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Construction of a Second-Countable Treon Space from a New Metric Structure

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Abstract: Bermejo Algebras, which consist of Algebra B and Treon Algebra, are non-associative and unital algebraic structures that introduce new complex entities. Recently, a Hausdorff space associated with the space where these algebras act was defined. We constructed a second-countable treon space utilizing a novel metric derived from these algebras. We defined topological spaces and established a metric to prove the second-countable property within Bermejo Algebras. Our findings contribute to the construction of manifolds in treon spaces by defining a countable base using the density of rationals in the real numbers.

Keywords: Bermejo Algebras; Treon Algebra; second-countable space; metric topology; manifolds

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Introduction

Bermejo Algebras (Algebra B and Treon Algebra) are two isomorphic, non-associative, and unital algebraic constructions recently described by Alejandro Bermejo [1,2]. From these algebras, Lie and Malcev algebras can be defined. Additionally, Bermejo Algebras give rise to complex structures distinct from $\mathbb{C}^2$ and quaternions [1–3].

In the elements of the Treon Algebra, conjugate products can be defined, which allow the establishment of inner products and norms [4] within their real components, without the need to equip the space with an inner product or a norm in the conventional manner. These quantities emerge simply by defining these conjugate products.

The spaces on which Bermejo Algebras are defined are Hausdorff spaces, including a particular quotient space defined with a norm structure as a fundamental element of the equivalence relation [4]. This analysis marked a starting point in the definition of manifolds in treon spaces.

However, for the definition of manifolds, besides being Hausdorff, the space must also be second-countable [5,6]. In this work, we deduce a space that meets this property, thereby contributing an element to the construction of manifolds for Bermejo Algebras.

Our study is based on the Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) [7,8]. This implies that the definition of the various numerical sets we use is grounded in ZFC. We define topological spaces, bases of topological spaces, and perform deductions to establish a metric that defines a topology, strictly based on well-defined numerical sets. With this, we demonstrate the second-countable property in topological spaces within Bermejo Algebras.

1. Theoretical Framework

1.1. Natural Numbers $\mathbb{N}$

The axioms of ZFC [7,8] enable us to construct a set with a structure that allows us to define the set of natural numbers, $\mathbb{N}$, as follows: $\mathbb{N} \equiv \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots\}$, such that $\emptyset$ is the empty set, $\emptyset \equiv 0$, $\{\emptyset\} \equiv 1$, $\{\{\emptyset\}\} \equiv 2$, and so on. Consequently, we define the *successor mapping* S such that $S : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \{n\}$.

Let $\mathbb{N}^* \equiv \mathbb{N} \setminus \{\emptyset\}$, and $m \in n$, the *predecessor mapping* of n, P, is defined as $P : \mathbb{N}^* \to \mathbb{N}, n \mapsto m$. Let $n \in \mathbb{N}^*$, the *n-power successor mapping*, S^n, is defined as $S^n \equiv S \circ S^{P(n)}$, such that $S^1 \equiv S$ and $S^0 \equiv \mathrm{id}_{\mathbb{N}}$, where $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}, n \mapsto n$. Thus, we define addition $+$ in the natural numbers as $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, (m, n) \mapsto m + n$, such that $m + n \equiv S^n(m)$ [9,10]. Note that the identity element of addition $+$ is 0, as $n + 0 = S^0(n) = n$.

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1.2. Integers $\mathbb{Z}$

We define the set of integers $\mathbb{Z}$ as a quotient set $\mathbb{Z} \equiv (\mathbb{N} \times \mathbb{N})/\sim$, such that the equivalence relation $\sim$ is defined as $(m, n) \sim (p, q) \Leftrightarrow m + q = n + p$. Thus, we do not have $\mathbb{N} \subset \mathbb{Z}$, as $\mathbb{N}$ cannot be expressed as $(\mathbb{N} \times \mathbb{N})/\sim$ [9,10]. However, we can define a bijective *embedding* $\xi : \mathbb{N} \hookrightarrow \mathbb{Z}, n \mapsto [(n, 0)]_\sim$, which allows incorporating $\mathbb{N}$ into $\mathbb{Z}$ to some extent. Note that $(n, 0) \sim (n + a, a), a \in \mathbb{N}$. This also allows us to define an inverse element for each $n \in \mathbb{N}$: $-n \equiv [(0, n)]_\sim \in \mathbb{Z}$.

We define addition $+$ in $\mathbb{Z}$, $+_\mathbb{Z}$, as $+_\mathbb{Z} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, [(m, n)]_\sim +_\mathbb{Z} [(p, q)]_\sim \equiv [(m + p, n + q)]_\sim$ [9,10]. Thus, a sum of the form $a +_\mathbb{Z} (-b)$, such that $a, b \in \mathbb{N}$, must necessarily be analyzed as $[(a, 0)]_\sim +_\mathbb{Z} [(0, b)]_\sim = [(a, b)]_\sim$ for it to be well-defined.

