$ and $A_2(z_2)$ are

$$\begin{aligned}A_1(z_1) = A(z_1,1) = \frac{n(z_1,1)}{d(z_1,1)} &= \frac{(n_{11}+n_{12})z_1 + n_{21}f_1(z_1) + n_{22}h_1(z_1)}{1+d_1-d_1 g_1(z_1)} , \\ A_2(z_2) = A(1,z_2) = \frac{n(1,z_2)}{d(1,z_2)} &= \frac{(n_{11}+n_{21})z_2 + n_{12}f_2(z_2) + n_{22}h_2(z_2)}{1+d_2-d_2 g_2(z_2)} .\end{aligned} \tag{17}$$

The mean arrival rates λ_1 and λ_2 are

$$\begin{aligned}\lambda_1 &\triangleq A_1'(1) = n_{11} + n_{12} + n_{21}f_1'(1) + n_{22}h_1'(1) + d_1 g_1'(1) \\ \lambda_2 &\triangleq A_2'(1) = n_{11} + n_{21} + n_{12}f_2'(1) + n_{22}h_2'(1) + d_2 g_2'(1) \ .\end{aligned} \tag{18}$$

We assume that $\lambda_1 < 1, \lambda_2 < 1$.

4. The Main Result

According to (12), the marginal system-content pgfs are given by

$$\begin{aligned}U(z_1,1) &= \frac{U(0,1)(z_1-1)A_1(z_1)}{z_1 - A_1(z_1)} = \frac{U(0,1)(z_1-1)n(z_1,1)}{k_1(z_1)} , \\ U(1,z_2) &= \frac{U(1,0)(z_2-1)A_2(z_2)}{z_2 - A_2(z_2)} = \frac{U(1,0)(z_2-1)n(1,z_2)}{k_2(z_2)} ,\end{aligned} \tag{19}$$

where

$$k_1(z_1) \triangleq z_1 d(z_1,1) - n(z_1,1) \quad , \quad k_2(z_2) \triangleq z_2 d(1,z_2) - n(1,z_2) \ . \tag{20}$$

We are now ready to formulate the main result of this paper.

Theorem 1:
In the stable parallel-queues system with joint arrival pgf

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{d(z_1,z_2)} = \frac{n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}h_1(z_1)h_2(z_2)}{1+d_1+d_2-d_1g_1(z_1)-d_2g_2(z_2)} \ , \tag{21}$$

where $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$ and $g_2(z_2)$ are arbitrary one-dimensional pgfs, and

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \quad , \quad n_{11}+n_{12}+n_{21}+n_{22} = 1 \ , \tag{22}$$

the steady-state joint system-content pgf $U(z_1,z_2)$ is given by

$$U(z_1,z_2) = \frac{n(z_1,z_2)U(z_1,1)U(1,z_2)}{n(z_1,1)n(1,z2)} = \frac{U(0,1)U(1,0)(z_1-1)(z_2-1)n(z_1,z_2)}{k_1(z_1)k_2(z_2)} \ , \tag{23}$$

provided that the functions $h_1(z_1)$ and $h_2(z_2)$ are genuine one-dimensional pgfs, given by

$$\begin{aligned} h_1(z_1) &= \alpha_1 z_1 + \beta_1 f_1(z_1) + \gamma_1 z_1 g_1(z_1) \ , \\ h_2(z_2) &= \alpha_2 z_2 + \beta_2 f_2(z_2) + \gamma_2 z_2 g_2(z_2) \ , \end{aligned} \tag{24}$$

where $\{\alpha_1,\beta_1,\gamma_1\}$ and $\{\alpha_2,\beta_2,\gamma_2\}$ are two sets of "normalized constants", i.e.,

$$\alpha_1+\beta_1+\gamma_1 = 1 \ , \ \alpha_2+\beta_2+\gamma_2 = 1 \ , \tag{25}$$

satisfying the restrictions

$$\beta_1 > 0 \ , \ ,\beta_2 > 0 \quad , \quad \gamma_1 = \frac{d_1\beta_1}{n_{21}} \geq 0 \ , \ \gamma_2 = \frac{d_2\beta_2}{n_{12}} \geq 0 \ . \tag{26}$$

4.1. Some Remarks on the Terms of Theorem 1

According to equations (24), Theorem 1 requires that $h_1(z_1)$ be a linear combination of z_1, $f_1(z_1)$ and $z_1g_1(z_1)$, and, similarly that $h_2(z_2)$ a linear combination of z_2, $f_2(z_2)$ and $z_2g_2(z_2)$, with coefficients that add up to 1. It is easily seen that, as required, this implies that $h_1(z_1)$ and $h_2(z_2)$ are normalized, i.e., $h_1(1) = h_2(1) = 1$. However, for arbitrary choices of the parameters β_1, β_2, d_1, d_2, n_{12} and n_{21}, the functions $h_1(z_1)$ and $h_2(z_2)$, as given in (24), could, in general, contain linear terms in z_1 or z_2 with a negative coefficient, which would prevent them from being genuine pgfs. Indeed, whereas the coefficients β_1, γ_1, β_2 and γ_2 are certainly nonnegative, this not necessarily the case for the coefficients of the linear terms in (24). It is clear that *sufficient* conditions to guarantee that $h_1(z_1)$ and $h_2(z_2)$ are valid pgfs are

$$\alpha_1 \geq 0 \ , \ \alpha_2 \geq 0 \ , \tag{27}$$

but these conditions are not *necessary*. In order to determine the linear terms in (24) completely, it is useful to decompose the functions $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$ and $g_2(z_2)$ as follows:

$$f_1(z_1) = f_{10} + f_{11}z_1 + z_1^2 v_1(z_1) \ , \ f_2(z_2) = f_{20} + f_{21}z_2 + z_2^2 v_2(z_2) \ , \tag{28}$$

and

$$g_1(z_1) = g_{10} + z_1 w_1(z_1) \ , \ g_2(z_2) = g_{20} + z_2 w_2(z_2) \ . \tag{29}$$

Substitution of (28) and (29) in (24) yields

$$\begin{aligned} h_1(z_1) &= \beta_1 f_{10} + (\alpha_1 + \beta_1 f_{11} + \gamma_1 g_{10})z_1 + (\beta_1 v_1(z_1) + \gamma_1 w_1(z_1))z_1^2 \\ h_2(z_2) &= \beta_2 f_{20} + (\alpha_2 + \beta_2 f_{21} + \gamma_2 g_{20})z_2 + (\beta_2 v_2(z_2) + \gamma_2 w_2(z_2))z_2^2 \ . \end{aligned} \tag{30}$$

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The above expressions make clear that $h_1(z_1)$ and $h_2(z_2)$ are genuine pgs *if and only if*

$$\alpha_1 + \beta_1 f_{11} + \gamma_1 g_{10} \geq 0 \ , \ \alpha_2 + \beta_2 f_{21} + \gamma_2 g_{20} \geq 0 \ , \tag{31}$$

which are milder conditions than (27).

4.2. Proving Theorem 1

In order to prove Theorem 1, we first establish a technical lemma. Let us define the bivariate function $e(z_1, z_2)$ as

$$\begin{aligned} e(z_1, z_2) \triangleq\ & n(0,1)n(1,0)n(z_1,z_2) + n(0,1)n(z_1,0)k_2(z_2) + n(1,0)n(0,z_2)k_1(z_1) \\ & + n(0,0)k_1(z_1)k_2(z_2) \ . \end{aligned} \tag{32}$$

Lemma 1: *The function $e(z_1, z_2)$ can be expressed as*

$$e(z_1, z_2) = n(0,1)n(1,0)z_1 z_2 d(z_1, z_2) \ .$$

Proof. Combining (24), (25) and (26), and due to the fact that β_1 and β_2 are assumed to be strictly positive, we can compute the functions $n_{21} f_1(z_1)$ and $n_{12} f_2(z_2)$ in terms of $g_1(z_1)$, $g_2(z_2)$, $h_1(z_1)$ and $h_2(z_2)$, as follows:

$$\begin{aligned} n_{21} f_1(z_1) &= \frac{\beta_1 z_1 \big(n_{21} + d_1(1 - g_1(z_1))\big) + n_{21}\big(z_1 - h_1(z_1)\big)}{\beta_1} \ , \\ n_{12} f_2(z_2) &= \frac{\beta_2 z_2 \big(n_{12} + d_2(1 - g_2(z_2))\big) + n_{12}\big(z_2 - h_2(z_2)\big)}{\beta_2} \ . \end{aligned} \tag{33}$$

Inserting (33) in (16), we then get

$$n(z_1, z_2) = n_{22}\big(h_1(z_1)h_2(z_2) - z_1 z_2\big) + z_1 z_2 d(z_1, z_2) - \frac{n_{21} z_2 \big(z_1 - h_1(z_1)\big)}{\beta_1} - \frac{n_{12} z_1 \big(z_2 - h_2(z_2)\big)}{\beta_2} \ , \tag{34}$$

where we have also used the definition (16) of $d(z_1, z_2)$.

Choosing either $z_1 = 0$ or $z_2 = 0$ in (34) yields

$$n(z_1, 0) = f_2(0)\big(n_{12} z_1 + \beta_2 n_{22} h_1(z_1)\big) \ , \ n(0, z_2) = f_1(0)\big(n_{21} z_2 + \beta_1 n_{22} h_2(z_2)\big) \ , \tag{35}$$

and, from this,

$$n(0,0) = f_1(0) f_2(0) n_{22} \beta_1 \beta_2 \ . \tag{36}$$

On the other hand, choosing either $z_1 = 1$ or $z_2 = 1$ in (34) leads to

$$\begin{aligned} n(z_1, 1) &= n_{22} h_1(z_1) + z_1\big(d(z_1, 1) - n_{22}\big) - \frac{n_{21}\big(z_1 - h_1(z_1)\big)}{\beta_1} \ , \\ n(1, z_2) &= n_{22} h_2(z_2) + z_2\big(d(1, z_2) - n_{22}\big) - \frac{n_{12}\big(z_2 - h_2(z_2)\big)}{\beta_2} \ , \end{aligned} \tag{37}$$

and, from this,

$$n(0,1) = f_1(0)\big(\beta_1 n_{22} + n_{21}\big) \ , \ n(0,1) = f_2(0)\big(\beta_2 n_{22} + n_{12}\big) \ . \tag{38}$$

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Using (37) and (38) in (20), we can express $k_1(z_1)$ and $k_2(z_2)$ as

$$\begin{aligned} k_1(z_1) &\triangleq z_1 d(z_1,1) - n(z_1,1) = \frac{\beta_1 n_{22} + n_{21}}{\beta_1}(z_1 - h_1(z_1)) = \frac{n(0,1)}{\beta_1 f_1(0)}(z_1 - h_1(z_1)) \ , \\ k_2(z_2) &\triangleq z_2 d(1,z_2) - n(1,z_2) = \frac{\beta_2 n_{22} + n_{12}}{\beta_2}(z_2 - h_2(z_2)) = \frac{n(1,0)}{\beta_2 f_2(0)}(z_2 - h_2(z_2)) \ . \end{aligned} \tag{39}$$

Substitution of (34), (35), (36) and (39) in (32) then leads to

$$\begin{aligned} e(z_1,z_2) = n(0,1)n(1,0)\bigg(& n_{22}(h_1(z_1)h_2(z_2) - z_1z_2) + z_1z_2 d(z_1,z_2) - \frac{n_{21}z_2(z_1 - h_1(z_1))}{\beta_1} \\ & - \frac{n_{12}z_1(z_2 - h_2(z_2))}{\beta_2}\bigg) + n(0,1)f_2(0)(n_{12}z_1 + \beta_2 n_{22} h_1(z_1))\frac{n(1,0)}{\beta_2 f_2(0)}(z_2 - h_2(z_2)) \\ & + n(1,0)f_1(0)(n_{21}z_2 + \beta_1 n_{22} h_2(z_2))\frac{n(0,1)}{\beta_1 f_1(0)}(z_1 - h_1(z_1)) \\ & + f_1(0)f_2(0)n_{22}\beta_1\beta_2\frac{n(0,1)}{\beta_1 f_1(0)}(z_1 - h_1(z_1))\frac{n(1,0)}{\beta_2 f_2(0)}(z_2 - h_2(z_2)) \ . \end{aligned} \tag{40}$$

It is then a matter of straightforward algebra to prove that all the terms in the above equation containing $h_1(z_1)$ and/or $h_2(z_2)$ compensate each other. The final result is

$$e(z_1,z_2) = n(0,1)n(1,0)z_1z_2 d(z_1,z_2) \ .$$

This concludes the proof of Lemma 1. □

Proof of Theorem 1

The proof of Theorem 1 consists of two steps. In a first step, we show that the function $U(z_1,z_2)$, defined in (23), is a genuine joint pgf. In the second step, we prove that, under the conditions of Theorem 1, $U(z_1,z_2)$ satisfies the functional Equation (9) of the system.

