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$2^m - 1$ as the Integer Formulation to Govern the Dynamics of Collatz-Type Sequences

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Abstract: It has been discovered that representing odd integers as modified binary expressions ending with $2^m - 1$ for $m \geq 1$ helps in understanding the dynamics of Collatz-type sequences. Starting with the original Collatz sequence $3n + 1$, it is found that when the odd step is applied to an odd integer ending with $2^m - 1$, an even integer that ends in $2^{m+1} + 2^m - 2$ is obtained, which is exactly once divisible by 2, unless the lowest index reduces to zero. This implies that the sequence alternates between odd and even steps m times. This governs the dynamics of the Collatz-type sequences because the value of m determines the number of times the integer can be divided by 2 in each even step. A shortcut method is then presented that gives the even integer after m odd-even steps are completed. This formulation also allows construction of odd integers to follow specific patterns of odd and even steps. The shortcut method for the modified Collatz sequence $5n + 1$ is also presented.

Keywords: Collatz; $3n + 1$

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1. Introduction

The Collatz problem [1–4], defines the following set of rules: If n is odd, multiply it by 3, and add 1. If n is even, it is divided by 2.

The associated Collatz conjecture states that every integer ultimately reduces to unity. To prove this conjecture, it must be shown not only that the sequence eventually cycles through 1, 4, 2, 1, but also that no integer diverges to infinitely larger integers [5,6].

While a complete proof may be impossible, this article attempts to understand the working of Collatz-type sequences. For this, the odd integers are expressed as modified binary expressions ending in $2^m - 1$ for $m \geq 1$. By examining the integers that result from this seed form, insights are gained into the patterns of the odd-even steps. The conditions that govern the progression of Collatz-type sequences are immediately made clear through the use of the modified binary form ending with $2^m - 1$. For this reason, $2^m - 1$ is stated as the governing integer formulation for Collatz-type sequences.

2. Behavior of Collatz Sequence $3n + 1$ with $2^m - 1$

Let n be an odd integer ending with $2^m - 1$. The term $2^m - 1$ is crucial because it governs the sequence's behavior. Although calculations will only explicitly show the evolution of this term, it is implied that the full odd integer, including higher index terms, is present but not explicitly written out.

Let $\mathcal{O}\{1\}, \mathcal{O}\{2\}, \ldots$ denote the resulting integer at the end of the odd step while $\mathcal{E}\{1\}, \mathcal{E}\{2\}, \ldots$ denote the resulting integer at the end of the even step. The modified binary expression of integers at the end of each step is:

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$$
\begin{aligned}
n &= 2^m - 1 \\
\mathcal{O}\{1\} &= (2+1)(2^m - 1) + 1 \\
&= 2^{m+1} + 2^m - 2 \\
\mathcal{E}\{1\} &= 2^m + 2^{m-1} - 1 \\
\mathcal{O}\{2\} &= (2+1)(2^m - 1) + 1 + (2+1)(2^{m-1}) \\
&= 2^{m+1} + 2^m - 2 + (2+1)(2^{m-1}) \\
\mathcal{E}\{2\} &= 2^m + 2^{m-1} - 1 + (2+1)(2^{m-2}) \\
\mathcal{O}\{3\} &= 2^{m+1} + 2^m - 2 + (2+1)(2^{m-1}) + (2+1)^2(2^{m-2}) \\
\mathcal{E}\{3\} &= 2^m + 2^{m-1} - 1 + (2+1)(2^{m-2}) + (2+1)^2(2^{m-3})
\end{aligned}
$$

The following observations are made:

- The number of divisions in each even step depends on m. If the lowest index of 2 vanishes, then the resulting integer is even, leading to an additional $\mathcal{E}$ step.
- The reduction in index at an $\mathcal{E}$ step is compensated by multiplying by $(2+1)$ at the $\mathcal{O}$ step. However, the reduction in index by an additional $\mathcal{E}$ step cannot be compensated and is carried forward for the remainder of the cycle.
- Once the first 2^0 is reached, additional $\mathcal{E}$ steps occur after fewer $\mathcal{OE}$ cycles since the lowest index is now less than m.
- After m additional $\mathcal{E}$ steps, the term 2^m is reduced to 2^0, and all lower index terms vanish. The value m is also deducted from higher indices.