1.3. Rational Numbers $\mathbb{Q}$

We define the set of rational numbers, $\mathbb{Q}$, as follows: $\mathbb{Q} \equiv (\mathbb{Z} \times \mathbb{Z}^*)/\sim$, where $\mathbb{Z}^* \equiv \mathbb{Z} \setminus \{\emptyset\}$, and the equivalence relation $\sim$ is given by $(a, b) \sim (c, d) \Leftrightarrow a \cdot_\mathbb{Z} d = b \cdot_\mathbb{Z} c$ [9,10].

Thus, we can define fractions $a/b \equiv [(a, b)]_\sim$. The product $\cdot_\mathbb{Z}$ is defined as $\cdot_\mathbb{Z} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, (a, b) \cdot_\mathbb{Z} (c, d) \equiv (a \cdot c + b \cdot d, a \cdot d + b \cdot c)$ [9,10], where $\cdot$ is the product in $\mathbb{N}$ and is trivially defined from addition in $\mathbb{N}$ as a recursion $m + m = S^m(m)$, that is, if we have $3 \cdot m = (m + m) + m = S^m(m) + m = S^m(S^m(m)) = (S^m \circ S^m)(m)$, for $4 \cdot m = ((m + m) + m) + m = (S^m(m) + m) + m = S^m(S^m(m)) + m = S^m(S^m(S^m(m))) = (S^m \circ S^m \circ S^m)(m)$, and for $n \cdot m = (S^m)_{(n-1)}(m)$.

To include $\mathbb{Z}$ within $\mathbb{Q}$, we similarly need a bijective embedding, which we define as $\zeta : \mathbb{Z} \hookrightarrow \mathbb{Q}, z \mapsto [(z, 1)]_\sim$.

The sum operation $+_\mathbb{Q}$ is defined as $+_\mathbb{Q} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}, [(a, b)]_\sim +_\mathbb{Q} [(c, d)]_\sim \equiv [(a \cdot_\mathbb{Z} d +_\mathbb{Z} b \cdot_\mathbb{Z} c, b \cdot_\mathbb{Z} d)]_\sim$, and the multiplication $\cdot_\mathbb{Q} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}, [(a, b)]_\sim \cdot_\mathbb{Q} [(c, d)]_\sim \equiv [(a \cdot_\mathbb{Z} d, b \cdot_\mathbb{Z} d)]_\sim$ [9,10].

1.4. Real Numbers $\mathbb{R}$

Consider a sequence $[[(a, b)]_\sim]_{n\in\mathbb{N}} \equiv (p_n)_{n\in\mathbb{N}} \in \mathbb{Q}$, defined through the mapping $\pi : \mathbb{N} \to \mathbb{Q}$, $n \mapsto p_n$. If $\forall\epsilon > 0 \exists N \in \mathbb{N}\, \forall n, m \geq N : |p_n - p_m| < \epsilon$, we say that the sequence is a *Cauchy sequence*. This sequence will be *convergent* if $\exists p_0 \in \mathbb{Q}\, \forall\epsilon > 0\, \exists N \in \mathbb{N}\, \forall n \geq N : |p_n - p_0| < \epsilon$. In this case, p_0 is the *limit* of the sequence $(p_n)_{n\in\mathbb{N}}$, and we say that the sequence converges to p_0, $(p_n)_{n\in\mathbb{N}} \to p_0$ [11–13].

We define the real numbers, $\mathbb{R}$, as a field $(\mathbb{R}, +, \cdot)$ equipped with a total order relation $\geq$, considering that: (1) $\mathbb{R}$ is constructed with the abelian groups $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$, (2) The order relation $\geq$ is compatible with the operations $+$ and $\cdot$, and with the Archimedean property ((1) and (2) are properties of $\mathbb{Q}$), and (3) Every Cauchy sequence is convergent, such that each element in $\mathbb{R}$ is uniquely determined by a limit point of a Cauchy sequence of rational elements [11–13]. This means we have a mapping $\delta : \mathbb{Q} \to \mathbb{R}$.

We want each element $r \in \mathbb{R}$ to be uniquely determined by a Cauchy sequence, but there are different Cauchy sequences that converge to the same limit. For example, using traditional numerical notation, the Cauchy sequences $a_n = 1 + \frac{1}{n}$ and $b_n = 1 - \frac{1}{n^2}$ converge to 1 as $n \to \infty$. Therefore, we do not have a one-to-one correspondence for the real element 1. This is solved with equivalence relations.