Step 1: $U(z_1,z_2)$ is a genuine joint pgf

Proof. In view of the relationship (13) between $U(z_1,z_2)$ and $Q(z_1,z_2)$ (and the corresponding marginal pgfs), Equation (23) can be rewritten as

$$U(z_1,z_2) = \frac{n(z_1,z_2)U(z_1,1)U(1,z_2)}{n(z_1,1)n(1,z_2)} = \frac{n(z_1,z_2)A(z_1,1)Q(z_1,1)A(1,z_2)Q(1,z_2)}{n(z_1,1)n(1,z_2)} \ .$$

Using (17), we then easily obtain

$$U(z_1,z_2) = \frac{n(z_1,z_2)Q(z_1,1)Q(1,z_2)}{d(z_1,1)d(1,z_2)} = n(z_1,z_2)Q(z_1,1)Q(1,z_2)\left(\frac{1}{d(z_1,1)}\right)\left(\frac{1}{d(1,z_2)}\right) \ ,$$

where the right-hand side is a product of five valid pgfs. Hence, $U(z_1,z_2)$ is a valid pgf as well. □

Step 2: $U(z_1,z_2)$ satisfies the functional equation

Proof. Combining (9) and (11), we can express the functional equation as

$$(z_1z_2 - A(z_1,z_2))U(z_1,z_2) = A(z_1,z_2)L(z_1,z_2) \ ,$$

and, from this,

$$A(z_1,z_2)(L(z_1,z_2) + U(z_1,z_2)) = z_1z_2U(z_1,z_2) \ .$$

Inserting the expression (15) for $A(z_1,z_2)$ then leads to

$$L(z_1,z_2)+U(z_1,z_2)=\frac{z_1z_2d(z_1,z_2)U(z_1,z_2)}{n(z_1,z_2)} \ . \tag{41}$$

The function $L(z_1,z_2)$ can be computed from (10) as

$$L(z_1,z_2)\triangleq(z_2-1)U(z_1,0)+(z_1-1)U(0,z_2)+(z_1-1)(z_2-1)U(0,0) \ , \tag{42}$$

where $U(z_1,0)$ and $U(0,z_2)$ can be derived from (23) as

$$U(z_1,0)=\frac{U(0,1)U(1,0)(z_1-1)n(z_1,0)}{n(1,0)\big(k_1(z_1)\big)} \ , \ U(0,z_2)=\frac{U(0,1)U(1,0)(z_2-1)n(0,z_2)}{n(0,1)\big(k_2(z_2)\big)} \ , \tag{43}$$

which also implies

$$U(0,0)=\frac{U(0,1)U(1,0)n(0,0)}{n(0,1)n(1,0)} \ . \tag{44}$$

Substitution of (43) and (44) in (45) then leads to

$$L(z_1,z_2)=\ell(z_1,z_2)b(z_1,z_2) \ , \tag{45}$$

where we have defined $\ell(z_1,z_2)$ and $b(z_1,z_2)$ as short-hand notations for

$$\ell(z_1,z_2)\triangleq n(0,1)n(z_1,0)k_2(z_2)+n(1,0)n(0,z_2)k_1(z_1)+n(0,0)k_1(z_1)k_2(z_2) \ ,$$

$$b(z_1,z_2)\triangleq\frac{U(0,1)U(1,0)(z_1-1)(z_2-1)}{n(0,1)n(1,0)k_1(z_1)k_2(z_2)} \ .$$

On the other hand, in view of (23), $U(z_1,z_2)$ can be expressed as

$$U(z_1,z_2)=n(0,1)n(1,0)n(z_1,z_2)b(z_1,z_2) \ . \tag{46}$$

Inserting (45) and (46) in (41), we then get

$$\big(\ell(z_1,z_2)+n(0,1)n(1,0)n(z_1,z_2)\big)b(z_1,z_2)=\big(n(0,1)n(1,0)z_1z_2d(z_1,z_2)\big)b(z_1,z_2) \ . \tag{47}$$

It thus suffices to show that the expressions between the large parentheses in the left-hand side and the right-hand side of (47) are equal. The expression in the left-hand side is exactly the function $e(z_1,z_2)$, defined in (32). Lemma 1 thus proves that (47) is fulfilled. This concludes the proof of Theorem 1. □

5. Subclasses A, B and C

Theorem 1 provides an explicit solution for the steady-state joint system-contents pgf $U(z_1,z_2)$, for any joint arrival pgf $A(z_1,z_2)$ that satisfies the shape specified in equations (21), (24), (25) and (26), whereby $h_1(z_1)$ and $h_2(z_2)$ are valid pgfs. We now define some interesting subclasses of arrival pgfs that correspond to specific choices of the (many) parameters that the model contains.

5.1. Subclass A: $d_1=0, d_2=0$

In this special case, $d(z_1,z_2)=1$ and the arrival pgf $A(z_1,z_2)$, defined in (14), simplifies to

$$A(z_1,z_2)=n(z_1,z_2)=n_{11}z_1z_2+n_{12}z_1f_2(z_2)+n_{21}z_2f_1(z_1)+n_{22}h_1(z_1)h_2(z_2) \ . \tag{48}$$

We define the *subclass* $\mathcal{A}$ of arrival pgfs $A(z_1, z_2)$ by the condition $d_1 = 0, d_2 = 0$. We notice that, within subclass $\mathcal{A}$, the pgfs $g_1(z_1)$ and $g_2(z_2)$ play no role anymore in $A(z_1, z_2)$. We also observe that, since $A(z_1, z_2) = n(z_1, z_2)$, the joint *queue-content* pgf $Q(z_1, z_2)$, given in (13), reduces to

$$Q(z_1, z_2) = \frac{U(z_1, z_2)}{n(z_1, z_2)} \ ,$$

and, hence, our main result (23) in Theorem 1 is equivalent to

$$Q(z_1, z_2) = Q(z_1, 1)Q(1, z_2) \ , \tag{49}$$

i.e., for arrival pgfs within subclass $\mathcal{A}$, the *queue contents* of both queues are mutually independent.

The parameters γ_1 and γ_2, defined in (26), are both equal to zero, and, hence, the functions $h_1(z_1)$ and $h_2(z_2)$, defined in (24), are given by

$$h_1(z_1) = (1 - \beta_1)z_1 + \beta_1 f_1(z_1) \ , \ h_2(z_2) = (1 - \beta_2)z_2 + \beta_2 f_2(z_2) \ . \tag{50}$$

Substitution of (50) in (48) then leads to

$$\begin{aligned} A(z_1, z_2) = n(z_1, z_2) = n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) \\ + n_{22}\big((1 - \beta_1)z_1 + \beta_1 f_1(z_1)\big)\big((1 - \beta_2)z_2 + \beta_2 f_2(z_2)\big) \ , \end{aligned} \tag{51}$$

which can be rewritten as

$$A(z_1, z_2) = n(z_1, z_2) = m_{11}z_1z_2 + m_{12}z_1f_2(z_2) + m_{21}z_2f_1(z_1) + m_{22}f_1(z_1)f_2(z_2) \ ,$$

if we define the new parameters m_{11}, m_{12}, m_{21} and m_{22} as

$$m_{11} \triangleq (1 - \beta_1)(1 - \beta_2)n_{22} \ , \ m_{12} \triangleq n_{12} + (1 - \beta_1)\beta_2 n_{22} \ ,$$

$$m_{21} \triangleq n_{21} + \beta_1(1 - \beta_2)n_{22} \ , \ m_{22} \triangleq \beta_1\beta_2 n_{22} \ .$$

It is easily seen that, if we choose $\beta_1 \leq 1$ and $\beta_2 \leq 1$, the sufficient conditions (27) are fulfilled, and the parameters m_{11}, m_{12}, m_{21} and m_{22} also represent a normalized set of probabilities, just as the original parameters n_{11}, n_{12}, n_{21} and n_{22}, i.e.,

$$m_{11}, m_{12}, m_{21}, m_{22} \geq 0 \quad , \quad m_{11} + m_{12} + m_{21} + m_{22} = 1 \ , \tag{52}$$

The marginal arrival pgfs $A_1(z_1)$ and $A_2(z_2)$, given in (17), reduce to

$$A_1(z_1) = (m_{11} + m_{12})z_1 + (m_{21} + m_{22})f_1(z_1) \ , \ A_2(z_2) = (m_{11} + m_{21})z_2 + (m_{12} + m_{22})f_2(z_2) \ , \tag{53}$$

whereas the marginal mean arrival rates, given in (18), simplify to

$$\lambda_1 = 1 - (m_{21} + m_{22})\big(1 - f_1'(1)\big] \ , \ \lambda_2 = 1 - (m_{12} + m_{22})\big(1 - f_2'(1)\big] \ . \tag{54}$$

Hence, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with

$$f_1'(1) < 1 \ , \ f_2'(1) < 1 \ .$$

The functions $k_1(z_1)$ and $k_2(z_2)$, defined in (20), reduce to

$$k_1(z_1) = (m_{21} + m_{22})\big(z_1 - f_1(z_1)\big) \ , \ k_2(z_2) = (m_{12} + m_{22})\big(z_2 - f_2(z_2)\big) \ ,$$

and, hence, the joint pgf $U(z_1, z_2)$ can be derived from (23) as

$$U(z_1, z_2) = \frac{M(z_1 - 1)(z_2 - 1)n(z_1, z_2)}{\big(z_1 - f_1(z_1)\big)\big(z_2 - f_2(z_2)\big)} \ , \tag{55}$$

where the constant M is defined as

$$M \triangleq \frac{U(0,1)U(1,0)}{(m_{21} + m_{22})(m_{12} + m_{22})} \ .$$

The only remaining unknown M in Equation (55) can be computed from the normalization condition $U(1,1) = 1$, which results in

$$M = \big(1 - f_1'(1)\big)\big(1 - f_2'(1)\big) \ . \tag{56}$$

A fully explicit expression for $U(z_1, z_2)$ then follows from (55) and (56).

In summary, we have thus proven the following corrolary of Theorem 1:

Corrolary A

In the stable parallel-queues system with joint arrival pgf

$$A(z_1, z_2) = m_{11} z_1 z_2 + m_{12} z_1 f_2(z_2) + m_{21} z_2 f_1(z_1) + m_{22} f_1(z_1) f_2(z_2) \ , \tag{57}$$

where $f_1(z_1)$ and $f_2(z_2)$ are arbitrary one-dimensional pgfs, and

$$m_{11}, m_{12}, m_{21}, m_{22} \geq 0 \quad , \quad m_{11} + m_{12} + m_{21} + m_{22} = 1 \ , \tag{58}$$

the steady-state queue contents of both queues are mutually independent, and the steady-state joint system-content pgf $U(z_1, z_2)$ is given by

$$U(z_1, z_2) = \frac{\big(1 - f_1'(1)\big)\big(1 - f_2'(1)\big)(z_1 - 1)(z_2 - 1)A(z_1, z_2)}{\big(z_1 - f_1(z_1)\big)\big(z_2 - f_2(z_2)\big)} \ . \tag{59}$$

Remark

If we define the *discriminant* D of $A(z_1, z_2)$ as $D \triangleq m_{12} m_{21} - m_{11} m_{22}$, then it is easily seen that if $D = 0$, the pgf $A(z_1, z_2)$ has a product form, i.e.,

$$A(z_1, z_2) = \frac{\big(m_{11} z_1 + m_{21} f_1(z_1)\big)\big(m_{11} z_2 + m_{12} f_2(z_2)\big)}{m_{11}} \ ,$$

and the arrivals of both customer types are mutually independent. It is clear from (59) that, in this case, the pgf $U(z_1, z_2)$ reduces to a product form too, i.e., the two steady-state *system contents* are also mutually independent.