Let the even integer obtained at the first even step after the m^{th} odd step be $\mathcal{E}^{(1)}\{m\}$.

$$
\begin{aligned}
\mathcal{E}^{(1)}\{m\} &= 2^m + 2^{m-1} + (2+1)(2^{m-2}) + \cdots + (2+1)^{m-1} - 1 \\
&= 3^m - 1
\end{aligned}
$$

Suppose the actual integer is $\sum_{n>m} 2^n + 2^m - 1$, then the integer obtained at $\mathcal{E}^{(1)}\{m\}$ becomes

$$
\mathcal{E}^{(1)}\{m\} = \left(\left(\frac{3}{2}\right)^m \sum_{n>m} 2^n\right) + 3^m - 1
$$

A few examples are given in Table 1.

Table 1. Examples of the even integer obtained at $\mathcal{E}^{(1)}\{m\}$ for various seed integers.

Integer	Modified binary	$\mathcal{E}^{(1)}\{m\}$	Integer value
7	$2^3 - 1$	$3^3 - 1$	26
19	$2^4 + 2^2 - 1$	$\left(\frac{3}{2}\right)^2 2^4 + 3^2 - 1$	44
34603007	$2^{25} + 2^{20} - 1$	$\left(\frac{3}{2}\right)^{20} 2^{25} + 3^{20} - 1$	115063885232
57343	$2^{15} + 2^{14} + 2^{13} - 1$	$\left(\frac{3}{2}\right)^{13} (2^{15} + 2^{14}) + 3^{13} - 1$	11160260

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3. Controlling Collatz Sequence Using $2^m - 1$

Since the behavior of the Collatz sequence is understood, it is now possible to estimate the starting integer based on a given pattern of $\mathcal{OE}$ cycles, or to estimate the cycle pattern based on the integer. An example of estimating the integer based on a given $\mathcal{OE}$ cycle is presented.

3.1. Estimating Integer Based on a Cycle Pattern

Suppose the following cycle pattern is desired:

$$\underbrace{\mathcal{OEE}}_{\alpha}\ \underbrace{\mathcal{OEE}}_{\beta}\ \underbrace{\mathcal{OEOEE}}_{\gamma}$$

The cycle is segmented into α, β, and γ blocks depending on termination by the extra $\mathcal{E}$ step. Let the integer be of the form $2^\gamma + 2^\beta + 2^\alpha - 1$. The value of α is visually determined as 1.

The residue of $2^\alpha - 1$ at the end of α block is one. The lowest index term at the end of α block determine the cycle pattern for the β block. The lowest index terms resulting from 2^β and 2^γ at the end of the α block are estimated using binary formulation in previous section.

$$\mathcal{OEE}(n) = \ldots + 2^{\alpha-1} + 2^{\alpha-2} + 2^{\beta-1} + 2^{\beta-2} + 1$$

The higher index terms are ignored. The integer formulation should end in $2^1 - 1$ since there is one $\mathcal{OE}$ cycle. Additionally, let $\gamma > \beta$, therefore, the lowest index term is $2^{\beta-2} + 1$ and this should be equivalent to $2^1 - 1$.

$$2^{\beta-2} + 1 \equiv 2^1 - 1$$
$$\beta \geq 4$$

In a similar manner, the lowest index at the end of β block are

$$\mathcal{OEEOEE}(n) = \ldots + 2^{\alpha-2} + 2^{\alpha-3} + 2^{\alpha-3} + 2^{\alpha-4} + 2^{\beta-2} + 2^{\beta-3} + 2^{\beta-3} + 2^{\beta-4} + 1$$
$$\mathcal{OEEOEE}(n) = \ldots + 2^{\alpha-1} + 2^{\alpha-4} + 2^{\beta-1} + 2^{\beta-4} + 1$$
$$\mathcal{OEEOEE}(n) = \ldots + 2^{\alpha-4} + 2^{\beta-4} + 1$$

