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Posted Date: 31 July 2024

doi: 10.20944/preprints202407.2480.v1

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# Consensus of T-S Fuzzy Fractional Order Singular Perturbation Multi-Agent Systems

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**Abstract:** This article focuses on the leader-following consensus of fuzzy fractional order singular perturbation multi-agent systems (FOSPMASs) with order in $(0, 2)$. By employing the T-S fuzzy modeling approach, a fuzzy FOSPMAS is constructed. Subsequently, a fuzzy observer-based controller is designed and the error system corresponding to each agent is derived. Through a series of equivalent transformations, the error system is decomposed into fuzzy singular fractional order systems (SFOSs). According to the admissibility of SFOSs, the consensus conditions of the fuzzy FOSPMAS are obtained based on linear matrix inequalities (LMIs) without equality constraint. Finally, the effectiveness of the criteria is verified through an RLC circuit model.

**Keywords:** fuzzy systems; singular perturbation systems; multi-agent systems; consensus

## 1. Introduction

In recent decades, the control towards multi-agent systems (MASs) has become a leading research subject, stemming from the superior efficiency of multiple agents collaborating to execute tasks compared to an individual agent [1]. MASs hold significant applications spanning various domains, including service robotics [2,3], hazardous environment detection [4], and unmanned aerial vehicle formation flying [5]. Consensus control of MASs is a fundamental and core issue based on tracking control [6,7]. A significant amount of research has emerged on the consensus of MASs [8–12]. Ren [8] constructed MASs with second-order integrator dynamics by analyzing the swarming model and designed a consensus protocol. Tian and Liu [9] attained two decentralized consensus conditions of MASs with diverse input and communication delays. Wen et al. [10] introduced an innovative protocol designed by using synchronous intermittent local feedback for second-order consensus of MASs. Zhang et al. [11] proposed event-trigger output feedback control approaches, enabling that all connected communication graphs reach consensus. Tan et al. [12] derived the consensus criteria for cyber-physical systems under sampled data control, employing suitable Lyapunov function. The above studies predominantly concentrate on achieving consensus of MASs with integer order, which is difficult to describe the actual systems in nature and industry.

Fractional order systems (FOSs) are capable of more accurately modeling and computing genetic and memory effects in various complex processes than integer order systems [13]. Similarly, the consensus of fractional order MASs (FOMASs) has attracted widespread interest [14–19]. Su and Ye proposed a control strategy with input delays to achieve the consensus of general linear and nonlinear FOMASs under event-triggered in [14,15], respectively. Yang et al. [16] considered the consensus of nonlinear distributed and input delayed FOMASs, and further explored the performance of FOMASs in terms of leader-following and leaderless global consensus in [17]. Hu et al. [18] developed an adaptive controller that employed an event-triggered scheme without Zeno behavior, aiming to realize the consensus of FOMASs. Bahrampour et al. [19] proposed new Lyapunov-based LMIs conditions to determine the state feedback controller gains on the distributed consensus control of heterogeneous FOMASs with interval uncertainties. However, many practical MASs exhibit multiple time-scale characteristics, which refer to the coupled coexistence of fast dynamics and slow dynamics. The design of controllers for these systems frequently encounters difficulties due to the presence of high dimensionality and pathological values [20,21].

