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Posted Date: 30 July 2024

doi: 10.20944/preprints202407.2192.v2

Keywords: collatz; 3x+1

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Crafting Integers That 'May' Diverge for Collatz Sequence

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Abstract: Let an odd integer be represented as $\sum_{n>m} 2^n + 2^m - 1$ for $m \geq 1$. To transform $2^m - 1$ according to the Collatz rules, the $3x + 1$ step is applied, resulting in the even integer $2^{m+1} + 2^m - 2$. This even integer, being exactly once divisible by 2, becomes the next odd integer $2^m + 2^{m-1} - 1$. This represents the growth phase, as exactly one even step follows an odd step. However, while the overall value of the integer increases, terms with decreasing indices are generated after each odd-even cycle. After m such cycles, a term 2^{m-m} (which equals 1) is generated, canceling the negative 1. Additional even steps are then performed until the integer becomes odd again, which occurs when the next smallest index becomes zero. The positive 1 thus obtained is written as $2^1 - 1$. If uninterrupted, the term $2^1 - 1$ has the trivial cycle $2^1 - 1 \to 2^2 \to 2^1 \to 2^1 - 1$. This represents the shrinkage phase, as two even steps follow an odd step. To interrupt the trivial cycle, a sequence such as $2^1 + 2^2 + \cdots$ must be available to combine with $2^1 - 1$ and produce $2^{\mathcal{M}} - 1$. This starts another growth phase that lasts for $\mathcal{M}$ odd-even cycles. Finally, there is an attempt to craft an integer that can escape the shrinkage phase in the Collatz dynamics.

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1. Introduction

The Collatz problem [1–4], defines the following set of rules: If x is odd, multiply it by 3, and add 1. If x is even, it is divided by 2.

The associated Collatz conjecture states that every integer ultimately reduces to unity. To prove this conjecture, it must be shown not only that the sequence eventually cycles through 1, 4, 2, 1, but also that no integer diverges to infinitely larger integers [5,6].

While a complete proof may be impossible, this article attempts to understand the working of Collatz-type sequences. For this, the odd integers are expressed as modified binary expressions $\sum_{n>m} 2^n + 2^m - 1$ for $m \geq 1$. By examining the integers that result from this seed form, insights are gained into the patterns of the odd-even steps. The conditions that govern the progression of Collatz-type sequences are immediately made clear through the use of the modified binary form ending with $2^m - 1$.

2. Behavior of Collatz Sequence $3n + 1$

Let x be an odd integer if the form $\sum_{n>m} 2^n + 2^m - 1$ for $m \geq 1$. The term $2^m - 1$ is crucial because it governs the sequence's behavior. Although calculations will only explicitly show the evolution of this term, it is implied that the full odd integer, including higher index terms, is present but not explicitly written out.

Let $\mathcal{O}\{1\}, \mathcal{O}\{2\}, \ldots$ denote the resulting integer at the end of the odd step while $\mathcal{E}\{1\}, \mathcal{E}\{2\}, \ldots$ denote the resulting integer at the end of the even step. The modified binary expression of integers at the end of each step is:

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$$
\begin{aligned}
x &= 2^m - 1 \\
\mathcal{O}\{1\} &= 2^{m+1} + 2^m - 2 \\
\mathcal{E}\{1\} &= 2^m + 2^{m-1} - 1 \\
\mathcal{O}\{2\} &= 2^{m+2} + 2^{m-1} - 2 \\
\mathcal{E}\{2\} &= 2^{m+1} + 2^{m-2} - 1 \\
\mathcal{O}\{3\} &= 2^{m+2} + 2^{m+1} + 2^{m-1} + 2^{m-2} - 2 \\
\mathcal{E}\{3\} &= 2^{m+1} + 2^m + 2^{m-2} + 2^{m-3} - 1 \\
\mathcal{O}\{4\} &= 2^{m+3} + 2^{m+1} + 2^{m-3} - 2 \\
\mathcal{E}\{4\} &= 2^{m+2} + 2^m + 2^{m-4} - 1
\end{aligned}
$$

The following observations are made:

- Under normal conditions, each $\mathcal{O}$ step is followed by exactly one $\mathcal{E}$ step.
- The value of integer increases after every $\mathcal{OE}$ step, since each $\mathcal{O}$ step is followed by exactly one $\mathcal{E}$ step. It is also evident from increase in number of terms and increase in the highest index value. This is termed as the growth phase of the integer.
- However, lower value indices also appear. The value of the lowest index decreases after every $\mathcal{OE}$ step.
- Consequently, after m $\mathcal{OE}$ steps, the lowest index becomes $m - m$, that is, zero.

Let the expression obtained after m $\mathcal{OE}$ steps be

$$
\mathcal{E}\{m\} = \cdots + 2^{m-q} + 2^{m-p} + 2^0 - 1 \tag{1}
$$

Where 2^{m-p} and 2^{m-q} are the next terms in order and $p, q < m$.

The integer obtained at (1) is even as 2^0 cancels the negative 1. The lowest common factor of this integer is 2^{m-p} and $(m-p)$ additional $\mathcal{E}$ steps are carried until the integer becomes odd.

$$
\begin{aligned}
\mathcal{E}\{m\}\mathcal{E}^{(m-p)} &= \cdots + 2^{p-q} + 1 \\
&= \cdots + 2^{p-q} + 2^1 - 1
\end{aligned}
$$

Here it is assumed that $2^{p-q} > 2^1$.