The higher index terms obtained from coalescing lower index terms are removed. The lowest index terms are $2^{\alpha-4} + 2^{\beta-4}$ and this should be equivalent to $2^2 - 1$ since the $\mathcal{OE}$ continues for two cycle.

$$2^{\alpha-4} + 2^{\beta-4} + 1 \equiv 2^2 - 1$$

If $\beta = 5$, the term $2^\beta + 1$ becomes equal to $2^2 - 1$. Therefore, the value of α is greater than 4. Some of the integers obtained for different values of α along with their cycle are given.

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3.1.1. $\alpha = 5$

The integer is $2^5 + 2^5 + 2^1 - 1 = 65$. The Collatz cycle is $\mathcal{OEEOEEOEEE}$. The cycle is correct till the β block but then differs for the γ block. It happens because the value of $alpha$ and β is taken equal that results in index coalescing.

3.1.2. $\alpha = 6$

The integer is $2^6 + 2^5 + 2^1 - 1 = 97$. The Collatz cycle is $\mathcal{OEEOEEOEOEOEE}$. The cycle is improvement over the previous one but still not correct. This occurs because $2^{\alpha-4} + 2^{\beta-4} + 1 = 2^2 + 2^2 - 1$. As one can see, the indices are same and coalesce to become $2^3 - 1$ for which the $\mathcal{OE}$ repeat three times.

3.1.3. $\alpha = 7$

The integer is $2^7 + 2^5 + 2^1 - 1 = 161$ and the Collatz cycle is $\mathcal{OEEOEEOEOEE}$ which is the desired cycle.

3.1.4. $\alpha = 8$

The integer is $2^8 + 2^5 + 2^1 - 1 = 289$ and the Collatz cycle is $\mathcal{OEEOEEOEOEE}$ which is also the desired cycle. Similarly, any integer $2^\alpha + 2^5 + 2^1 - 1$ will follow the desired cycle for any $\alpha \geq 7$.

4. Application to Collatz-Type $5n + 1$ Sequence

Consider the modified binary expression of integers obtained when $5n + 1$ is applied to $2^m - 1$:

$$
\begin{aligned}
n &= 2^m - 1 \\
\mathcal{O}\{1\} &= (2^2 + 1)(2^m - 1) + 1 \\
&= 2^{m+2} + 2^m - 2^2 \\
\mathcal{E}^{(1)}\{1\} &= 2^{m+1} + 2^{m-1} - 2 \\
\mathcal{E}^{(2)}\{1\} &= 2^m + 2^{m-2} - 1 \\
\mathcal{O}\{2\} &= (2^2 + 1)(2^m - 1) + 1 + (2^2 + 1)(2^{m-2}) \\
&= 2^{m+2} + 2^m - 2^2 + (2^2 + 1)(2^{m-2}) \\
\mathcal{E}^{(1)}\{2\} &= 2^{m+1} + 2^{m-1} - 2 + (2^2 + 1)(2^{m-3}) \\
\mathcal{E}^{(2)}\{2\} &= 2^m + 2^{m-2} - 1 + (2^2 + 1)(2^{m-4}) \\
\mathcal{O}\{3\} &= 2^{m+2} + 2^m - 2^2 + (2^2 + 1)(2^{m-2}) + (2^2 + 1)^2(2^{m-4}) \\
\mathcal{E}^{(1)}\{3\} &= 2^{m+1} + 2^{m-1} - 2 + (2^2 + 1)(2^{m-3}) + (2^2 + 1)^2(2^{m-5}) \\
\mathcal{E}^{(2)}\{3\} &= 2^m + 2^{m-2} - 1 + (2^2 + 1)(2^{m-4}) + (2^2 + 1)^2(2^{m-6})
\end{aligned}
$$

The following observations are made:

- If m is odd then the pattern $\mathcal{OE}^{(1)}\mathcal{E}^{(2)}$ repeats for $\left(\frac{m-1}{2}\right)$ times. The integer obtained at $\mathcal{E}^{(1)}\{\frac{m+1}{2}\}$ is odd and a $\mathcal{O}$ step follows, resulting in the pattern terminating with $\mathcal{OE}^{(1)}\mathcal{O}$.
- If m is even then the pattern $\mathcal{OE}^{(1)}\mathcal{E}^{(2)}$ repeats for $\left(\frac{m}{2} - 1\right)$ times. The integer obtained at $\mathcal{E}^{(2)}\{\frac{m}{2}\}$ is even and another $\mathcal{E}$ step follows, resulting in the pattern terminating with $\mathcal{OE}^{(1)}\mathcal{E}^{(2)}\mathcal{E}^{(3)}$. The actual number of $\mathcal{E}$ steps depend on the integer.

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4.1. Case 1: m Is Odd

Suppose m is odd and the $\mathcal{O}\{\frac{m+1}{2}+1\}$ step occurs after the $\mathcal{E}^{(1)}\{\frac{m+1}{2}\}$ step. The even integer obtained at $\mathcal{O}\{\frac{m+1}{2}+1\}$ is

$$\begin{aligned}\mathcal{O}\left\{\frac{m+1}{2}+1\right\} &= (2^2+1)(2^{m+1}-2)+1 \\ &+ (2^2+1)(2^{m-1})+(2^2+1)^2(2^{m-3})+\cdots+(2^2+1)^{\frac{m+1}{2}} \\ &= 2^{m+3}+2^{m+1} \\ &+ \underline{(2^2+1)(2^{m-1})+(2^2+1)^2(2^{m-3})+\cdots+(2^2+1)^{\frac{m+1}{2}}}-9 \\ &= 2^{m+3}+2^{m+1}+5^{\frac{m+3}{2}}-5\cdot 2^{m+1}-9 \\ &= 5^{\frac{m+3}{2}}-9\end{aligned}$$

As before, if the actual integer is $\sum_{n>m} 2^n + 2^m - 1$, then the integer obtained at $\mathcal{O}\left\{\frac{m+1}{2}+1\right\}$ becomes

$$\mathcal{O}\left\{\frac{m+1}{2}+1\right\} = \left(10\cdot\left(\frac{5}{2^2}\right)^{\frac{m+1}{2}}\sum_{n>m}2^n\right)+5^{\frac{m+3}{2}}-9$$

A few examples are given in Table 2.

Table 2. Values of even integer obtained at $\mathcal{O}\left\{\frac{m+1}{2}+1\right\}$ for various seed integers with odd m.

Integer	Modified binary	$\mathcal{O}\left\{\frac{m+1}{2}+1\right\}$	Integer value
7	2^3-1	$5^{\frac{3+3}{2}}-9$	116
95	2^6+2^5-1	$10\cdot\left(\frac{5}{2^2}\right)^{\frac{5+1}{2}}2^6+5^{\frac{5+3}{2}}-9$	1866
57343	$2^{15}+2^{14}+2^{13}-1$	$10\cdot\left(\frac{5}{2^2}\right)^{\frac{13+1}{2}}(2^{15}+2^{14})+5^{\frac{13+3}{2}}-9$	2734366

4.2. Case 2: m Is Even

Let z additional $\mathcal{E}$ steps occur after $\mathcal{E}^{(2)}\{\frac{m}{2}\}$. The resulting odd integer will be

$$\mathcal{E}^{(2+z)}\left\{\frac{m}{2}\right\} = \frac{1}{2^z}\left(2^m+2^{m-2}-1+(2^2+1)(2^{m-4})+\cdots+(2^2+1)^{\frac{m}{2}-1}\right)$$