Singular perturbation systems (SPSs) have multiple time-scale and inherently pathological dynamical properties [22–24]. SPSs with certain parasitic parameter $\varepsilon$ are modeled to describe real systems. In power system modeling, $\varepsilon$ is used to represent transient phenomena in machine reactors or voltage regulators [25]. In industrial control systems, it signifies small time constants between control and response [26]. Numerous scholars have intensively studied SPSs [27–32]. On one hand, two commonly employed strategies for solving control problems of SPSs are the quasi-steady-state method [27] and the block diagonalization method [28], which decompose the system into slow and fast subsystems. But these methods rely on the assumption that the fast subsystem matrix is nonsingular and are not applicable to non-standard SPSs that cannot be easily decomposed. On the other hand, Yang et al. [29], Gao et al. [30] and Liu et al. [31] proposed the integral sliding mode control method for full-order SPSs with mismatched disturbances, uncertainty and nonlinear input, respectively. Their methods are based on a full-order model, which significantly eliminates the need to decompose the system. Furthermore, techniques such as Lyapunov functions and LMIs are also applied to system analysis. Fridman [32] derived the LMIs criteria for the stability of SPSs for delay proportional to $\varepsilon$ and delay independent of $\varepsilon$, respectively. Additionally, for singular perturbation MASs (SPMASs), both Ben Rejeb et al. [33] and Tognetti et al. [34] designed the decentralized controllers, enabling systems to synchronize and ensuring global performance. Xu et al. [35] presented the sliding-mode controller with memory output for addressing consensus of SPMASs in finite time. Zhang et al. [36] achieved global Mittag-Leffler consensus tracking for fractional SPMASs modeled by discontinuous function with nondecreasing property. However, in practical application, the exact value of the parameter $\varepsilon$ is often difficult to obtain directly. By analyzing the background information of specific problems in depth, the reasonable change range of $\varepsilon$ is effectively estimated. Given $\varepsilon$ in known interval , the design of controllers for achieving consensus of nonlinear FOSPMASs remains an open problem in the field of control theory.

T-S fuzzy models possess the capability of approximating nonlinear dynamics, thereby the well-established control methods for linear systems is extended to the analysis and design of nonlinear systems. Therefore, numerous scholars have done extensive research endeavors focusing on T-S fuzzy SPSs [37–40]. Yang and Zhang [37] proposed a design method of state feedback controller depending on $\varepsilon$ for T-S fuzzy SPSs. Chen et al. [38] focused on nonlinear SPSs and presented two novel methods to design static output feedback $H_\infty$ controller based on LMIs. Visavakitcharoen et al. [39] designed an event-triggered controller based on integral feedback for nonlinear SPSs with a fuzzy model. Zhang and Han [40] proposed two diverse feedback controllers aiming to attain the stabilization criteria of fuzzy FOSPSs with order $\alpha \in (0,1)$. Nevertheless, the research on the consensus control of fuzzy fractional order singular perturbation MASs (FOSPMASs) is still relatively limited.

Inspired by previous discussions, this paper focuses on filling this research gap. The following is an overview of the main contributions of this research:

1. To provide a more accurate portrayal of complex systems in practice, a T-S fuzzy FOSPMAS with $\alpha \in (0,2)$ is formulated to reduce the difficulty of directly studying nonlinear systems. Compared to integer order systems, the constructed model exhibits more enhanced accuracy and complexity. A fuzzy FOSPS with error as a variable is derived by designing a fuzzy observer-based controller.
2. The fuzzy FOSPS is analyzed by transforming it into a fuzzy SFOS using the system augmentation method. In comparison to the existing work [41], the proposed approach not only relaxes the assumption that the fast subsystem matrix must be nonsingular, but also avoids the ill-conditioned issue arising from the parameter $\varepsilon$.
3. The consensus conditions for fuzzy FOSPMASs with $\alpha \in (0,1)$ and $[1,2)$ are formulated in this study for any $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$, where $\underline{\varepsilon}$ and $\overline{\varepsilon}$ are given lower and upper boundaries, respectively. The results are presented based on LMIs without equality constraints, reducing solution difficulties. It is demonstrated through an RLC circuit model that the proposed methods are effective in practice.

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The remaining parts are structured in the following manner: Section 2 provides foundational definitions in graph theory and correlative lemmas. The establishment of system model and the primary findings on the consensus of FOSPMASs are detailed in Section 3. Section 4 presents two practical examples. Lastly, Section 5 summarizes the study.

**Notations:** $X > 0$ and $X \geqslant 0$ signify that the matrix $X$ is positive definite and positive semi-definite, respectively. $X^T$ stands for transpose of the matrix $X$, and $\text{sym}\{X\} = X + X^T$. $\text{spec}\,(E, A)$ is the spectrum of $\det(s^\alpha E - A) = 0$. Symbol $*$ is the symmetric element of a matrix. $\otimes$ denotes the Kronecker product. For $\alpha \in (0, 2)$, $a = \sin\left(\alpha\frac{\pi}{2}\right), b = \cos\left(\alpha\frac{\pi}{2}\right)$, $\Theta$ denotes $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. $\text{diag}(\cdot)$ represents a diagonal matrix. $\lceil\alpha\rceil$ stands for rounding $\alpha$ up to the nearest integer.