5.2. Subclass B: $\alpha_1 = 0, \alpha_2 = 0$

The requirements (27) represent sufficient conditions to guarantee that the functions $h_1(z_1)$ and $h_2(z_2)$ are valid pgfs. A trivial way to satisfy (27) is to choose

$$\alpha_1 = 0 \ , \ \alpha_2 = 0 \ . \tag{60}$$

We define the *subclass B* of arrival pgfs $A(z_1, z_2)$ by the condition (60). From (25) and (26), it follows that (60) is equivalent with

$$\gamma_1 = \frac{d_1}{d_1 + n_{21}} \ , \ \gamma_2 = \frac{d_2}{d_2 + n_{12}} \ . \tag{61}$$

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Substitution of (60) in (24) then yields

$$h_1(z_1) = (1-\gamma_1)f_1(z_1) + \gamma_1 z_1 g_1(z_1) \;,\; h_2(z_2) = (1-\gamma_2)f_2(z_2) + \gamma_2 z_2 g_2(z_2) \;. \quad (62)$$

Equation (62) implies that

$$h_1'(1) = (1-\gamma_1)f_1'(1) + \gamma_1[1+g_1'(1)] \;,\; h_2'(1) = (1-\gamma_2)f_2'(1) + \gamma_2[1+g_2'(1)] \;,$$

so that the mean arrival rates, given in (18), can be expressed as

$$\begin{aligned} \lambda_1 &= (n_{11}+n_{12}+\gamma_1 n_{22}) + (n_{21}+(1-\gamma_1)n_{22})f_1'(1) + (d_1+\gamma_1 n_{22})g_1'(1) \;, \\ \lambda_2 &= (n_{11}+n_{21}+\gamma_2 n_{22}) + (n_{12}+(1-\gamma_2)n_{22})f_2'(1) + (d_2+\gamma_2 n_{22})g_2'(1) \;. \end{aligned} \quad (63)$$

The functions $k_1(z_1)$ and $k_2(z_2)$, defined in (20), are given by

$$\begin{aligned} k_1(z_1) &= \frac{n_{21}+(1-\gamma_1)n_{22}}{1-\gamma_1}\left([1-\gamma_1 g_1(z_1)]z_1 - (1-\gamma_1)f_1(z_1)\right) \;, \\ k_2(z_2) &= \frac{n_{12}+(1-\gamma_2)n_{22}}{1-\gamma_2}\left([1-\gamma_2 g_2(z_2)]z_2 - (1-\gamma_2)f_2(z_2)\right) \;. \end{aligned} \quad (64)$$

The joint pgf $U(z_1, z_2)$ can be derived from (23) and (64) as

$$U(z_1,z_2) = \frac{M(z_1-1)(z_2-1)n(z_1,z_2)}{\left([1-\gamma_1 g_1(z_1)]z_1 - (1-\gamma_1)f_1(z_1)\right)\left([1-\gamma_2 g_2(z_2)]z_2 - (1-\gamma_2)f_2(z_2)\right)} \;, \quad (65)$$

where the constant M has been defined as

$$M \triangleq \frac{U(0,1)U(1,0)(1-\gamma_1)(1-\gamma_2)}{(n_{21}+(1-\gamma_1)n_{22})(n_{12}+(1-\gamma_2)n_{22})} \;.$$

The only remaining unknown M in Equation (88) can be computed from the normalization condition $U(1,1) = 1$, which results in

$$M = \left((1-\gamma_1)[1-f_1'(1)] - \gamma_1 g_1'(1)\right)\left((1-\gamma_2)[1-f_2'(1)] - \gamma_2 g_2'(1)\right) \;. \quad (66)$$

A fully explicit expression for $U(z_1, z_2)$ then follows from (69) and (70).

Summarizing again, we have thus proven the following corrolary of Theorem 1:

Corrolary *B*

In the stable parallel-queues system with joint arrival pgf

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1+d_1+d_2-d_1 g_1(z_1) - d_2 g_2(z_2)} \;, \quad (67)$$

where $n(z_1, z_2)$ is defined as

$$\begin{aligned} n(z_1,z_2) \triangleq\; & n_{11}z_1z_2 + n_{12}z_1 f_2(z_2) + n_{21}z_2 f_1(z_1) \\ & + n_{22}\left((1-\gamma_1)f_1(z_1) + \gamma_1 z_1 g_1(z_1)\right)\left((1-\gamma_2)f_2(z_2) + \gamma_2 z_2 g_2(z_2)\right) \;, \end{aligned} \quad (68)$$

with $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$, $g_2(z_2)$ arbitrary one-dimensional pgfs, and

$$n_{11}, n_{12}, n_{21}, n_{22} \geq 0 \;,\; n_{11}+n_{12}+n_{21}+n_{22} = 1 \;,\; \gamma_1 \triangleq \frac{d_1}{d_1+n_{21}} \;,\; \gamma_2 \triangleq \frac{d_2}{d_2+n_{12}} \;,$$

the steady-state joint system-content pgf $U(z_1,z_2)$ is given by

$$U(z_1,z_2) = \frac{M(z_1-1)(z_2-1)n(z_1,z_2)}{\left([1-\gamma_1 g_1(z_1)]z_1-(1-\gamma_1)f_1(z_1)\right)\left([1-\gamma_2 g_2(z_2)]z_2-(1-\gamma_2)f_2(z_2)\right)} \ , \tag{69}$$

with

$$M \triangleq \left((1-\gamma_1)[1-f_1'(1)]-\gamma_1 g_1'(1)\right)\left((1-\gamma_2)[1-f_2'(1)]-\gamma_2 g_2'(1)\right) \ . \tag{70}$$

5.3. Subclass C: No Linear Terms in $h_1(z_1)$ and $h_2(z_2)$

The requirements (31) are *necessary and sufficient* conditions in order for $h_1(z_1)$ and $h_2(z_2)$ to represent genuine pgfs. In this subsection, we examine the extreme case whereby the inequalities in (31) are replaced by equalities, i.e., where

$$\alpha_1+\beta_1 f_{11}+\gamma_1 g_{10}=0 \ , \ \alpha_2+\beta_2 f_{21}+\gamma_2 g_{20}=0 \ . \tag{71}$$

In view of (25) and (26), (71) can be rewritten as

$$(1-\alpha_1)(n_{21}f_{11}+d_1 g_{10})+\alpha_1(n_{21}+d_1)=0 \ , \ (1-\alpha_2)(n_{12}f_{21}+d_2 g_{20})+\alpha_2(n_{12}+d_2)=0 \ . \tag{72}$$

Solving (72) for α_1 and α_2, we find

$$\alpha_1 = -\frac{n_{21}f_{11}+d_1 g_{10}}{n_{21}(1-f_{11})+d_1(1-g_{10})} \ , \ \alpha_2 = -\frac{n_{12}f_{21}+d_2 g_{20}}{n_{12}(1-f_{21})+d_2(1-g_{20})} \ . \tag{73}$$

In these circumstances, due to (30), the functions $h_1(z_1)$ and $h_2(z_2)$ can be expressed as

$$\begin{aligned} h_1(z_1) &= \frac{n_{21}f_{10}+z_1^2\left(n_{21}v_1(z_1)+d_1 w_1(z_1)\right)}{n_{21}(1-f_{11})+d_1(1-g_{10})} \ , \\ h_2(z_2) &= \frac{n_{12}f_{20}+z_2^2\left(n_{12}v_2(z_2)+d_2 w_2(z_2)\right)}{n_{12}(1-f_{21})+d_2(1-g_{20})} \ , \end{aligned} \tag{74}$$

and, hence, contain no linear terms in z_1 and z_2, respectively.

In order to further simplify the expressions, let us consider the (further) special case where

$$v_1(z_1)=w_1(z_1) \ , \ v_2(z_2)=w_2(z_2) \ . \tag{75}$$

Of course, we then also have $v_1(1)=w_1(1), v_2(1)=w_2(1)$. From (28) and (29), we readily obtain

$$v_1(1)=1-(f_{10}+f_{11}) \ , \ v_2(1)=1-(f_{20}+f_{21}) \ , \ w_1(1)=1-g_{10} \ , \ w_2(1)=1-g_{20} \ ,$$

and, hence, $v_1(1)=w_1(1), v_2(1)=w_2(1)$ implies

$$f_{11}=g_{10}-f_{10} \ , \ f_{21}=g_{20}-f_{20} \ .$$

We now choose to additionally simplify the model by assuming

$$g_{10}=f_{10}\triangleq 1-\omega_1 \ , \ g_{20}=f_{20}\triangleq 1-\omega_2 \quad \Leftrightarrow \quad f_{11}=0 \ , f_{20}=0 \ , \tag{76}$$

where we have introduced the new parameters ω_1 and ω_2, which are valid probabilities.

By definition, we refer to arrival pgfs $A(z_1,z_2)$ (of the form considered in Theorem 1) as pgfs of *subclass C* if and only if they comply with the conditions (71), (75) and (76).

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Using (75) and (76), we obtain the following expressions for $h_1(z_1)$ and $h_2(z_2)$ from (74):

$$h_1(z_1) = \frac{n_{21}(1-\omega_1) + (n_{21}+d_1)z_1^2 v_1(z_1)}{n_{21}+d_1\omega_1} \ , \ h_2(z_2) = \frac{n_{12}(1-\omega_2) + (n_{12}+d_2)z_2^2 v_2(z_2)}{n_{12}+d_2\omega_2} \ , \tag{77}$$

From their definitions in (28) and (29), it follows that the functions $v_1(z_1)$ and $v_2(z_2)$ only contain powers of z_1 and z_2 with nonnegative coefficients, but are not necessarily normalized. It is useful to replace them by new functions, say $c_1(z_1)$ and $c_2(z_2)$, that do satisfy a normalization condition, and, hence, are genuine pgfs, as follows:

$$c_1(z_1) \triangleq \frac{v_1(z_1)}{v_1(1)} = \frac{v_1(z_1)}{\omega_1} \ , \ c_2(z_2) \triangleq \frac{v_2(z_2)}{v_2(1)} = \frac{v_2(z_2)}{\omega_2} \ . \tag{78}$$

All the defining functions of our model can then be expressed in terms of the pgfs $c_1(z_1)$ and $c_2(z_2)$ as follows:

$$f_1(z_1) = 1-\omega_1+\omega_1 z_1^2 c_1(z_1) \ , \ f_2(z_2) = 1-\omega_2+\omega_2 z_2^2 c_2(z_2) \ , \tag{79}$$

$$g_1(z_1) = 1-\omega_1+\omega_1 z_1 c_1(z_1) \ , \ g_2(z_2) = 1-\omega_2+\omega_2 z_2 c_2(z_2) \ , \tag{80}$$

$$h_1(z_1) = 1-\theta_1+\theta_1 z_1^2 c_1(z_1) \ , \ h_2(z_2) = 1-\theta_2+\theta_2 z_2^2 c_2(z_2) \ , \tag{81}$$

where we have defined the probabilities θ_1 and θ_2 as

$$\theta_1 \triangleq \frac{(n_{21}+d_1)\omega_1}{n_{21}+d_1\omega_1} \ , \ \theta_2 \triangleq \frac{(n_{12}+d_2)\omega_2}{n_{12}+d_2\omega_2} \ . \tag{82}$$

The arrival pgf $A(z_1,z_2)$ can be determined by substitution of (79), (80) and (81) in (21):

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1+d_1\omega_1(1-z_1c_1(z_1))+d_2\omega_2(1-z_2c_2(z_2))} \ , \tag{83}$$

where

$$\begin{aligned} n(z_1,z_2) \triangleq\ & n_{11}z_1z_2 + n_{12}z_1\left(1-\omega_2+\omega_2 z_2^2 c_2(z_2)\right) + n_{21}z_2\left(1-\omega_1+\omega_1 z_1^2 c_1(z_1)\right) \\ & + n_{22}\left(1-\theta_1+\theta_1 z_1^2 c_1(z_1)\right)\left(1-\theta_2+\theta_2 z_2^2 c_2(z_2)\right) \ . \end{aligned} \tag{84}$$