Let $((p_n)_{n\in\mathbb{N}})_i$ be Cauchy sequences, which we denote as $\left((p_n)_{n\in\mathbb{N}}\right)_{i\in\mathcal{C}}$. We define the set $\mathcal{R} \equiv \{(p_n)_{n\in\mathbb{N}} \in \mathbb{Q} : \forall n \in \mathbb{N} : p_n \in \mathbb{Q} \land (p_n)_{n\in\mathbb{N}} \Leftrightarrow \left((p_n)_{n\in\mathbb{N}}\right)_{i\in\mathcal{C}}\}$. Denoting by superscripts $a, b, c, \ldots$ the Cauchy sequences that are distinct from each other, for two arbitrary elements $(p_n^a)_{n\in\mathbb{N}}, (p_n^b)_{n\in\mathbb{N}} \in \mathcal{R}$ we define: $(p_n^a)_{n\in\mathbb{N}} \sim (p_n^b)_{n\in\mathbb{N}} \Leftrightarrow (p_n^a - p_n^b)_{n\in\mathbb{N}} \to 0$. Understanding by 0 the rational equivalence class $[(0, 1)]_\sim$. With this, we define the equivalence class $[(p_n)_{n\in\mathbb{N}}]_\sim \Leftrightarrow \{(p_n^a)_{n\in\mathbb{N}} : (p_n^a)_{n\in\mathbb{N}} \sim (p_n^a)_{n\in\mathbb{N}}\}$, which determines a bijective correspondence $[(p_n)_{n\in\mathbb{N}}]_\sim \mapsto r \in \mathbb{R}$. Therefore, we say that $\mathbb{R}$ is defined as $\mathbb{R} \equiv \mathcal{R}/\sim = \{[(p_n)_{n\in\mathbb{N}}]_\sim : (p_n)_{n\in\mathbb{N}} \in \mathcal{R}\}$, where the operations and the order relation are well-defined in the sense that $[(p_n^a)_{n\in\mathbb{N}}]_\sim + [(p_n^b)_{n\in\mathbb{N}}]_\sim \equiv [(p_n^a + p_n^b)_{n\in\mathbb{N}}]_\sim$, $[(p_n^a)_{n\in\mathbb{N}}]_\sim \cdot [(p_n^b)_{n\in\mathbb{N}}]_\sim \equiv [(p_n^a \cdot p_n^b)_{n\in\mathbb{N}}]_\sim$, and $[(p_n^a)_{n\in\mathbb{N}}]_\sim < [(p_n^b)_{n\in\mathbb{N}}]_\sim \Leftrightarrow \exists\delta > 0 \exists N \in \mathbb{N} \forall n \geq N : p_n^b - p_n^a > \delta$ [11–13].

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1.4.1. Completeness of $\mathbb{R}$

We say that a number a is an *upper bound* of a set U if a is greater than or equal to any element of U [14]. On the other hand, a *supremum* of a set U is the smallest upper bound [15]. A field F is *ordered* if it satisfies three properties $\forall a, b, c \in F$: (1) Trichotomy: $a > b \veebar a = b \veebar a > b$ ($\veebar$ denotes the exclusive-or operator), (2) Compatibility with addition: $a \leq b \Rightarrow a + c \leq b + c$, and (3) Compatibility with multiplication: $a \leq b \wedge c \geq 0 \Rightarrow a \cdot c \leq b \cdot c$ [16]. Now, an ordered field is *complete* if any non-empty subset that is bounded above (has an upper bound) has a supremum in the field [16].

The real numbers are a complete ordered field, therefore, if we have any subset of real numbers that is bounded above, then that subset has a supremum in the real numbers. The rational numbers $\mathbb{Q}$, on the other hand, are not complete because there are subsets of $\mathbb{Q}$ that have upper bounds but do not have a supremum in $\mathbb{Q}$. For example, the set $U = \{y \in \mathbb{Q} : y^2 < 2\}$ has an upper bound, can be any element of $\mathbb{Q}$ greater than or equal to all elements of U. In our example, two upper bounds can be 1.7 and 2, as their squares are greater than 2. However, the supremum of U is a real number, in this case is $\sqrt{2}$, since $\sqrt{2}$ is the smallest real number greater than or equal to all elements of U. But, $\sqrt{2} \notin \mathbb{Q}$. Thus, although U is bounded above in $\mathbb{Q}$, it does not have a supremum in $\mathbb{Q}$.

The lack of a supremum in $\mathbb{Q}$ for certain bounded above sets means that $\mathbb{Q}$ is not complete. In contrast, in $\mathbb{R}$, any non-empty subset bounded above has a supremum that is also in $\mathbb{R}$.