## 2. Preliminaries

### 2.1. Graph Theory

Consider the case of an MAS comprising a single leader and $N$ followers. The information exchanged between $N$ agents is presented by an undirected graph $\mathcal{G}$. The Laplace matrix of the graph $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ij} = \begin{cases} -a_{ij}, i = j \\ \sum_{j=1}^{N} a_{ij}, i \neq j \end{cases}$ and $a_{ij}$ is the element of weighted adjacency matrix $\mathcal{A}$ of graph $\mathcal{G}$. $a_{ij} > 0$ means that follower $i$ communicates with follower $j$, otherwise $a_{ij} = 0$. When $\mathcal{G}$ is undirected, $\mathcal{A}$ is symmetric. Similarly, $h_i$ represents the communication between the leader and follower $i$, and $h_i > 0$ means follower $i$ receives information from the leader, or else $h_i = 0$.

### 2.2. Preliminary Lemmas

Consider a continuous linear singular FOS (SFOS) with $\alpha \in (0, 2)$ described by

$$ED^\alpha x(t) = Ax(t), \tag{1}$$

where $A, E \in \mathbb{R}^{n\times n}$ are the system matrices, $\text{rank}(E) = m < n$. $x(t) \in \mathbb{R}^n$ represents the state. $D^\alpha$ denotes the Caputo fractional order derivative, defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} \int_0^t \frac{f^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha+1-\lceil\alpha\rceil}} d\tau,$$

where $\Gamma(\cdot)$ is the Euler Gamma function. Indicate (1) by the triple $(E, A, \alpha)$.

When $E = I$, system (1) is simplified to a normal FOS

$$D^\alpha x(t) = Ax(t). \tag{2}$$

**Lemma 1.** *[42] System (2) is stable iff* $|\arg(\text{spec}(A))| > \alpha\frac{\pi}{2}$.

**Lemma 2.** *[43] Choose two nonsingular matrices U and V such that*

$$UEV = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}, UAV = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \tag{3}$$

*then system (1) is regular, impulse-free and stable, defined admissible, iff* $A_4$ *is nonsingular and*

$$\left|\arg\left(\text{spec}\left(A_1 - A_2A_4^{-1}A_3\right)\right)\right| > \alpha\frac{\pi}{2}.$$

**Lemma 3.** *[44] System (1) is admissible with* $\alpha \in [1, 2)$ *iff there exists a matrix* $P \in \mathbb{R}^{n\times n}$ *satisfying*

$$EP = P^TE^T \geqslant 0, \ \text{sym}\{\Theta \otimes AP\} < 0.$$

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**Lemma 4.** *[43] System (1) is admissible with $\alpha \in (0,1)$ iff there exist two matrices $X, Y \in \mathbb{R}^{n\times n}$ satisfying*

$$\begin{bmatrix} EX & EY \\ -EY & EX \end{bmatrix} = \begin{bmatrix} X^T E^T & -Y^T E^T \\ Y^T E^T & X^T E^T \end{bmatrix} \geqslant 0,\ \mathrm{sym}\{A(aX - bY)\} < 0.$$

**Lemma 5.** *[45] Given a symmetric constant matrix $Z$ and constant matrices $U, V$, the inequality*

$$Z + UFV + V^T F^T U^T < 0$$

*holds for all $F$ satisfying $F^T F \leqslant S$ iff there exist some $\rho > 0$ such that*

$$Z + \begin{bmatrix} \rho^{-1} V^T & \rho U \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \rho^{-1} V \\ \rho U^T \end{bmatrix} < 0.$$

## 3. Main Results

### 3.1. System Model Description

Consider an MAS consisting of a leader and $N$ followers, and the dynamic of each agent is modeled by a T-S fuzzy FOSPS. This nonlinear system is described by the fuzzy rules as follows:

Rule $k$: IF $\xi_1(t)$ is $\Pi_{k1}$ and $\cdots$ and $\xi_p(t)$ is $\Pi_{kp}$,

THEN the dynamic description of each agent is written as

$$\begin{cases} E(\varepsilon) D^\alpha x_i(t) = A_k x_i(t) + B_k u_i(t) \\ y_i(t) = C_k x_i(t) \end{cases} \tag{4}$$

$$\begin{cases} E(\varepsilon) D^\alpha x_0(t) = A_k x_0(t) \\ y_0(t) = C_k x_0(t) \end{cases} \tag{5}$$

where

$$E(\varepsilon) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix},\ n_1 + n_2 = n,\ i = 1, 2, \cdots, N.$$