The odd step produces an even integer given by

$$\begin{aligned}\mathcal{O}\left\{\frac{m}{2}+1\right\} &= \frac{1}{2^z}\Big((2^2+1)(2^m-1)\Big)+1 \\ &+ \frac{1}{2^z}\left((2^2+1)(2^{m-2})+(2^2+1)^2(2^{m-4})+\cdots+(2^2+1)^{\frac{m}{2}}\right) \\ &= \frac{1}{2^z}\left(5^{\frac{m}{2}+1}-5\right)+1\end{aligned}$$

As before, if the actual integer is $\sum_{n>m} 2^n + 2^m - 1$, then the integer obtained at $\mathcal{O}\{\frac{m}{2}+1\}$ becomes

$$\mathcal{O}\left\{\frac{m}{2}+1\right\} = \left(\frac{5}{2^z}\cdot\left(\frac{5}{2^2}\right)^{\frac{m}{2}}\sum_{n>m}2^n\right)+\frac{1}{2^z}\left(5^{\frac{m}{2}+1}-5\right)+1$$

The variable z is chosen such that $\mathcal{O}\left\{\frac{m}{2}+1\right\}$ is even.
A few examples are given in Table 3.

Table 3. Values of even integer obtained at $\mathcal{O}\left\{\frac{m}{2}+1\right\}$ for various seed integers with even m. (*6 is obtained when $\mathcal{O}$ is applied to 1.)

Integer	Modified binary	$\mathcal{O}\left\{\frac{m+1}{2}+1\right\}$	z	Integer value
3	2^2-1	$\frac{1}{2^z}\left(5^{\frac{2}{2}+1}-5\right)+1$	2	6*
79	2^6+2^4-1	$\left(\frac{5}{2^z}\cdot\left(\frac{5}{2^2}\right)^{\frac{4}{2}}2^6\right)+\frac{1}{2^z}\left(5^{\frac{4}{2}+1}-5\right)+1$	2	156
53247	$2^{15}+2^{14}+2^{12}-1$	$\left(\frac{5}{2^z}\cdot\left(\frac{5}{2^2}\right)^{\frac{12}{2}}(2^{15}+2^{14})\right)+\frac{1}{2^z}\left(5^{\frac{12}{2}+1}-5\right)+1$	2	253906

5. Conclusion

It has been recently discovered that expressing odd integers in the alternate binary form 2^m-1 aids in understanding the workings of Collatz-type sequences. The understanding leads to formulations that generate the even integer obtained after m odd-even steps are completed. For the original Collatz sequence $3n+1$, the said even integer is given by $\left(\left(\frac{3}{2}\right)^m\sum_{n>m}2^n\right)+3^m-1$. This formulation allows for construction on odd integers that follow a specific odd-even step pattern.

For the modified Collatz-type sequence $5n+1$, the formulation for the even integer depends on m. If m is odd, the even integer is $\left(10\cdot\left(\frac{5}{2^2}\right)^{\frac{m+1}{2}}\sum_{n>m}2^n\right)+5^{\frac{m+3}{2}}-9$. However, when m is even, the even integer is $\left(\frac{5}{2^z}\cdot\left(\frac{5}{2^2}\right)^{\frac{m}{2}}\sum_{n>m}2^n\right)+\frac{1}{2^z}\left(5^{\frac{m}{2}+1}-5\right)+1$. Here, the maximum value of z is such that the integer remains even.

References

1. Lagarias, J.C. The 3x+ 1 problem: An annotated bibliography (1963–1999). *The ultimate challenge: the 3x* **2003**, *1*, 267–341.
2. Lagarias, J.C. The 3x+ 1 problem: An annotated bibliography. *preprint* **2004**.
3. Lagarias, J.C. The 3x+ 1 problem: An annotated bibliography, II (2000-2009). *arXiv preprint math/0608208* **2006**.
4. Lagarias, J.C. *The ultimate challenge: The 3x+ 1 problem*; American Mathematical Soc., 2010.
5. Terras, R. A stopping time problem on the positive integers. *Acta Arithmetica* **1976**, *3*, 241–252.
6. Tao, T. Almost all orbits of the Collatz map attain almost bounded values. Forum of Mathematics, Pi. Cambridge University Press, 2022, Vol. 10, p. e12.