The marginal mean arrival rates can be computed from (18), which results in

$$\begin{aligned} \lambda_1 &= 1 - \frac{n_{21}+n_{22}+\omega_1 d_1}{n_{21}+\omega_1 d_1}\left(n_{21}+\omega_1 d_1 - \omega_1(n_{21}+d_1)(2+c_1'(1))\right) \\ \lambda_2 &= 1 - \frac{n_{12}+n_{22}+\omega_2 d_2}{n_{12}+\omega_2 d_2}\left(n_{12}+\omega_2 d_2 - \omega_2(n_{12}+d_2)(2+c_2'(1))\right) \ . \end{aligned} \tag{85}$$

Consequently, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with

$$c_1'(1) < \frac{n_{21}-\omega_1(2n_{21}+d_1)}{\omega_1(n_{21}+d_1)} \ , \ c_2'(1) < \frac{n_{12}-\omega_2(2n_{12}+d_2)}{\omega_2(n_{12}+d_2)} \ .$$

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The functions $k_1(z_1)$ and $k_2(z_2)$ that constitute the denominator of $U(z_1,z_2)$ can be derived from (39):

$$\begin{aligned} k_1(z_1) &= \frac{n_{21}+n_{22}+\omega_1 d_1}{n_{21}+\omega_1 d_1}\left(n_{21}(z_1-1)+\omega_1(n_{21}+d_1 z_1)-\omega_1(n_{21}+d_1)z_1^2 c_1(z_1)\right) \ , \\ k_2(z_2) &= \frac{n_{12}+n_{22}+\omega_2 d_2}{n_{12}+\omega_2 d_2}\left(n_{12}(z_2-1)+\omega_2(n_{12}+d_2 z_2)-\omega_2(n_{12}+d_2)z_2^2 c_2(z_2)\right) \ , \end{aligned} \tag{86}$$

where we have also used (82). Introducing the notations $V_1(z_1)$ and $V_2(z_2)$ as

$$\begin{aligned} V_1(z_1) &\triangleq \frac{z_1-1}{n_{21}(z_1-1)+\omega_1(n_{21}+d_1 z_1)-\omega_1(n_{21}+d_1)z_1^2 c_1(z_1)} \ , \\ V_2(z_2) &\triangleq \frac{z_2-1}{n_{12}(z_2-1)+\omega_2(n_{12}+d_2 z_2)-\omega_2(n_{12}+d_2)z_2^2 c_2(z_2)} \ , \end{aligned} \tag{87}$$

we can compute the joint pgf $U(z_1,z_2)$ from (23) and (86) as

$$U(z_1,z_2) = MV_1(z_1)V_2(z_2)n(z_1,z_2) \ , \tag{88}$$

where the constant M has been defined as

$$M \triangleq \frac{U(0,1)U(1,0)(n_{21}+\omega_1 d_1)(n_{12}+\omega_2 d_2)}{(n_{21}+n_{22}+\omega_1 d_1)(n_{12}+n_{22}+\omega_2 d_2)} \ .$$

As before, the remaining unknown M can be determined by invoking the normalization condition $U(1,1)=1$, which results in

$$M = \left(n_{21}+\omega_1 d_1-\omega_1(n_{21}+d_1)(2+c_1'(1))\right)\left(n_{12}+\omega_2 d_2-\omega_2(n_{12}+d_2)(2+c_2'(1))\right) \ . \tag{89}$$

A fully explicit expression for $U(z_1,z_2)$ then follows from (88), (98) and (89).

Summarizing again, we have thus proven the following corrolary of Theorem 1:

Corrolary *C*

In the stable parallel-queues system with joint arrival pgf

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1+d_1\omega_1(1-z_1c_1(z_1))+d_2\omega_2(1-z_2c_2(z_2))} \ , \tag{90}$$

where

$$\begin{aligned} n(z_1,z_2) \triangleq\ & n_{11}z_1z_2+n_{12}z_1(1-\omega_2+\omega_2 z_2^2 c_2(z_2))+n_{21}z_2(1-\omega_1+\omega_1 z_1^2 c_1(z_1)) \\ & +n_{22}(1-\theta_1+\theta_1 z_1^2 c_1(z_1))(1-\theta_2+\theta_2 z_2^2 c_2(z_2)) \ , \end{aligned} \tag{91}$$

with $c_1(z_1)$ and $c_2(z_2)$ arbitrary one-dimensional pgfs, and

$$n_{11},n_{12},n_{21},n_{22} \geq 0 \quad , \quad n_{11}+n_{12}+n_{21}+n_{22}=1 \ ,$$

$$0 \leq \omega_1,\omega_2 \leq 1 \quad , \quad \theta_1 \triangleq \frac{(n_{21}+d_1)\omega_1}{n_{21}+d_1\omega_1} \ , \ \theta_2 \triangleq \frac{(n_{12}+d_2)\omega_2}{n_{12}+d_2\omega_2} \ ,$$

the steady-state joint system-content pgf $U(z_1,z_2)$ is given by

$$U(z_1,z_2) = MV_1(z_1)V_2(z_2)n(z_1,z_2) \ , \tag{92}$$

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where

$$V_1(z_1) \triangleq \frac{z_1 - 1}{n_{21}(z_1-1) + \omega_1(n_{21}+d_1 z_1) - \omega_1(n_{21}+d_1)z_1^2 c_1(z_1)} \;,$$
$$V_2(z_2) \triangleq \frac{z_2 - 1}{n_{12}(z_2-1) + \omega_2(n_{12}+d_2 z_2) - \omega_2(n_{12}+d_2)z_2^2 c_2(z_2)} \;, \tag{93}$$

and

$$M \triangleq \Big(n_{21} + \omega_1 d_1 - \omega_1(n_{21}+d_1)(2+c_1'(1))\Big)\Big(n_{12} + \omega_2 d_2 - \omega_2(n_{12}+d_2)(2+c_2'(1))\Big) \;. \tag{94}$$

6. Special Cases within Subclasses A, B and C

In this section we consider several instances of subclasses A, B and C, whereby specific choices are made for the various parameters and functions appearing in the formulations of Corrolaries A, B and C.

6.1. Special Cases within Subclass A

6.1.1. At Most One Arrival per Slot in Queue 1

Here we choose

$$f_1(z_1) = 1 - \sigma_1 + \sigma_1 z_1 \;,\; f_1'(1) = \sigma_1 \;, \tag{95}$$

which implies that the pgf $A(z_1, z_2)$, given in (57), reduces to

$$A(z_1,z_2) = \big((m_{21}\sigma_1 + m_{11})z_2) + (m_{22}\sigma_1 + m_{12})f_2(z_2)\big)z_1 + (1-\sigma_1)(m_{21}z_2 + m_{22}f_2(z_2) \;,$$

which is clearly linear in z_1, meaning that queue 1 receives at most one arrival per slot. The marginal arrival pgf $A_2(z_2)$ and the mean arrival rate λ_2 follow from (53) and (54) as

$$A_2(z_2) = (m_{11}+m_{21})z_2 + (m_{12}+m_{22})f_2(z_2) \;,\; \lambda_2 = 1 - (m_{12}+m_{22})[1 - f_2'(1)] \;,$$

from which we can deduce that

$$f_2(z_2) = \frac{A_2(z_2) - (m_{11}+m_{21})z_2}{m_{12}+m_{22}} \;,\; 1 - f_2'(1) = \frac{1-\lambda_2}{m_{12}+m_{22}} \;. \tag{96}$$

According to Corrolary A, the pgf $U(z_1, z_2)$ can be obtained from (59) by substitution of (95), i.e.,

$$U(z_1,z_2) = \frac{(1-\sigma_1)(1-f_2'(1))(z_1-1)(z_2-1)A(z_1,z_2)}{\big(z_1 - (1-\sigma_1+\sigma_1 z_1)\big)\big(z_2 - f_2(z_2)\big)} = \frac{(1-f_2'(1))(z_2-1)A(z_1,z_2)}{z_2 - f_2(z_2)} \;.$$

Owing to (96) and (52), this can be rewritten as

$$U(z_1,z_2) = \frac{(1-\lambda_2)(z_2-1)A(z_1,z_2)}{z_2 - A_2(z_2)} \;.$$

This particular result is well-known. We first established it through an alternative, more direct, approach in our earlier short paper [5]. It is interesting that we retrieve it here as a very simple special case of our more general results.

6.1.2. The Case $m_{12} = m_{21} = 0$

Again in our earlier paper [5], we stated (without proof) the following theorem. Later, we also provided a formal proof in [11].

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Theorem 2: *If $V(z_1,z_2)$ denotes the joint system-content pgf in a parallel-queues system with joint arrival pgf $E(z_1,z_2)$, and a new arrival pgf $A(z_1,z_2)$ is defined as*

$$A(z_1,z_2) \triangleq (1-\nu)z_1z_2 + \nu E(z_1,z_2) \ , \quad \textit{where} \quad 0 < \nu \leq 1 \ , \tag{97}$$

then the joint system-content pgf $U(z_1,z_2)$ corresponding with arrival pgf $A(z_1,z_2)$ is given by

$$U(z_1,z_2) = \frac{V(z_1,z_2)A(z_1,z_2)}{E(z_1,z_2)} \ .$$

Specifically, if the arrivals of both types are mutually independent in the original system, i.e., if $E(z_1,z_2)$ has a product form, $E(z_1,z_2) = E_1(z_1)E_2(z_2)$, then $V(z_1,z_2)$ has a product form too, i.e., $V(z_1,z_2) = V(z_1,1)V(1,z_2)$, with, similar to (12),

$$V(z_1,1) = \frac{(1-E_1'(1))(z_1-1)E_1(z_1)}{z_1-E_1(z_1)} \ , \ V(1,z_2) = \frac{(1-E_2'(1))(z_2-1)E_2(z_2)}{z_2-E_2(z_2)} \ , \tag{98}$$

and (99) reduces to

$$U(z_1,z_2) = \frac{V(z_1,1)V(1,z_2)A(z_1,z_2)}{E_1(z_1)E_2(z_2)} = \frac{(1-E_1'(1))(1-E_2'(1))(z_1-1)(z_2-1)A(z_1,z_2)}{(z_1-E_1(z_1))(z_2-E_2(z_2))} \ . \tag{99}$$

It is remarkable that we can easily retrieve this property as a simple instance of our subclass-A results, if we choose

$$m_{11} = 1-\nu, m_{12} = m_{21} = 0, m_{22} = \nu, f_1(z_1) = E_1(z_1), f_2(z_2) = E_2(z_2) \ .$$

Indeed, equations (57) and (59) from the formulation of Corrolary A are then equivalent with equations (97) and (99) from the formulation of Theorem 2. We do emphasize that Theorem 2 was proven to be valid also if $E(z_1,z_2)$ does not have a product form.

6.1.3. Geometric f-Distributions

Here, we choose *geometric* distributions with respective mean values σ_1 and σ_2 for the pgfs $f_1(z_1)$ and $f_2(z_2)$:

$$f_1(z_1) = \frac{1}{1+\sigma_1-\sigma_1z_1} \ , \ f_2(z_2) = \frac{1}{1+\sigma_2-\sigma_2z_2} \ , \ f_1'(1) = \sigma_1 \ , \ f_2'(1) = \sigma_2 \ . \tag{100}$$

The arrival pgf $A(z_1,z_2)$ then follows from (57) as

$$A(z_1,z_2) = \frac{F(z_1,z_2)}{(1+\sigma_1-\sigma_1z_1)(1+\sigma_2-\sigma_2z_2)} \ ,$$

where $F(z_1,z_2)$ is a quadratic polynomial in both z_1 and z_2, defined as

$$F(z_1,z_2) \triangleq m_{22} + m_{12}z_1(1+\sigma_1-\sigma_1z_1) + m_{21}z_2(1+\sigma_2-\sigma_2z_2) \\ + m_{11}z_1z_2(1+\sigma_1-\sigma_1z_1)(1+\sigma_2-\sigma_2z_2) \ .$$

The marginal mean arrival rates follow from (54):

$$\lambda_1 = 1-(m_{21}+m_{22})(1-\sigma_1) \ , \ \lambda_2 = 1-(m_{12}+m_{22})(1-\sigma_2) \ . \tag{101}$$

The stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are therefore equivalent with $\sigma_1 < 1, \sigma_2 < 1$.