2. Theoretical Development

2.1. Base of the Treon Topological Space (Λ, T_Λ)

Bermejo defined the set Λ of r-treonspheres [4] as:

$$\Lambda \equiv \{p_i \in \text{Preim}_{\langle \cdot^2 \rangle} \subset X : \sqrt{\text{Re}\langle p^2 \rangle_i} = r_i \wedge r_i > 0\}$$

where $\text{Re}\langle p^2 \rangle_i$ is the real component of the operation $\langle p^2 \rangle \equiv p \odot p^{*(i,j)}$, where $\odot$ is the product in algebra B, and $*_{(i,j)}$ is the double complex conjugation of a treon [4]. $\text{Preim}_{\langle \cdot^2 \rangle}$ denotes the preimage of the composition mapping $H = h \circ \langle \cdot^2 \rangle$, such that [4]:

$$H : \text{Preim}_{\langle \cdot^2 \rangle} \to \mathbb{R}^3,$$

$$p \mapsto \vec{p},$$

which takes treon p preimages of the operation $\langle \cdot^2 \rangle$, and assigns them elements of the vector space $\mathbb{R}^3$.

The mapping h that composes it is defined as [4]:

$$h : N \to \mathbb{R}^3,$$

$$\langle p^2 \rangle \mapsto \vec{p},$$

such that $N \subset X$, with X being the total treon set. N is defined as [4]:

$$N \equiv \{p \in X : p = \langle p^2 \rangle\}.$$

On the other hand, the mapping $\langle \cdot^2 \rangle$ is defined as [4]:

$$\langle \cdot^2 \rangle : \Lambda \subset X \to N \subset X,$$

$$p \mapsto \langle p^2 \rangle.$$

Let T_Λ be a topology associated with Λ, we will have an arbitrary treon topological space (Λ, T_Λ) [4]. Then, we assert that the collection of open subsets $B \subseteq T_\Lambda$ constitutes a *basis* for T_Λ if [17,18]:

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$$\forall U \in T_\Lambda \, \exists (\beta_i)_{i \in I} : \beta_i \in B \wedge \bigcup_{i \in I} \beta_i = U,$$

where the index i belongs to an arbitrary set I.

2.2. Treon Topology Induced by the Bermejian Metric

In algebra B, the Bermejian inner product is defined from the product $p_i \odot p_j^{*(i,j)}$ [2,4], denoted $\langle p_i, p_j \rangle$:

$$\langle p_i, p_j \rangle = (p_i \diamond p_j, -p_{i1}p_{j2} + p_{i2}p_{j1} - p_{i3}p_{j2}, -p_{i1}p_{j3} + p_{i3}p_{j1} - p_{i3}p_{j2}),$$

where $p_i \diamond p_j \equiv p_{i1}p_{j1} + p_{i2}p_{j2} + p_{i3}p_{j3}$. We should not confuse the subscript notation i, j with the superscript $*_{(i,j)}$, which involves the double conjugate of the treon.

For the case $\langle p_i, p_i \rangle \equiv \langle p^2 \rangle$, this product yields:

$$\langle p_i^2 \rangle = (\|p_i\|^2, 2p_{i1}p_{i2} + p_{i2}p_{i3}, 2p_{i1}p_{i3} + p_{i3}p_{i2}),$$

where $\|p_i\|^2 \equiv p_{i1}^2 + p_{i2}^2 + p_{i3}^2$.

If for all $p_i = (p_{i1}, p_{i2}, p_{i3})$ and $p_j = (p_{j1}, p_{j2}, p_{j3})$, we define a difference between treons as:

$$d_{ij} \equiv p_i - p_j = (p_{i1}, p_{i2}, p_{i3}) - (p_{j1}, p_{j2}, p_{j3}),$$

we will have as a result:

$$d_{ij} = (p_{i1} - p_{j1}, p_{i2} - p_{j2}, p_{i3} - p_{j3}).$$

Executing the Bermejian inner product $\langle d_{ij}^2 \rangle$:

$$\langle d_{ij}^2 \rangle = (\|d_{ij}\|^2, 2d_{ij1}d_{ij2} + d_{ij2}d_{ij3}, 2d_{ij1}d_{ij3} + d_{ij3}d_{ij2}).$$

Therefore, we can define a mapping $g_{ij} \equiv g(p_i, p_j)$ as:

$$g_{ij} : \Lambda \times \Lambda \to \mathbb{R},$$

$$g_{ij} \equiv \sqrt{\mathrm{Re}\langle d_{ij}^2 \rangle}.$$

This mapping has the following properties for all $p_i, p_j, p_k \in \Lambda$:

1. $g_{ij} \geq 0$,
2. $g_{ij} = 0 \iff i = j$,
3. $g_{ij} = g_{ji}$,
4. $g_{ij} \leq g_{ik} + g_{kj}$.

These properties define the mapping g_{ij} as a metric, which we denote as the *Bermejian metric*.