The integer ends in $2^1 - 1$, for which the trivial cycle is $2^1 - 1 \to 2^2 \to 2^1 \to 2^1 - 1$. Since two $\mathcal{E}$ steps follow one $\mathcal{O}$ step in the trivial cycle, the value of the integer decreases and this is termed as the shrinkage phase. If the term $2^1 - 1$ is undisturbed, the shrinkage phase follows the following trivial cycle:

$$
\begin{aligned}
\mathcal{O} &= \cdots + 2^{p-q} + 2^2 \\
\mathcal{OE} &= \cdots + 2^{p-q-1} + 2 \\
\mathcal{OEE} &= \cdots + 2^{p-q-2} + 2 - 1
\end{aligned}
$$

Following three scenarios are possible:

- Case 1: The value of 2^{p-q} is 2^2, in which case the integer ends in 2^3. Three even steps follow and the resulting 2^0 is expressed as $2^1 - 1$.

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- Case 2: The value of 2^{p-q-1} is reduced to 1, in which case, the terms $2^{p-q-1} + 2$ are expressed as $2^2 - 1$.
- Case 3: The value of 2^{p-q-2} is reduced to 2, in which case, the terms $2^{p-q-2} + 2 - 1$ are expressed as $2^2 - 1$, which is same as case 2.

It is seen that the dynamics of Collatz is such that the additional $\mathcal{E}$ steps occur after a fewer $\mathcal{OE}$ cycle once the first 2^0 term is reached. The trivial cycle that is set in, causes the next smallest index to reduce to zero, leading to a domino effect. All the three scenarios discussed above lead back to trivial cycle and, consequently, the shrinkage phase.

There are two ways an integer can escape the said dynamics of the Collatz:

- m is infinite.
- Alternatively, if every time the binary expression of an integer ends in positive 1 and is re-written as $2^1 - 1$, there is a sequence of $2^1 + 2^2 + 2^3 + \cdots$ that combines with 2^1 to produce an index larger than m.

The next section will focus on the latter method.

3. Crafting Integers That 'May' Diverge

Let the seed integer be of the form $x = 2^b + 2^a - 1$. As a starting point, let $a = 1$. The integer obtained after applying $\mathcal{OEE}$ is:

$$\mathcal{OEE}(x) = 2^{b-1} + 2^{b-2} + 2 - 1$$

If the terms $2^{b-1} + 2^{b-2}$ are set to $4 + 2$ for $b = 3$, the integer obtained after $\mathcal{OEE}$ becomes $\mathcal{OEE}(x) = 2^3 - 1$. That is, the lowest index increased from 1 to 3, instead of decreasing. The seed integer in this case is 9.

Next, let the seed integer be $x = 2^c + 2^3 + 2^1 - 1$. The integer obtained after applying $\mathcal{OEE}$ is $2^3 - 1$ which is followed by $\mathcal{OEOEOEE}$. Therefore, integer obtained after applying $\mathcal{OEEOEOEOEE}$ is:

$$\begin{aligned}
\mathcal{OEEOEOEOEE}(x) &= \mathcal{OEEOEOEOEE}(2^a) + 2^4 + 2^3 + 2^1 \\
&= 2^{a+1} + 2^{a-1} + 2^{a-5} + 2^4 + 2^3 + 2^1 \\
\mathcal{OEEOEOEOEEE}(x) &= 2^a + 2^{a-2} + 2^{a-6} + 2^3 + 2^2 + 2 - 1
\end{aligned}$$

The integer is reduced to odd value in last step. If a is let equal to 7, the resulting integer becomes $2^7 + 2^5 + 2^4 - 1$. Here again, the lowest index increased in value from 1 to 3 and to finally 4. The seed integer in this case is 137.

Higher indices can be calculated in a similar manner. Changing the value of a will yield different set of seed integers. Another example of such an integer is 27 where a is equal to 2

The integers crafted in this manner undergo multiple growth phases. However, all growth phases ultimately lead to a shrinkage phase due to the sequential nullification of indices. This implies that to defy the Collatz dynamics and avoid the shrinkage phase, the seed integer must possess an infinite number of engineered indices that continuously restart the growth phase indefinitely.

4. Conclusion

The transformation process of the odd integer $\sum_{n>m} 2^n + 2^m - 1$ under the Collatz rules of $3x + 1$ can be broken down into the following regimes:

 doi:10.20944/preprints202407.2192.v2

- **Growth phase**: This phase occurs when the odd integer does not have $2^1 - 1$ as the ending term. One odd step is followed by exactly one even step. The value of the highest index, as well as the number of terms in the expression, increases. However, terms with decreasing indices are also produced.
- **End index nullification**: After m odd-even cycles, a term of the form 2^{m-m} is produced, which cancels the negative 1. The resulting integer is even, and division by 2 continues until the next smallest index reduces to zero, making the integer odd again.
- **Trivial cycle & Shrinkage phase**: The index that reduces to zero is written as $2^1 - 1$. If left undisturbed, $2^1 - 1$ generates the trivial cycle $2^1 - 1 \rightarrow 2^2 \rightarrow 2^1 \rightarrow 2^1 - 1$, which has one odd step followed by two even steps. This causes the value of the integer to shrink.

The dynamics of the Collatz sequence allow for the creation of integers where the smallest index increases before entering the trivial cycle. Examples of such integers include 9, 137, and 27. Although these crafted integers undergo multiple growth phases, all growth phases eventually lead to a shrinkage phase. This suggests that for an integer to diverge to infinity in the Collatz $3x + 1$ sequence, it must be infinite.

Data Availability Statement: Data availability is not applicable to this article as no new data were created or analysed in this study.

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