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The system-content pgf $U(z_1, z_2)$ can be obtained by using (100) in (59):

$$U(z_1, z_2) = \frac{(1-\sigma_1)(1-\sigma_2)F(z_1, z_2)}{(1-\sigma_1 z_1)(1-\sigma_2 z_2)} \ , \tag{102}$$

a remarkably simple expression. The zeroes of the denominator are

$$z_1 = \frac{1}{\sigma_1} > 1 \ , \ z_2 = \frac{1}{\sigma_2} > 1 \ .$$

We return to this special case further in the paper.

6.1.4. Binomial f-Distributions

Here, we choose *binomial* distributions of order 2, again with respective mean values σ_1 and σ_2, for the pgfs $f_1(z_1)$ and $f_2(z_2)$:

$$f_1(z_1) = (1 - \frac{\sigma_1}{2} - \frac{\sigma_1}{2} z_1)^2 \ , \ f_2(z_2) = (1 - \frac{\sigma_2}{2} - \frac{\sigma_2}{2} z_2)^2 \ , \ f_1'(1) = \sigma_1 \ , \ f_2'(1) = \sigma_2 \ . \tag{103}$$

The arrival pgf $A(z_1, z_2)$ follows from (57) as

$$A(z_1, z_2) = m_{11} z_1 z_2 + m_{12} z_1 (1 - \frac{\sigma_2}{2} - \frac{\sigma_2}{2} z_2)^2 + m_{21} z_2 (1 - \frac{\sigma_1}{2} - \frac{\sigma_1}{2} z_1)^2$$
$$+ m_{22} (1 - \frac{\sigma_1}{2} - \frac{\sigma_1}{2} z_1)^2 (1 - \frac{\sigma_2}{2} - \frac{\sigma_2}{2} z_2)^2 \ ,$$

and is a quadratic polynomial in both z_1 and z_2. Again, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with $\sigma_1 < 1, \sigma_2 < 1$.

The system-content pgf $U(z_1, z_2)$ can be obtained by using (103) in (59):

$$U(z_1, z_2) = \frac{16(1-\sigma_1)(1-\sigma_2)A(z_1, z_2)}{((2-\sigma_1)^2 - \sigma_1^2 z_1)((2-\sigma_2)^2 - \sigma_2^2 z_2)} \ ,$$

again a rather simple expression. The zeroes of the denominator are

$$z_1 = \left(\frac{2-\sigma_1}{\sigma_1}\right)^2 > 1 \ , \ z_2 = \left(\frac{2-\sigma_2}{\sigma_2}\right)^2 > 1 \ .$$

6.1.5. Batch-2-Geometric f-Distributions

Here, we choose *batch-2-geometric* distributions with respective mean values σ_1 and σ_2 for the pgfs $f_1(z_1)$ and $f_2(z_2)$:

$$f_1(z_1) = \frac{2}{2 + \sigma_1 - \sigma_1 z_1^2} \ , \ f_2(z_2) = \frac{2}{2 + \sigma_2 - \sigma_2 z_2^2} \ , \ f_1'(1) = \sigma_1 \ , \ f_2'(1) = \sigma_2 \ . \tag{104}$$

The terminology *batch-2-geometric* reflects the fact that a random variable with this distribution can only take values equal to geometrically distributed multiples of the batch-size 2. The arrival pgf $A(z_1, z_2)$ follows from (57) as

$$A(z_1, z_2) = \frac{F(z_1, z_2)}{(2 + \sigma_1 - \sigma_1 z_1^2)(2 + \sigma_2 - \sigma_2 z_2^2)} \ ,$$

where $F(z_1, z_2)$ is a cubic polynomial in both z_1 and z_2, defined as

$$F(z_1, z_2) \triangleq 4m_{22} + 2m_{12} z_1 (2 + \sigma_1 - \sigma_1 z_1^2) + 2m_{21} z_2 (2 + \sigma_2 - \sigma_2 z_2^2)$$
$$+ m_{11} z_1 z_2 (2 + \sigma_1 - \sigma_1 z_1^2)(2 + \sigma_2 - \sigma_2 z_2^2) \ .$$

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Once again, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with $\sigma_1 < 1, \sigma_2 < 1$.

The system-content pgf $U(z_1, z_2)$ can be obtained by using (104) in (59):

$$U(z_1,z_2) = \frac{(1-\sigma_1)(1-\sigma_2)F(z_1,z_2)}{\big(2-\sigma_1 z_1(1+z_1)\big)\big(2-\sigma_2 z_2(1+z_2)\big)} \;,$$

a remarkably simple expression. The zeroes of the denominator lie outside the unit disks $\{z_1 : |z_1| \leq 1\}$ and $\{z_2 : |z_2| \leq 1\}$ in the complex z_1-plane and z_2-plane, respectively, and are given by

$$z_1 = \frac{\sqrt{\sigma_1(\sigma_1+8)} - \sigma_1}{2\sigma_1} > 1 \quad \text{and} \quad z_1 = -\frac{\sqrt{\sigma_1(\sigma_1+8)} + \sigma_1}{2\sigma_1} < -1$$

$$z_{2a} = \frac{\sqrt{\sigma_2(\sigma_2+8)} - \sigma_2}{2\sigma_2} > 1 \quad \text{and} \quad z_2 = -\frac{\sqrt{\sigma_2(\sigma_2+8)} + \sigma_2}{2\sigma_2} < -1 \;.$$

6.2. Special Cases within Subclass B

In subclass A, the bivariate arrival pgf $A(z_1, z_2)$ is completely determined by the univariate pgfs $f_1(z_1)$ and $f_2(z_2)$, and the pgfs $g_1(z_1)$ and $g_2(z_2)$ play no role. In subclass B, however, all the defining one-dimensional pgfs contribute to $A(z_1, z_2)$. In order to specifically examine the effect of $g_1(z_1)$ and $g_2(z_2)$, we consider two examples where $f_1(z_1) = f_2(z_2) = 1$, combined with different choices for $g_1(z_1)$ and $g_2(z_2)$.

6.2.1. The Case $f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1, g_2(z_2) = z_2$

Here, we choose

$$f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1, g_2(z_2) = z_2 \;,\; f_1'(1) = f_2'(1) = 0, g_1'(1) = 1, g_2'(1) = 1 \;. \tag{105}$$

The arrival pgf $A(z_1, z_2)$ follows from (67) and (68) as

$$A(z_1,z_2) = \frac{n_{11}z_1z_2 + n_{12}z_1 + n_{21}z_2 + n_{22}(1-\gamma_1+\gamma_1 z_1^2)(1-\gamma_2+\gamma_2 z_2^2)}{1+d_1+d_2-d_1z_1-d_2z_2} \;. \tag{106}$$

In view of (63), the marginal mean arrival rates are

$$\lambda_1 = 1 - \frac{1-2\gamma_1}{1-\gamma_1}\big(n_{21} + (1-\gamma_1)n_{22}\big) \;,\; \lambda_2 = 1 - \frac{1-2\gamma_2}{1-\gamma_2}\big(n_{12} + (1-\gamma_2)n_{22}\big) \;,$$

and, hence, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with $\gamma_1 < 1/2, \gamma_2 < 1/2$.

The joint system-content pgf $U(z_1, z_2)$ can be obtained by using (105) in (68), (69) and (70), which results in

$$U(z_1,z_2) = \frac{(1-2\gamma_1)(1-2\gamma_2)\big(n_{11}z_1z_2 + n_{12}z_1 + n_{21}z_2 + n_{22}[1-\gamma_1+\gamma_1 z_1^2][1-\gamma_2+\gamma_2 z_2^2]\big)}{(1-\gamma_1-\gamma_1 z_1)(1-\gamma_2-\gamma_2 z_2)} \;. \tag{107}$$

The zeroes of the denominator are given by

$$z_1 = \frac{1-\gamma_1}{\gamma_1} > 1 \;,\; z_2 = \frac{1-\gamma_2}{\gamma_2} > 1 \;.$$

Remark

It is worth mentioning that a special instance of this case was treated in our recent paper [6]. There, we considered a *parallel-queues* system, whereby the total number of arrivals per slot (of both customer types together) has a *shifted geometric* distribution with pgf $C(z)$ and mean value $q \geq 1$, i.e.,

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$C(z) = z/[q-(q-1)z]$, and new arrivals are routed independently and probabilistically to queue 1 or 2 with probabilities p and $1-p$ respectively, implying that the joint arrival pgf $A(z_1,z_2)$ is given by

$$A(z_1,z_2) = C(pz_1 + (1-p)z_2) = \frac{pz_1 + (1-p)z_2}{q-(q-1)pz_1 - (q-1)(1-p)z_2} \ . \tag{108}$$

In appendix A of [6], we formally proved that the joint system-content pgf $U(z_1,z_2)$ for this system is

$$U(z_1,z_2) = \frac{(\kappa_1-1)(\tau_1-1)\big(pz_1+(1-p)z_2\big)}{(\kappa_1-z_1)(\tau_1-z_2)} \ , \text{ with } \kappa_1 \triangleq \frac{1-p}{p(q-1)} \ , \tau_1 \triangleq \frac{p}{(1-p)(q-1)} \ . \tag{109}$$

The proof in [6] was a (rather complicated) *constructive* proof, whereby we explicitly solved the functional Equation (9),

$$K(z_1,z_2)U(z_1,z_2) = A(z_1,z_2)L(z_1,z_2) \ ,$$

by expressing that the unknown function $L(z_1,z_2)$ should vanish for all (z_1,z_2) in the area of convergence of $U(z_1,z_2)$ for which the kernel $K(z_1,z_2)$ vanishes. This allowed us to determine the boundary functions $U(z_1,0)$ and $U(0,z_2)$, and, from this, the function $L(z_1,z_2)$, and, eventually, the pgf $U(z_1,z_2)$, as given in (109).

The function $A(z_1,z_2)$ in (108) is clearly of the form (106) considered in the current subsection, provided we choose

$$n_{11} = 0 \ , \ n_{12} = p \ , \ n_{21} = 1-p \ , \ n_{22} = 0 \ , \ d_1 = p(q-1) \ , \ d_2 = (1-p)(q-1) \ . \tag{110}$$

We now show that the solution (109) can be easily retrieved from the results in the current subsection.

Proof. Using (110) in the definitions (61) of our current parameters γ_1 and γ_2 leads to

$$\gamma_1 \triangleq \frac{d_1}{d_1+n_{21}} = \frac{p(q-1)}{p(q-1)+1-p} \ , \ \gamma_2 \triangleq \frac{d_2}{d_2+n_{12}} = \frac{(1-p)(q-1)}{(1-p)(q-1)+p} \ ,$$

and, from this,

$$\frac{1-\gamma_1}{\gamma_1} = \frac{1-p}{p(q-1)} = \kappa_1 \ , \ \frac{1-\gamma_2}{\gamma_2} = \frac{p}{(1-p)(q-1)} = \tau_1 \ , \tag{111}$$

and

$$\frac{1-2\gamma_1}{\gamma_1} = \frac{1-p}{p(q-1)} = \kappa_1 - 1 \ , \ \frac{1-2\gamma_2}{\gamma_2} = \frac{p}{(1-p)(q-1)} = \tau_1 - 1 \ . \tag{112}$$

Inserting (110) in (107) yields

$$U(z_1,z_2) = \frac{(1-2\gamma_1)(1-2\gamma_2)\big(pz_1+(1-p)z_2\big)}{(1-\gamma_1-\gamma_1 z_1)(1-\gamma_2-\gamma_2 z_2)} \ . \tag{113}$$

Division of both the numerator and the denominator of the above expression by $\gamma_1\gamma_2$ and substitution of (111) and (112) then clearly shows that (113) is identical to (109). □

We have thus been able, once again, to recover a specific existing result as a particular case of the results of the current paper.

6.2.2. The Case $f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1^2, g_2(z_2) = z_2^2$

Here, we choose

$$f_1(z_1) = f_2(z_2) = 1, g_1(z_1) = z_1^2, g_2(z_2) = z_2^2 \ , \ f_1'(1) = f_2'(1) = 0, g_1'(1) = 2, g_2'(1) = 2 \ . \tag{114}$$

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The arrival pgf $A(z_1,z_2)$ follows from (67) and (68) as

$$A(z_1,z_2) = \frac{n_{11}z_1z_2 + n_{12}z_1 + n_{21}z_2 + n_{22}(1-\gamma_1+\gamma_1 z_1^3)(1-\gamma_2+\gamma_2 z_2^3)}{1+d_1+d_2-d_1z_1^2-d_2z_2^2} \ .$$

The marginal mean arrival rates can be deduced from (63):

$$\lambda_1 = 1 - \frac{1-3\gamma_1}{1-\gamma_1}\big(n_{21}+(1-\gamma_1)n_{22}\big) \ , \ \lambda_2 = 1 - \frac{1-3\gamma_2}{1-\gamma_2}\big(n_{12}+(1-\gamma_2)n_{22}\big) \ ,$$

and, hence, the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with $\gamma_1 < 1/3, \gamma_2 < 1/3$.