2.3. Countable Base for the Treon Space (Λ, T_Λ)

Let the pair (Λ, T_Λ) be our treon topological space, let $W \subseteq \Lambda$ not necessarily $W \in T_\Lambda$, and let any $U \in T_\Lambda$ be an open set in T_Λ:

We say that p is an *accumulation point* of W [17,18], p_{acc}, if:

$$\forall U \in T, p \in U : U \setminus \{p\} \cap W \neq \emptyset.$$

Thus, W' is the *derived set* of W if:

$$W' = \{p \in \Lambda : p = p_{\text{acc}}\},$$

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and thus, we say that $\overline{W}$ is the *closure* of W if:

$$\overline{W} = W \cup W'.$$

Let (Λ, T_Λ), with $W \subseteq \Lambda$. We say that W is *dense* in Λ if and only if $\overline{W} = \Lambda$ [19,20]. Additionally, we say that W is countable if $|W| \leq |\mathbb{N}|$ [20], so there exists a bijective function $f : \mathbb{N} \to W$. We say that W is *uncountable* simply if it is not countable. Considering this: $\mathbb{R}$ is uncountable, and $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.

Consequently, we can define a countable basis for a topology using the Bermejian metric as follows:

Let (Λ, T_Λ) be a topological space induced by the Bermejian metric, denoted (Λ, g_{ij}). We define the basis B of $T_\Lambda \equiv g_{ij}$ as the collection of balls with radius ϵ centered at a point $p_0 = (p_{01}, p_{02}, p_{03})$:

$$B = \{B_\epsilon(p_0) : p_0 \in \Lambda, \epsilon \in \mathbb{R}, \epsilon > 0\},$$

where the radius ϵ corresponds to metrics fixed at points p_0 to p_i, such that $\epsilon \equiv g_{0i} = r_{0i}$.

Understanding that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, we can define:

$$B = \{B_\epsilon(p_0) : p_0 \in \Lambda_{\mathbb{Q}^3}, \epsilon \in \mathbb{Q}, \epsilon > 0\},$$

such that B is a countable base of T_Λ.

Thus, we say that our treon topological space (Λ, T_Λ) is second-countable, since there exists a countable base B of T_Λ. Henceforth, understand $(\Lambda, T_\Lambda) \equiv (\Lambda_{\mathbb{Q}^3}, T_{\Lambda_{\mathbb{Q}^3}}) \equiv (\Lambda_{\mathbb{Q}^3}, g_{ij})$.

To ensure that (Λ, T_Λ) is second-countable implied ensuring that there exists a countable base B, which in turn implied verifying that $\Lambda_{\mathbb{Q}^3}$ is indeed countably infinite.

2.3.1. Proof of the Countability of $\Lambda_{\mathbb{Q}^3}$

Proof that Λ Is Infinitely Uncountable

Λ is an infinitely uncountable set because a set of treons $p = (p_1, p_2, p_3)$ has components $p_i \in \mathbb{R}$. We demonstrate this through a proof by contradiction.

Proof by contradiction: We start by assuming Λ is countable and will arrive at a contradiction.

If Λ is countable, then $\forall (p_i, 0, 0) \in W \subset \Lambda$, such that $p_i \in (0,1) \subset \mathbb{R}$, implies that W is countable. We have denoted the open interval as $(0,1)$.

If W is countable, then there exists a bijective function f:

$$f : \mathbb{N} \to W,$$

$$1 \mapsto p_1 = 0.p_{11}p_{12}p_{1m},$$

$$2 \mapsto p_2 = 0.p_{21}p_{22}p_{2m},$$

$$n \mapsto p_n = 0.p_{nm}.$$

Thus, $\forall p_n \in (0,1) \subset \mathbb{R} \exists n \in \mathbb{N} : n \mapsto p_n$.

We can construct a $\hat{p}_\alpha \in (0,1)$ such that $p_{nm} \Rightarrow n = m$. Therefore, $\hat{p}_\alpha$ is different from any p_n. Hence, $\forall \hat{p}_\alpha \in (0,1) \nexists n \in \mathbb{N} : n \mapsto p_n$. This generates a contradiction.

We conclude that $(0,1) \subset \mathbb{R}$ is infinitely uncountable; by extension, $\mathbb{R}$ is infinitely uncountable, $(p_i, 0, 0)$ is infinitely uncountable, and any $(p_i, p_j, p_k) \in \Lambda$, such that $p_i, p_j, p_k \in \mathbb{R}$, implies that Λ is infinitely uncountable.

We need the components of a treon to be elements of a countable set. In this sense, we chose the set $\mathbb{Q}$, which is a countably infinite set, as the set from which we will take the components of the treon space.

Construction of a Countably Infinite Set $\Lambda_{\mathbb{Q}^3}$

We can define a set of elements $p_n \in \mathbb{Q}, \forall n \in \mathbb{N}$ such that we have an arbitrary treon of the form $(p_n, 0, 0) \in Y \subset \Lambda_{\mathbb{Q}^3}$. Thus:

$$p_n = \{[(m,n)]_\sim \in \mathbb{Q} : \forall(m \in \mathbb{Z} \wedge n \in \mathbb{Z}^*) \Rightarrow [(m,n)]_\sim \in (\mathbb{Z} \times \mathbb{Z}^*)/\sim\},$$

where $\sim$ is defined as $(a,b) \sim (c,d) \Leftrightarrow a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$.