$x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^l$ and $y_i(t) \in \mathbb{R}^v$ represent the state, control input and output of follower $i$, respectively. $x_0(t) \in \mathbb{R}^n$ and $y_0(t) \in \mathbb{R}^v$ represent the state and output of leader. The system matrices $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times l}$, and $C_k \in \mathbb{R}^{v\times n}$ are constant. Additionally, $\Pi_{kj}$ are the fuzzy sets of the premise variables $\xi_j(t)$, where $j = 1, \cdots, p$ and $k = 1, \cdots, r$. Here, $r$ represents the number of rules.

According to procedures of defuzzification, the global model of the T-S fuzzy FOSPMAS (4)-(5) is derived:

$$\begin{cases} E(\varepsilon) D^\alpha x_i(t) = \sum_{k=1}^r \eta_k(\xi)[A_k x_i(t) + B_k u_i(t)] \\ y_i(t) = \sum_{k=1}^r \eta_k(\xi) C_k x_i(t) \end{cases} \tag{6}$$

$$\begin{cases} E(\varepsilon) D^\alpha x_0(t) = \sum_{k=1}^r \eta_k(\xi) A_k x_0(t) \\ y_0(t) = \sum_{k=1}^r \eta_k(\xi) C_k x_0(t) \end{cases} \tag{7}$$

where $\eta_k(\xi) = \frac{\prod_{j=1}^p \omega_{kj}(\xi_j(t))}{\sum_{k=1}^r \prod_{j=1}^p \omega_{kj}(\xi_j(t))}$ is the weighting function, and $\omega_{kj}(\xi_j(t))$ is the membership function, satisfying $\eta_k(\xi(t)) \geqslant 0$, $\sum_{k=1}^r \eta_k(\xi(t)) = 1$, $\prod_{j=1}^p \omega_{kj}(\xi_j(t)) \geqslant 0$ and $\sum_{k=1}^r \prod_{j=1}^p \omega_{kj}(\xi_j(t)) > 0$ .

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Utilizing the complete state information for controller design is often challenging owing to economic constraints and measurement limitations. To address this issue and design a consensus protocol for the fuzzy FOSPMAS (6)-(7), a fuzzy observer is formulated as

$$\begin{cases} E(\varepsilon)D^{\alpha}\hat{x}_i(t) = \sum_{k=1}^{r}\eta_k(\xi)A_k\hat{x}_i(t) + \sum_{k=1}^{r}\eta_k(\xi)B_k u_i(t) + z_i(t) \\ z_i(t) = \sum_{s=1}^{r}\eta_s(\xi)W_s\left[\sum_{j=1}^{N}a_{ij}(\tilde{y}_i(t)-\tilde{y}_j(t)) + h_i\tilde{y}_i(t)\right] \end{cases} \tag{8}$$

where $\hat{x}_i(t)$ signifies the estimated state of the follower $i$, and $\tilde{y}_i(t) = y_i(t) - \sum_{k=1}^{r}\eta_k(\xi)C_k\hat{x}_i(t)$ represents the error between the actual output $y_i$ and the weighted sum of estimated outputs. Furthermore, $W_s \in \mathbb{R}^{n\times v}$ denotes the gain matrix.

To achieve the consensus of (6)-(7), a distributed control protocol based on (8) is designed:

$$u_i(t) = \sum_{q=1}^{r}\eta_q(\xi)K_q\left[\sum_{j=1}^{N}a_{ij}(\hat{x}_i(t)-\hat{x}_j(t)) + h_i(\hat{x}_i(t)-x_0(t))\right]. \tag{9}$$