The joint system-content pgf $U(z_1,z_2)$ can be obtained by using (114) in (68), (69) and (70), which results in

$$U(z_1,z_2) = \frac{(1-3\gamma_1)(1-3\gamma_2)\big(n_{11}z_1z_2+n_{12}z_1+n_{21}z_2+n_{22}[1-\gamma_1+\gamma_1z_1^3][1-\gamma_2+\gamma_2z_2^3]\big)}{\big(1-\gamma_1-\gamma_1z_1(1+z_1)\big)\big(1-\gamma_2-\gamma_2z_2(1+z_2)\big)} \ . \tag{115}$$

Again, it is easy to show that the zeroes of the denominator lie outside the unit disks $\{z_1 : |z_1| \le 1\}$ and $\{z_2 : |z_2| \le 1\}$ in the complex z_1-plane and z_2-plane, respectively. We return to this special case in more detail further in the paper.

6.3. Special Cases within Subclass C

In order to simplify the expressions in this subsection, we first make the following assumptions:

$$n_{11} = 0 \ , \ n_{12} = n_{21} = d_1 = d_2 = d \ , \ n_{22} = 1-2d \ , \ \omega_1 = \omega_2 = \frac{1}{4} \ . \tag{116}$$

According to (90) and (91), the arrival pgf is given by

$$A(z_1,z_2) = \frac{n(z_1,z_2)}{1+\frac{d}{4}\big(2-z_1c_1(z_1)-z_2c_2(z_2)\big)} \ ,$$

where

$$n(z_1,z_2) = \frac{d}{4}\Big(z_1\big(3+z_2^2c_2(z_2)\big)+z_2\big(3+z_1^2c_1(z_1)\big)\Big) + \frac{1-2d}{25}\big(3+z_2^2c_2(z_2)\big)\big(3+z_1^2c_1(z_1)\big) \ . \tag{117}$$

The marginal mean arrival rates are

$$\lambda_1 = 1 - \frac{4-3d}{20}\big(1-2c_1'(1)\big) \ , \ \lambda_2 = 1 - \frac{4-3d}{20}\big(1-2c_2'(1)\big) \ ,$$

which implies that the stability conditions $\lambda_1 < 1, \lambda_2 < 1$ are equivalent with $c_1'(1) < 1/2, c_2'(1) < 1/2$. From (93) and (94), we get

$$V_1(z_1) = \frac{4(z_1-1)}{d\big(5z_1-2z_1^2c_1(z_1)-3\big)} \ , \ V_2(z_2) = \frac{4(z_2-1)}{d\big(5z_2-2z_2^2c_2(z_2)-3\big)} \ ,$$

and

$$M = \frac{d^2}{16}\big(1-2c_1'(1)\big)\big(1-2c_2'(1)\big) \ .$$

It then follows from (92) that the system-content pgf is given by

$$U(z_1,z_2) = \frac{\big(1-2c_1'(1)\big)\big(1-2c_2'(1)\big)(z_1-1)(z_2-1)n(z_1,z_2)}{\big(3-5z_1+2z_1^2c_1(z_1)\big)\big(3-5z_2+2z_2^2c_2(z_2)\big)} \ ,$$

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We now make a number of different choices for the pgfs $c_1(z_1)$ and $c_2(z_2)$.

6.3.1. Bernoulli c-Distributions

Here, we choose Bernoulli distributions with parameters σ_1 and σ_2 for the pgfs $c_1(z_1)$ and $c_2(z2)$:

$$c_1(z_1) = 1 - \sigma_1 + \sigma_1 z_1 \ , \ c_2(z_2) = 1 - \sigma_2 + \sigma_2 z_2 \ , \ c_1'(1) = \sigma_1 \ , \ c_2'(1) = \sigma_2 \ .$$

The pgf $U(z_1, z_2)$ then reduces to

$$U(z_1, z_2) = \frac{(1-2\sigma_1)(1-2\sigma_2)n(z_1, z_2)}{(3 - 2z_1 - 2\sigma_1 z_1^2)(3 - 2z_2 - 2\sigma_2 z_2^2)} \ . \tag{118}$$

We come back to this special case further in this paper.

6.3.2. Geometric c-Distributions

Here, we choose geometric distributions with respective mean values σ_1 and σ_2 for the pgfs $c_1(z_1)$ and $c_2(z2)$:

$$c_1(z_1) = \frac{1}{1+\sigma_1 - \sigma_1 z_1} \ , \ c_2(z_2) = \frac{1}{1+\sigma_2 - \sigma_2 z_2} \ , \ c_1'(1) = \sigma_1 \ , \ c_2'(1) = \sigma_2 \ . \tag{119}$$

The pgf $U(z_1, z_2)$ then reduces to

$$U(z_1, z_2) = \frac{(1-2\sigma_1)(1-2\sigma_2)(1+\sigma_1-\sigma_1 z_1)(1+\sigma_2-\sigma_2 z_2)n(z_1, z_2)}{(3(1+\sigma_1) - (2+5\sigma_1)z_1)(3(1+\sigma_2) - (2+5\sigma_2)z_2)} \ .$$

The zeroes of the denominator are given by

$$z_1 = \frac{3(1+\sigma_1)}{2+5\sigma_1} > 1 \ , \ z_2 = \frac{3(1+\sigma_2)}{2+5\sigma_2} > 1 \ .$$

6.3.3. Negative Binomial c-Distributions

Here, we choose negative binomial distributions of order two for the pgfs $c_1(z_1)$ and $c_2(z_2)$:

$$c_1(z_1) = \frac{4}{(2+\sigma_1 - \sigma_1 z_1)^2} \ , \ c_2(z_2) = \frac{4}{(2+\sigma_2 - \sigma_2 z_2)^2} \ , \ c_1'(1) = \sigma_1 \ , \ c_2'(1) = \sigma_2 \ . \tag{120}$$

In this case, the pgf $U(z_1, z_2)$ is given by

$$U(z_1, z_2) = W_1(z_1) W_2(z_2) n(z_1, z_2) \ ,$$

where

$$W_1(z_1) \triangleq \frac{(1-2\sigma_1)(2+\sigma_1-\sigma_1 z_1)^2}{(3(2+\sigma_1)^2 - 4(2+\sigma_1)(1+2\sigma_1)z_1 + 5\sigma_1^2 z_1^2)} \ ,$$

$$W_2(z_2) \triangleq \frac{(1-2\sigma_2)(2+\sigma_2-\sigma_2 z_2)^2}{(3(2+\sigma_2)^2 - 4(2+\sigma_2)(1+2\sigma_2)z_2 + 5\sigma_2^2 z_2^2)} \ .$$

Again, the zeroes of the denominator are the solutions of quadratic equations and can be computed explicitly; also, it is not difficult to show that they lie outside the unit disks in the z_1-plane and the z_2-plane; their exact expressions are omitted, as they are of no particular importance at this stage.

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7. Inverting the Joint pgf $U(z_1, z_2)$

In this section, we focus on the derivation of the steady-state joint *probability mass function* (pmf) $u(m,n)$ of the system contents in queues 1 and 2, which is defined as

$$u(m,n) \triangleq \lim_{k\to\infty} \text{Prob}[u_{1,k} = m, u_{2,k} = n] \ ,$$

and is related to the joint pgf $U(z_1, z_2)$ by the equation

$$U(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} u(m,n) z_1^m z_2^n \ .$$

In an earlier paper [11], we proved (with slightly different notations) the following useful theorem to determine $u(m,n)$ from $U(z_1,z_2)$ for "interior states" (m,n) in the state space.

Theorem 3

If the joint pgf $U(z_1,z_2)$ is a rational *function of z_1 and z_2 of the form*

$$U(z_1,z_2) = \frac{B(z_1,z_2)}{r_1(z_1)r_2(z_2)} = \frac{B(z_1,z_2)}{\prod_{i=1}^{L_1}(z_1-\kappa_i)\prod_{j=1}^{L_2}(z_2-\tau_j)} \ , \tag{121}$$

where the numerator $B(z_1,z_2)$ is a bivariate polynomial of degree K_1 in z_1 and K_2 in z_2, and the denominator is a product of two univariate functions $r_1(z_1)$ and $r_2(z_2)$, having only zeroes of multiplicity 1, and the numerator and the denominator are mutually prime, then threshold values m_0 and n_0 can be defined as $m_0 \triangleq max(0, K_1 - L_1 + 1), n_0 \triangleq max(0, K_2 - L_2 + 1)$, such that for $m \geq m_0, n \geq n_0$, the pmf $u(m,n)$ is given by a finite linear combination of bivariate geometric terms, i.e.,

$$u(m,n) = \sum_{i=1}^{L_1}\sum_{j=1}^{L_2} \mu_{i,j}\left(\frac{1}{\kappa_i}\right)^m\left(\frac{1}{\tau_j}\right)^n \ , \ m\geq m_0, n\geq n_0 \ , \tag{122}$$

where

$$\mu_{i,j} \triangleq \frac{B(\kappa_i,\tau_j)}{\kappa_i\tau_j r_1'(\kappa_i)r_2'(\tau_j)} \ . \tag{123}$$

7.1. Some Comments

In all the examples that we have considered in this paper, we have chosen rational functions for the constituting one-dimensional pgfs $f_1(z_1)$, $f_2(z_2)$, $g_1(z_1)$, $g_2(z_2)$, $h_1(z_1)$, $h_2(z_2)$ of the joint arrival pgf $A(z_1,z_2)$, defined in (21). This implies that the pgf $U(z_1,z_2)$, given in (23) by

$$U(z_1,z_2) = \frac{U(0,1)U(1,0)(z_1-1)(z_2-1)n(z_1,z_2)}{k_1(z_1)k_2(z_2)} \ , \tag{124}$$

is a rational bivariate function whose denominator is a product of two univariate functions, and can therefore be expressed in the form required to apply Theorem 3.

By definition, the quantities κ_i and τ_j, occurring in (122), are the zeroes of $r_1(z_1)$ (or $k_1(z_1)$) and $r_2(z_2)$ (or $k_2(z_2)$). According to (122), the geometric decay rates of the system-content distribution are the inverse values of these zeroes, i.e., the ith decay rate for queue 1 is equal to $1/\kappa_i$, and the jth decay rate for queue 2 is given by $1/\tau_j$. Each bivariate geometric term in $u(m,n)$ thus corresponds to a couple (κ_i, τ_j) of zeroes of $r_1(z_1)$ and $r_2(z_2)$, but the opposite is not necessarily true, since, for some i and j, it may happen that the coefficient $\mu_{i,j}$ in Equation (122) is zero. According to (123), this situation occurs if $B(\kappa_i,\tau_j) = 0$. If this is the case for one or more couples (κ_i,τ_j), the number of nonzero bivariate geometric terms in $u(m,n)$ is lower than the product $L1 \times L2$.

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7.2. Specific Examples

In this subsection, we apply Theorem 3 in a number of examples of arrival pgfs $A(z_1, z_2)$ belonging to subclasses A, B and C, as defined before.