Accordingly, we redefine the component p_n as:

$$p_n = \{[(m,n)]_\sim \in \mathbb{Q} : \forall(m \in \mathbb{Z} \wedge n \in \mathbb{N}^* \subset \mathbb{Z}^*)\},$$

where we necessarily consider in $\mathbb{N}^* \subset \mathbb{Z}^*$ the bijective embedding mapping ξ:

$$\xi : \mathbb{N} \hookrightarrow \mathbb{Z},$$

$$n \mapsto [(n,0)]_\sim,$$

which allows us to have the elements of $\mathbb{N}$ well-defined as elements within $\mathbb{Z}$.

To avoid confusion between the equivalence relation that defines $\mathbb{Z} \equiv (\mathbb{N} \times \mathbb{N})/\sim$ with the equivalence relation that defines $\mathbb{Q} \equiv (\mathbb{Z} \times \mathbb{Z}^*)/\sim$, we denote:

$$\mathbb{Z} \equiv (\mathbb{N} \times \mathbb{N})/\sim_{\mathbb{Z}},$$

$$\mathbb{Q} \equiv (\mathbb{Z} \times \mathbb{Z}^*)/\sim_{\mathbb{Q}}.$$

Thus:

$$p_n = \{[(m, [(n,0)]_{\sim_{\mathbb{Z}}})]_{\sim_{\mathbb{Q}}} \in \mathbb{Q} : \forall(m \in \mathbb{Z} \wedge n \in \mathbb{N}^* \Rightarrow [(n,0)]_{\sim_{\mathbb{Z}}} \in \mathbb{N}^* \subset \mathbb{Z}^*)\}.$$

Therefore:

$$\begin{aligned} p_1 &= \left\{\left[\left(m, [(1,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}} \in \mathbb{Q} : m \in \mathbb{Z} \wedge 1 \equiv [(1,0)] \in \mathbb{N}^* \subset \mathbb{Z}^*\right\} \\ &= \left\{\dots, \left[\left(-1, [(1,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}}, \left[\left(0, [(1,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}}, \left[\left(1, [(1,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}}, \dots\right\}, \end{aligned}$$

$$\begin{aligned} p_2 &= \left\{\left[\left(m, [(2,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}} \in \mathbb{Q} : m \in \mathbb{Z} \wedge 2 \equiv [(2,0)] \in \mathbb{N}^* \subset \mathbb{Z}^*\right\} \\ &= \left\{\dots, \left[\left(-1, [(2,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}}, \left[\left(0, [(2,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}}, \left[\left(1, [(2,0)]_{\sim_{\mathbb{Z}}}\right)\right]_{\sim_{\mathbb{Q}}}, \dots\right\}, \end{aligned}$$

and so on, such that:

$$\mathbb{Q} = \bigcup_{i=1}^{n} p_i.$$

Note that $\mathbb{Q} \equiv (\mathbb{Z} \times \mathbb{Z}^*)/\sim$ is equivalent to $\mathbb{Q} \equiv (\mathbb{Z} \times \mathbb{N}^*)/\sim$ considering the mapping ξ; this is because there exists a bijective functional correspondence π:

$$\pi : (\mathbb{Z} \times \mathbb{Z}^*)/\sim \to (\mathbb{Z} \times \mathbb{N}^*)/\sim,$$

$$[(-a,-b)]_\sim \mapsto [(a,b)]_\sim,$$

$$[(c,-d)]_\sim \mapsto [(-c,d)]_\sim.$$

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Considering that $\mathbb{Z}$ is a countably infinite set, because we can perform a correspondence $f : \mathbb{N} \to \mathbb{Z}, \quad 1 \mapsto 0, \quad 2 \mapsto 1, \quad 3 \mapsto -1, \quad 4 \mapsto 2, \quad 5 \mapsto -2, \ldots,$ we assert that each set $p_n = [(m, [(n, 0)]_{\sim_\mathbb{Z}})]_{\sim_\mathbb{Q}}$, for $n \in \mathbb{N}^*$ and $m \in \mathbb{Z}$, is countable: Since a countable union of countable sets remains countable, the union set of all p_n is also countable [17–20].

Thus, $(p_n, 0, 0)$ is countable and, by extension: $\forall (p_{n1}, p_{n2}, p_{n3}) \in \Lambda_{\mathbb{Q}^3}$, such that $p_{ni} \in \mathbb{Q}$, implies that $\Lambda_{\mathbb{Q}^3}$ is a countably infinite set.

2.3.2. Proof that $\Lambda_{\mathbb{Q}^3}$ is Dense in Λ

The fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ implies that $\Lambda_{\mathbb{Q}^3}$ is dense in Λ, since for any pair of different treons $(p_a, 0, 0), (p_b, 0, 0) \in \Lambda$, with $p_a, p_b \in \mathbb{R}$, there exists a $(p_k, 0, 0) \in \Lambda_{\mathbb{Q}^3}$, with $p_k \in \mathbb{Q}$ such that $p_a < p_k < p_b$. This procedure can be applied to each component of a treon, or collectively to all components. It is crucial to note that we do not assume an order relation for treons in Λ, as such an assumption would be inconsistent in a three-dimensional space of complex entities. We assert that, since treons are defined by their components, the density of these components induces a corresponding "density" in the treon.

Proof

For all $(p_a, 0, 0), (p_b, 0, 0) \in \Lambda$, such that $p_a, p_b \in \mathbb{R}, p_a < p_b \Rightarrow 0 < p_b - p_a$.

For all $a \in \mathbb{Z}_{>0}$, we have an arbitrary element of $\mathbb{Q}$ $[(a, p_b - p_a)]_{\sim_\mathbb{Q}} > 0$.

The Archimedean property [21] states that for any field F, with $b \in F$:

$$\forall b > 0 \exists n \in \mathbb{N} : b < n,$$

and this allows us to ensure that $[(a, p_b - p_a)]_{\sim_\mathbb{Q}} < [(n, [(1, 0)]_{\sim_\mathbb{Z}})]_{\sim_\mathbb{Q}}$.

From now on, understanding the axiomatic foundation of the base, we can simplify the notation of the set $\mathbb{N}$, such that $[(1, 0)] \in \mathbb{N} \equiv 1$, therefore:

$$[(a, p_b - p_a)]_{\sim_\mathbb{Q}} < [(n, [(1, 0)]_{\sim_\mathbb{Z}})]_{\sim_\mathbb{Q}} \equiv [(a, p_b - p_a)]_{\sim_\mathbb{Q}} < [(n, 1)]_{\sim_\mathbb{Q}}.$$

Then:

$$[(a, 1)]_{\sim_\mathbb{Q}} < [(n \cdot (p_b - p_a), 1)]_{\sim_\mathbb{Q}},$$

$$[(a + n \cdot p_a, 1)]_{\sim_\mathbb{Q}} < [(n \cdot p_b, 1)]_{\sim_\mathbb{Q}}.$$

Recognizing this, we can further simplify the notation without compromising the rigor of the proof:

$$[(a + n \cdot p_a, 1)]_{\sim_\mathbb{Q}} < [(n \cdot p_b, 1)]_{\sim_\mathbb{Q}} \equiv a + n \cdot p_a < n \cdot p_b.$$

Let $p_\mathbb{Z} \in \mathbb{Z}$ be the integer part of a number $n \cdot p$, such that $p \in \mathbb{R}$ and $n \in \mathbb{N}$. It always holds that $p_\mathbb{Z} \leq n \cdot p < p_\mathbb{Z} + a$, for $a \in \mathbb{Z}_{>1}$.

Taking $p_\mathbb{Z} \leq n \cdot p_a$, we have $p_\mathbb{Z} + a \leq n \cdot p_a + a$.

Therefore:

$$p_\mathbb{Z} \leq n \cdot p_a < p_\mathbb{Z} + a \leq n \cdot p_a + a < n \cdot p_b.$$

Hence:

$$n \cdot p_a < p_\mathbb{Z} + a < n \cdot p_b.$$

Since $(p_\mathbb{Z} + a) \in \mathbb{Z}$; we denote $p_k \equiv p_\mathbb{Z} + a$,. Consequently, we have:

$$n \cdot p_a < p_k < n \cdot p_b.$$

Now, resuming our notation:

$$[(n \cdot p_a, 1)]_{\sim_\mathbb{Q}} < [(p_k, 1)]_{\sim_\mathbb{Q}} < [(n \cdot p_b, 1)]_{\sim_\mathbb{Q}}.$$

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Multiplying by $[(1,n)]_{\sim_{\mathbb{Q}}}$, we obtain:

$$[(n \cdot p_a, n)]_{\sim_{\mathbb{Q}}} < [(p_k, n)]_{\sim_{\mathbb{Q}}} < [(n \cdot p_b, n)]_{\sim_{\mathbb{Q}}},$$

$$[(p_a, 1)]_{\sim_{\mathbb{Q}}} < [(p_k, n)]_{\sim_{\mathbb{Q}}} < [(p_b, 1)]_{\sim_{\mathbb{Q}}}.$$

Since $p_a, p_b \in \mathbb{R}$, the equivalence classes $[(p_a,1)]_{\sim_{\mathbb{Q}}}$ and $[(p_b,1)]_{\sim_{\mathbb{Q}}}$ will be real. And since $[(p_k,n)]_{\sim_{\mathbb{Q}}} \in \mathbb{Q}$, this implies that between any pair of elements of the set $\mathbb{R}$ we can always find an element of the set $\mathbb{Q}$.