7.2.1. An Example within Subclass A

In this example, we revisit the model of subsection 6.1.3. The system-content pgf $U(z_1, z_2)$ is given in (102). The parameters and functions, appearing in the formulation of Theorem 3 are $K_1 = K_2 = 2, L_1 = L_2 = 1$, and

$$B(z_1, z_2) = (1-\sigma_1)(1-\sigma_2)F(z_1, z_2) \;\; , \;\; r_1(z_1) = 1 - \sigma_1 z_1 \;\; , \;\; r_2(z_2) = 1 - \sigma_2 z_2 \;\; ,$$

with

$$F(z_1, z_2) \triangleq m_{22} + m_{12} z_1 (1 + \sigma_1 - \sigma_1 z_1) + m_{21} z_2 (1 + \sigma_2 - \sigma_2 z_2) \\ + m_{11} z_1 z_2 (1 + \sigma_1 - \sigma_1 z_1)(1 + \sigma_2 - \sigma_2 z_2) \;\; .$$

The zeroes of $r_1(z_1)$ and $r_2(z_2)$ are

$$\kappa_1 = \frac{1}{\sigma_1} > 1 \;\; , \;\; \tau_1 = \frac{1}{\sigma_2} > 1 \;\; .$$

The coefficient $\mu_{1,1}$ can be computed from (123) as

$$\mu_{1,1} = (1-\sigma_1)(1-\sigma_2) \;\; .$$

Finally, the pmf $u(m, n)$ for interior states (m, n) follows from (122) as

$$u(m, n) = (1-\sigma_1)(1-\sigma_2)\left(\frac{1}{\sigma_1}\right)^m \left(\frac{1}{\sigma_2}\right)^n \;\; , \;\; m \geq 2, n \geq 2 \;\; .$$

7.2.2. An Example within Subclass B

Here, we consider a symmetric instance of the model of subsection 6.2.2, with the following specific parameter choices:

$$n_{12} = n_{21} = n_0, d_1 = d_2 = d, \gamma_1 = \gamma_2 = \gamma \;\; .$$

The joint system-content pgf $U(z_1, z_2)$ can be obtained from (115):

$$U(z_1, z_2) = \frac{(1-3\gamma)^2 \left(n_{11} z_1 z_2 + n_0 z_1 + n_0 z_2 + n_{22}[1 - \gamma + \gamma z_1^3][1 - \gamma + \gamma z_2^3]\right)}{\left(1 - \gamma - \gamma z_1(1+z_1)\right)\left(1 - \gamma - \gamma z_2(1+z_2)\right)} \;\; .$$

We can apply Theorem 3 with $K_1 = K_2 = 3, L_1 = L_2 = 2$,

$$B(z_1, z_2) \triangleq (1-3\gamma)^2 \left(n_{11} z_1 z_2 + n_0 z_1 + n_0 z_2 + n_{22}[1 - \gamma + \gamma z_1^3][1 - \gamma + \gamma z_2^3]\right) \;\; ,$$

and

$$r_1(z_1) \triangleq 1 - \gamma - \gamma z_1(1+z_1) \;\; , \;\; r_2(z_2) \triangleq 1 - \gamma - \gamma z_2(1+z_2) \;\; .$$

The zeroes of $r_1(z_1)$ and $r_2(z_2)$ lie outside the unit disks $\{z_1 : |z_1| \leq 1\}$ and $\{z_2 : |z_2| \leq 1\}$ in the complex z_1-plane and z_2-plane, respectively, and are given by

$$\kappa_1 = \tau_1 = \frac{\sqrt{\gamma(4-3\gamma)} - \gamma}{2\gamma} > 1 \;\; , \;\; \kappa_2 = \tau_2 = -\frac{\sqrt{\gamma(4-3\gamma)} + \gamma}{2\gamma} < -1 \;\; .$$

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The coefficients $\mu_{i,j}$ can be computed from (123):

$$\mu_{1,1} = \mu\frac{1-\gamma+(1-4n_0)(1+\sqrt{\gamma(4-3\gamma)})}{2-\gamma-\sqrt{\gamma(4-3\gamma)}} \;,\; \mu_{2,2} = \mu\frac{1-\gamma+(1-4n_0)(1-\sqrt{\gamma(4-3\gamma)})}{2-\gamma+\sqrt{\gamma(4-3\gamma)}}$$
$$\mu_{1,2} = \mu_{2,1} = \mu\frac{1-\gamma-n_0(2-3\gamma)}{1-\gamma} \;,\quad \text{where} \quad \mu \triangleq \frac{(1-3\gamma)^2}{\gamma(4-3\gamma)} \;. \tag{125}$$

In general, all these coefficients are nonzero, and the linear combination in (122) contains four terms:

$$u(m,n) = \mu_{1,1}(\frac{1}{\kappa_1})^{m+n} + \mu_{12}\Big((\frac{1}{\kappa_1})^m(\frac{1}{\kappa_2})^n + (\frac{1}{\kappa_2})^m(\frac{1}{\kappa_1})^n\Big) + \mu_{2,2}(\frac{1}{\kappa_2})^{m+n} \;,\; m \geq 2, n \geq 2 \;.$$

Careful study shows that it is impossible to choose the parameter n_0, appearing in (125), in such a way that the coefficients $\mu_{1,1}$, $\mu_{1,2}$ or $\mu_{2,1}$ are zero, but there does exist a value of n_0 such that $\mu_{2,2}$ vanishes; the required n_0-value is

$$n_0 = \frac{2-\gamma+\sqrt{\gamma(4-3\gamma)}}{4(1+\sqrt{\gamma(4-3\gamma)})} > 0 \;. \tag{126}$$

This is an acceptable value since it implies that

$$n_{11} + n_{22} = 1 - n_{12} - n_{21} = 1 - 2n_0 = \frac{\gamma+\sqrt{\gamma(4-3\gamma)}}{2(1+\sqrt{\gamma(4-3\gamma)})} > 0$$

So, in case n_0 is chosen in accordance with (126), the linear combination in (122) contains only three bivariate geometric terms:

$$u(m,n) = \mu_{1,1}(\frac{1}{\kappa_1})^{m+n} + \mu_{12}\Big((\frac{1}{\kappa_1})^m(\frac{1}{\kappa_2})^n + (\frac{1}{\kappa_2})^m(\frac{1}{\kappa_1})^n\Big) \;,\; m \geq 2, n \geq 2 \;,$$

and does not contain a term with two negative decay rates.

7.2.3. An Example within Subclass C

We now go back to the model in subsection 6.3.1. Here, the system-content pgf $U(z_1, z_2)$ is given by (118). We can apply Theorem 3 provided we choose $K_1 = K_2 = 3, L_1 = L_2 = 2$,

$$B(z_1,z_2) = (1-2\sigma_1)(1-2\sigma_2)n(z_1,z_2) \;,\; r_1(z_1) = (3-2z_1-2\sigma_1 z_1^2) \;,\; r_2(z_2) = (3-2z_2-2\sigma_1 z_2^2) \;,$$

where, owing to (117), $n(z_1, z_2)$ is given by

$$n(z_1,z_2) = \frac{d}{4}\Big(z_1\big(3+z_2^2(1-\sigma_2+\sigma_2 z_2)\big) + z_2\big(3+z_1^2(1-\sigma_1+\sigma_1 z_1)\big)\Big)$$
$$+ \frac{1-2d}{25}\big(3+z_1^2(1-\sigma_1+\sigma_1 z_1)\big)\big(3+z_2^2(1-\sigma_2+\sigma_2 z_2)\big) \;. \tag{127}$$

The zeroes of $r_1(z_1)$ and $r_2(z_2)$ lie outside the unit disks $\{z_1 : |z_1| \leq 1\}$ and $\{z_2 : |z_2| \leq 1\}$ in the complex z_1-plane and z_2-plane, respectively, and are given by

$$\kappa_1 = \frac{\sqrt{1+6\sigma_1}-1}{2\sigma_1} > 1 \;,\; \kappa_2 = -\frac{\sqrt{1+6\sigma_1}+1}{2\sigma_1} < -1$$
$$\tau_1 = \frac{\sqrt{1+6\sigma_2}-1}{2\sigma_2} > 1 \;,\; \tau_2 = -\frac{\sqrt{1+6\sigma_2}+1}{2\sigma_2} < -1 \;.$$

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The coefficients $\mu_{i,j}$ can be computed from (123):

$$\mu_{1,1} = \mu\frac{4 + d(\sigma_1\kappa_1 + \sigma_2\tau_1 - 1)}{(1 + 2\sigma_1\kappa_1)(1 + 2\sigma_2\tau_1)} \ , \ \mu_{1,2} = \mu\frac{4 + d(\sigma_1\kappa_1 + \sigma_2\tau_2 - 1)}{(1 + 2\sigma_1\kappa_1)(1 + 2\sigma_2\tau_2)} \ ,$$

$$\mu_{2,1} = \mu\frac{4 + d(\sigma_1\kappa_2 + \sigma_2\tau_1 - 1)}{(1 + 2\sigma_1\kappa_2)(1 + 2\sigma_2\tau_1)} \ , \ \mu_{2,2} = \mu\frac{4 + d(\sigma_1\kappa_2 + \sigma_2\tau_2 - 1)}{(1 + 2\sigma_1\kappa_2)(1 + 2\sigma_2\tau_2)} \ ,$$

where we have defined μ as

$$\mu \triangleq \frac{(1 - 2\sigma_1)(1 - 2\sigma_2)}{16} \ .$$

Again, in general, all these coefficients are nonzero, and the linear combination in (122) contains four terms:

$$u(m,n) = \mu_{1,1}\left(\frac{1}{\kappa_1}\right)^m\left(\frac{1}{\tau_1}\right)^n + \mu_{12}\left(\frac{1}{\kappa_1}\right)^m\left(\frac{1}{\tau_2}\right)^n + \left(\frac{1}{\kappa_2}\right)^m\left(\frac{1}{\tau_1}\right)^n + \mu_{2,2}\left(\frac{1}{\kappa_2}\right)^m\left(\frac{1}{\tau_2}\right)^n \ , \ m \geq 2, n \geq 2 \ . \tag{128}$$

Let d_{ij} denote the value of d that makes $\mu_{i,j}$ zero, then we can easily compute the following values:

$$d_{11} = \frac{8}{4 - \sqrt{1 + 6\sigma_1} - \sqrt{1 + 6\sigma_2}} \ , \ d_{12} = \frac{8}{4 - \sqrt{1 + 6\sigma_1} + \sqrt{1 + 6\sigma_2}} \ ,$$

$$d_{21} = \frac{8}{4 + \sqrt{1 + 6\sigma_1} - \sqrt{1 + 6\sigma_2}} \ , \ d_{22} = \frac{8}{4 + \sqrt{1 + 6\sigma_1} + \sqrt{1 + 6\sigma_2}} \ .$$

Taking into account the stability conditions $\sigma_1 < 1/2, \sigma_2 < 1/2$, as we have shown in subsection 6.3.1, it is readily seen that all these d-values are positive, as required, but none of them is lower than $1/2$, which is also necessary, because in this model, according to (116), $n_{22} = 1 - 2d$ and needs to be positive. We conclude that, in this particular case, the pmf $u(m,n)$ always contains exactly four bivariate geometric terms, as shown in (128).

8. Concluding REMARKS

This paper has considered the steady-state queueing analysis of a system of two *parallel* discrete-time single-server queues with mutually interdependent arrivals, characterized by the joint arrival pgf $A(z_1,z_2)$. We have identified a very broad, multi-parameter, generic, class of arrival pgfs $A(z_1,z_2)$ for which we were able to determine explicit analytic solutions for the joint system-content pgf $U(z_1,z_2)$. We think this is the main virtue of the paper. It is also interesting to observe that our results encompass most of the previously known results for this kind of system, which is known to be hard to analyze.

Although the class of arrival pgfs $A(z_1,z_2)$ examined in this paper is very broad, it still has its limitations, which are mainly due to the shape of the arrival pgf, i.e., Equation (14),

$$A(z_1,z_2) = \frac{n_{11}z_1z_2 + n_{12}z_1f_2(z_2) + n_{21}z_2f_1(z_1) + n_{22}h_1(z_1)h_2(z_2)}{1 + d_1 + d_2 - d_1g_1(z_1) - d_2g_2(z_2)} \ ,$$

and the requirement that the pgfs appearing in the above expression should be related as stated in equations (24) or (33), which can be rewritten as

$$\beta_1n_{21}(z_1 - f_1(z_1)) + \beta_1d_1z_1(1 - g_1(z_1)) + n_{21}(z_1 - h_1(z_1)) = 0 \ ,$$

$$\beta_2n_{12}(z_2 - f_2(z_2)) + \beta_2d_2z_2(1 - g_2(z_2)) + n_{12}(z_2 - h_2(z_2)) = 0 \ .$$

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Since the parameters β_1 and β_2 need to be strictly positive – we need this in the proof of Lemma 1 – we can thus not have a constant numerator for $A(z_1, z_2)$ without the requirement that the denominator be also constant. Hence, a seemingly simple arrival pgf like

$$A(z_1, z_2) = \frac{1}{1 + d_1 + d_2 - d_1 z_1 - d_2 z_2} \tag{129}$$

is not a special case of our model. So far, we have never seen a solution for the *"global geometric"* arrival pgf in (129), and the current paper also does not provide one.