Using our logical notation, consider the direction of one of the components of a treon between an arbitrary pair of treons $(p_a, 0, 0)$ and $(p_b, 0, 0) \in \Lambda$, denoted as $([(p_a,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R}, 0, 0)$ and $([(p_b,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R}, 0, 0)$, respectively, there always exists a $(p_k, 0, 0) \in \Lambda_{\mathbb{Q}^3}$, denoted as $([(p_k,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{Q}, 0, 0)$.

Extending this analysis to the topological analysis, we have: Let an arbitrary element $[(p_0,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R}$ and let $[(r,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R}$, such that $r > 0$:

$$[(p_0, 1)]_{\sim_{\mathbb{Q}}} - [(r, 1)]_{\sim_{\mathbb{Q}}} < [(p_0, 1)]_{\sim_{\mathbb{Q}}}.$$

Therefore, as for each pair $[(p_a,1)]_{\sim_{\mathbb{Q}}}, [(p_b,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R}$, with $[(p_a,1)]_{\sim_{\mathbb{Q}}} < [(p_b,1)]_{\sim_{\mathbb{Q}}}$, there exists a $[(p_k,n)]_{\sim_{\mathbb{Q}}}$ such that $[(p_a,1)]_{\sim_{\mathbb{Q}}} < [(p_k,n)]_{\sim_{\mathbb{Q}}} < [(p_b,1)]_{\sim_{\mathbb{Q}}}$, we have:

$$\forall [(p_0, 1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R} \exists [(p_k, n)]_{\sim_{\mathbb{Q}}} : [(p_0, 1)]_{\sim_{\mathbb{Q}}} - [(r, 1)]_{\sim_{\mathbb{Q}}} < [(p_k, n)]_{\sim_{\mathbb{Q}}} < [(p_0, 1)]_{\sim_{\mathbb{Q}}}.$$

This implies that:

$$[(p_k, n)]_{\sim_{\mathbb{Q}}} \in (([(p_0, 1)]_{\sim_{\mathbb{Q}}} - [(r, 1)]_{\sim_{\mathbb{Q}}}, [(p_0, 1)]_{\sim_{\mathbb{Q}}} + [(r, 1)]_{\sim_{\mathbb{Q}}}) \setminus \{[(p_0, 1)]_{\sim_{\mathbb{Q}}}\}) \cap \mathbb{Q},$$

where $([(p_0,1)]_{\sim_{\mathbb{Q}}} - [(r,1)]_{\sim_{\mathbb{Q}}}, [(p_0,1)]_{\sim_{\mathbb{Q}}} + [(r,1)]_{\sim_{\mathbb{Q}}})$ denotes an open interval.

Therefore:

$$(([(p_0, 1)]_{\sim_{\mathbb{Q}}} - [(r, 1)]_{\sim_{\mathbb{Q}}}, [(p_0, 1)]_{\sim_{\mathbb{Q}}} + [(r, 1)]_{\sim_{\mathbb{Q}}}) \setminus \{[(p_0, 1)]_{\sim_{\mathbb{Q}}}\}) \cap \mathbb{Q} \neq \emptyset.$$

Hence, $[(p_0,1)]_{\sim_{\mathbb{Q}}}$ is an accumulation point of $\mathbb{Q}$.

Since we had chosen an arbitrary element $[(p_0,1)]_{\sim_{\mathbb{Q}}} \in \mathbb{R}$, this implies that all $[(p_0,1)]_{\sim_{\mathbb{Q}}}$ reals are accumulation points of $\mathbb{Q}$. Then: $\mathbb{Q}' = \mathbb{R}$, and therefore, as $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$, we can conclude that $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

Conclusions

We have constructed a metric space within the context of the treon space without the need to explicitly define a traditional metric. Instead, we derived the metric from the Bermejian product, where it is implicitly defined. This approach enabled us to obtain a metric space, which facilitated the definition of epsilon-balls centered at an arbitrary treon. Using these balls, we were able to construct a basis for the topology induced by the Bermejian metric.

We defined a countable basis by leveraging the density and countability of rational numbers within the real numbers.

The theoretical development we presented confirmed that the treon topological space was indeed second-countable, as there existed a countable basis.

The construction of a countable basis will provide a powerful tool for the topological and geometric analysis of spaces defined by Bermejian Algebras, opening new possibilities for the definition and study of manifolds in these spaces. Second-countability will be a fundamental property that ensures our topological structures are manageable from an analytical and geometric perspective, facilitating the construction of more complex structures such as differentiable manifolds.

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