Future work could go in several directions. We may try to further extend the class of arrival pgfs which lead to explicit solutions for the *parallel-queues* system, dealt with in this paper, but we may also consider other types of coupled queues, such as the (other) ones mentioned in the introduction section of this paper.

References

1. Jaffe, S. The equilibrium distribution for a clocked buffered switch. *Probability in the Engineering and Informational Sciences* **1992**, *6*, 425–438.
2. Boxma, O.; van Houtum, G. The compensation approach applied to a 2x2 switch. *Probability in the Engineering and Informational Sciences* **1993**, *7*, 471–493.
3. Cohen, J. On the determination of the stationary distribution of a symmetric clocked buffered switch. In *Proc. ITC-15, Teletraffic Contributions for the Information Age*; Ramaswami, V.; Wirth, P., Eds.; Elsevier, 1997; Vol. 2a, pp. 297–307.
4. Adan, I.; Boxma, O.; Resing, J. Queueing models with multiple waiting lines. *Queueing Systems* **2001**, *37*, 65–98.
5. Bruneel, H. Some thoughts on the analysis of coupled queues. *Queueing Systems* **2022**, *100*, 185–187.
6. Bruneel, H.; Devos, A. Asymptotic behavior of a system of two coupled queues when the content of one queue is very high. *Queueing Systems* **2023**, *105*, 189–232.
7. Cohen, J. Analysis of the asymmetrical shortest two-server queueing model. *Journal of Applied Mathematics and Stochastic Analysis* **1998**, *11*, 115–162.
8. Dimitriou, I. Analysis of the symmetric join the shortest orbit queue. *Operations Research Letters* **2021**, *49*, 23–29.
9. Adan, I.; Boxma, O.; Kapodistria, S.; Kulkarni, V. The shorter queue polling model. *Annals of Operations Research* **2016**, *241*, 167–200.
10. Wright, P.E. Two parallel processors with coupled inputs. *Advances in Applied Probability* **1992**, *24*, 986–1007.
11. Bruneel, H.; Devos, A. Coupled queues whose stationary joint content distribution is a finite sum of bivariate geometric terms. *submitted* **2024**.
12. Takagi, H. Queuing analysis of polling models. *ACM Computing Surveys (CSUR)* **1988**, *20*, 5–28.
13. Levy, H.; Sidi, M. Polling systems: applications, modeling, and optimization. *IEEE Transactions on Communications* **1990**, *38*, 1750–1760.
14. Vishnevskii, V.; Semenova, O. Mathematical methods to study the polling systems. *Automation and Remote Control* **2006**, *67*, 173–220.
15. de Haan, R.; Boucherie, R.; van Ommeren, J. A polling model with an autonomous server. *Queueing Systems* **2009**, *62*, 279–308.
16. Al Hanbali, A.; de Haan, R.; Boucherie, R.; van Ommeren, J. Time-limited polling systems with batch arrivals and phase-type service times. *Annals of Operations Research* **2012**, *198*, 57–82.
17. Saxena, M.; Boxma, O.; Kapodistria, S.; Nunez Queija, R. Two queues with random time-limited polling. *Probability and Mathematical Statistics* **2017**, *37*, 257–289.
18. Borst, S.; Boxma, O. Polling: past, present, and perspective. *Top* **2018**, *26*, 335–369.
19. Vishnevsky, V.; Semenova, O. Polling systems and their application to telecommunication networks. *Mathematics* **2021**, *9*, 117.
20. Eisenberg, M. Two queues with alternating service. *SIAM Journal on Applied Mathematics* **1979**, *36*, 287–303.

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21. Coffman, E.; Fayolle, G.; Mitrani, I. Two queues with alternating service periods. Proceedings of the 12th IFIP WG 7.3 International Symposium on Computer Performance Modelling, Measurement and Evaluation, 1987, pp. 227–239.
22. Feng, W.; Kowada, M.; Adachi, K. A two-queue model with Bernoulli service schedule and switching times. *Queueing Systems* **1998**, *30*, 405–434.
23. Devos, A.; Fiems, D.; Walraevens, J.; Bruneel, H. An approximate analysis of a Bernoulli alternating service model. International Conference on Queueing Theory and Network Applications. Springer, 2019, pp. 314–329.
24. Devos, A.; Walraevens, J.; Fiems, D.; Bruneel, H. Approximations for the performance evaluation of a discrete-time two-class queue with an alternating service discipline. *Annals of Operations Research* **2020**, pp. 1–27.
25. Devos, A.; Walraevens, J.; Fiems, D.; Bruneel, H. Heavy-Traffic Comparison of a Discrete-Time Generalized Processor Sharing Queue and a Pure Randomly Alternating Service Queue. *Mathematics* **2021**, *9*, 2723.
26. Devos, A. Analysis of a two-class queueing model with randomly alternating service. PhD thesis, Ghent University, 2022.
27. Devos, A.; Walraevens, J.; Fiems, D.; Bruneel, H. Analysis of a discrete-time two-class randomly alternating service model with Bernoulli arrivals. *Queueing Systems* **2020**, *96*, 133–152.
28. Devos, A.; De Muynck, M.; Bruneel, H.; Walraevens, J. A product-form solution for a two-class Geo/Geo/D/1 queue with random routing and randomly alternating service. EAI International Conference on Performance Evaluation Methodologies and Tools. Springer, 2022, pp. 81–95.
29. Walraevens, J.; Steyaert, B.; Bruneel, H. Delay characteristics in discrete-time GI-G-1 queues with non-preemptive priority queueing discipline. *Performance Evaluation* **2002**, *50*, 53–75.
30. Walraevens, J.; .; Steyaert, B.; Bruneel, H. Performance analysis of a single-server ATM queue with a priority scheduling. *Computers & Operations Research* **2003**, *30*, 1807–1829.
31. Walraevens, J.; Steyaert, B.; Bruneel, H. Performance analysis of a GI-Geo-1 buffer with a preemptive resume priority scheduling discipline. *European Journal of Operational Research* **2004**, *157*, 130–151.
32. Walraevens, J.; Steyaert, B.; Bruneel, H. A packet switch with a priority scheduling discipline: Performance analysis. *Telecommunication Systems* **2005**, *28*, 53–77.
33. Walraevens, J.; Steyaert, B.; Moeneclaey, M.; Bruneel, H. Delay analysis of a HOL priority queue. *Telecommunication Systems* **2005**, *30*, 81–98.
34. Walraevens, J.; Fiems, D.; Bruneel, H. The discrete-time preemptive repeat identical priority queue. *Queueing Systems* **2006**, *53*, 231–243.
35. Maertens, T.; Walraevens, J.; Bruneel, H. On priority queues with priority jumps. *Performance Evaluation* **2006**, *63*, 1235–1252.
36. Walraevens, J.; Steyaert, B.; Bruneel, H. A preemptive repeat priority queue with resampling: Performance analysis. *Annals of Operations Research* **2006**, *146*, 189–202.
37. Maertens, T.; Walraevens, J.; Bruneel, H. A modified HOL priority scheduling discipline: performance analysis. *European Journal of Operational Research* **2007**, *180*, 1168–1185.
38. Walraevens, J.; Wittevrongel, S.; Bruneel, H. A discrete-time priority queue with train arrivals. *Stochastic models* **2007**, *23*, 489–512.
39. Maertens, T.; Walraevens, J.; Bruneel, H. Priority queueing systems: from probability generating functions to tail probabilities. *Queueing Systems* **2007**, *55*, 27–39.
40. Maertens, T.; Walraevens, J.; Bruneel, H. Performance comparison of several priority schemes with priority jumps. *Annals of Operations Research* **2008**, *162*, 109–125.
41. Walraevens, J.; Steyaert, B.; Bruneel, H. Analysis of a discrete-time preemptive resume priority buffer. *European Journal of Operational Research* **2008**, *186*, 182–201.
42. Walraevens, J.; Fiems, D.; Bruneel, H. Time-dependent performance analysis of a discrete-time priority queue. *Performance Evaluation* **2008**, *65*, 641–652.
43. Walraevens, J.; Maertens, T.; Bruneel, H. A semi-preemptive priority scheduling discipline: performance analysis. *European Journal of Operational Research* **2013**, *224*, 324–332.
44. Walraevens, J.; Bruneel, H.; Fiems, D.; Wittevrongel, S. Delay analysis of multiclass queues with correlated train arrivals and a hybrid priority/FIFO scheduling discipline. *Applied Mathematical Modelling* **2017**, *45*, 823–839.

45. De Clercq, S.; Walraevens, J. Delay analysis of a two-class priority queue with external arrivals and correlated arrivals from another node. *Annals of Operations Research* **2020**, *293*, 57–72.
46. Walraevens, J.; Van Giel, T.; De Vuyst, S.; Wittevrongel, S. Asymptotics of waiting time distributions in the accumulating priority queue. *Queueing Systems* **2022**, *101*, 221–244.
47. Konheim, A.; Meilijson, I.; Melkman, A. Processor-sharing of 2 parallel lines. *Journal of Applied Probability* **1981**, *18*, 952–956. doi:10.2307/3213071.
48. Parekh, A.; Gallager, R. A generalized processor sharing approach to flow control in integrated services networks: the single-node case. *IEEE/ACM transactions on networking* **1993**, *1*, 344–357.
49. Núnez-Queija, R. Sojourn times in a processor sharing queue with service interruptions. *Queueing systems* **2000**, *34*, 351–386.
50. Walraevens, J.; van Leeuwaarden, J.; Boxma, O. Power series approximations for generalized processor sharing systems. *Queueing Systems* **2010**, *66*, 107–130.
51. Vanlerberghe, J.; Walraevens, J.; Maertens, T.; Bruneel, H. A procedure to approximate the mean queue content in a discrete-time generalized processor sharing queue with Bernoulli arrivals. *Performance Evaluation* **2019**, *134*, 102001.
52. Gail, H.; Grover, G.; Guérin, R.; Hantler, S.; Rosberg, Z.; Sidi, M. Buffer size requirements under longest queue first. *Performance Evaluation* **1993**, *18*, 133–140.
53. Pedarsani, R.; Walrand, J. Stability of multiclass queueing networks under longest-queue and longest-dominating-queue scheduling. *Journal of Applied Probability* **2016**, *53*, 421–433.
54. Perel, E.; Perel, N.; Yechiali, U. A polling system with 'Join the shortest-serve the longest'policy. *Computers & Operations Research* **2020**, *114*, 104809.
55. Perel, E.; Perel, N.; Yechiali, U. A 3-queue polling system with join the shortest-serve the longest policy. *Indagationes Mathematicae* **2023**, *34*, 1101–1120.
56. Van Leeuwaarden, J.; Resing, J. A tandem queue with coupled processors: computational issues. *Queueing Systems* **2005**, *51*, 29–52.
57. Resing, J.; Ormeci, L. A tandem queueing model with coupled processors. *Operations Research Letters* **2003**, *31*, 383–389. doi:10.1016/S0167-6377(03)00046-4.
58. Malyshev, V. An analytical method in the theory of two-dimensional positive random walks. *Siberian Mathematical Journal* **1972**, *13*, 917–929.
59. Cohen, J. Analysis of a two-dimensional algebraic nearest-neighbour random walk (queue with paired services). Technical report, CWI, Amsterdam, 1994.
60. Cohen, J. On a class of two-dimensional nearest-neighbour random walks. *Journal of Applied Probability* **1994**, *31*, 207–237.
61. Fayolle, G.; Malyshev, V.; Iasnogorodski, R. *Random walks in the quarter-plane*; Vol. 40, Springer, 1999.
62. Adan, I.; van Leeuwaarden, J.; Raschel, K. The compensation approach for walks with small steps in the quarter plane. *Combinatorics Probability & Computing* **2013**, *22*, 161–183. doi:10.1017/S0963548312000594.
63. Cohen, J. On the asymmetric clocked buffered switch. *Queueing Systems* **1998**, *30*, 385–404.
64. Bruneel, H.; Kim, B. *Discrete-time models for communication systems including ATM*; Kluwer Academic Publisher: Boston, 1993.