Let $x_{ei}(t) = x_i(t) - x_0(t)$ and $\hat{x}_{ei}(t) = x_i(t) - \hat{x}_i(t)$. By substituting (9) into (6) and subtracting (6) from (8), the error system is written as

$$\begin{aligned} E(\varepsilon)D^{\alpha}x_{ei}(t) = &\sum_{k=1}^{r}\eta_k(\xi)A_k x_{ei}(t) + \sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q \\ &\cdot\left[\sum_{j=1}^{N}a_{ij}\big(x_{ei}(t)-x_{ej}(t)-(\hat{x}_{ei}(t)-\hat{x}_{ej}(t))\big) + h_i(x_{ei}(t)-\hat{x}_{ei}(t))\right], \end{aligned} \tag{10}$$

$$\begin{aligned} E(\varepsilon)D^{\alpha}\hat{x}_{ei}(t) = &\sum_{k=1}^{r}\eta_k(\xi)A_k\hat{x}_{ei}(t) - \sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)W_sC_k \\ &\cdot\left[\sum_{j=1}^{N}a_{ij}(\hat{x}_{ei}(t)-\hat{x}_{ej}(t)) + h_i\hat{x}_{ei}(t)\right]. \end{aligned} \tag{11}$$

Let $\bar{x}_e(t) = \begin{bmatrix} x_e^T(t) & \hat{x}_e^T(t)\end{bmatrix}^T$, where

$$x_e(t) = \begin{bmatrix} x_{e1}^T(t), & \cdots, & x_{eN}^T(t)\end{bmatrix}^T, \quad \hat{x}_e(t) = \begin{bmatrix}\hat{x}_{e1}^T(t), & \cdots, & \hat{x}_{eN}^T(t)\end{bmatrix}^T.$$

The compact form of system (10)-(11) is

$$\begin{aligned} (I_N\otimes E(\varepsilon))D^{\alpha}x_e(t) = &\left(I_N\otimes\sum_{k=1}^{r}\eta_k(\xi)A_k\right)x_e(t) \\ &+ M\otimes\left(\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q\right)(x_e(t)-\hat{x}_e(t)), \end{aligned} \tag{12}$$

$$(I_N\otimes E(\varepsilon))D^{\alpha}\hat{x}_e(t) = \left(I_N\otimes\sum_{k=1}^{r}\eta_k(\xi)A_k\right)\hat{x}_e(t) - M\otimes\left(\sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)W_sC_k\right)\hat{x}_e(t). \tag{13}$$

where $M = \mathcal{L} + \text{diag}(h_1, h_2, \cdots, h_N)$.

By combining (12)-(13), the error system is described as

$$\bar{E}(\varepsilon)D^{\alpha}\bar{x}_e(t) = \bar{A}\bar{x}_e(t), \tag{14}$$

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where

$$\bar{E}(\varepsilon)=\begin{bmatrix} I_N\otimes E(\varepsilon) & 0\\ 0 & I_N\otimes E(\varepsilon)\end{bmatrix},\ \bar{A}=\begin{bmatrix}\bar{A}_1 & \bar{A}_2\\ 0 & \bar{A}_4\end{bmatrix},$$
$$\bar{A}_1=I_N\otimes\sum_{k=1}^{r}\eta_k(\xi)A_k+M\otimes\left(\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q\right),$$
$$\bar{A}_2=-M\otimes\left(\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q\right),$$
$$\bar{A}_4=I_N\otimes\sum_{k=1}^{r}\eta_k(\xi)A_k-M\otimes\left(\sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)W_sC_k\right).$$

### 3.2. Equivalent Transformations

In this section, equivalence conditions of the consensus of fuzzy FOSPMASs are derived by addressing the stability problem of system (14).

Based on graph theory, it is known that matrix $M$ is positive definite. Therefore, an orthogonal matrix $V$ exists such that $V^TMV=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_N)$ and all eigenvalues $\lambda_i$ of the matrix $M$ possess positive real parts, where $i=1,2,\cdots,N$.

Let $\tilde{X}_e(t)=\begin{bmatrix}X_e(t)^T & \hat{X}_e(t)^T\end{bmatrix}^T$, where $X_e=(V^T\otimes I_N)x_e(t)$, $\hat{X}_e=(V^T\otimes I_N)\hat{x}_e(t)$. According to the properties of Kronecker product, system (14) is transformed into the subsequent form:

$$\tilde{E}(\varepsilon)D^\alpha\tilde{X}_{ei}(t)=\tilde{A}\tilde{X}_{ei}(t), \tag{15}$$

where

$$\tilde{E}(\varepsilon)=\begin{bmatrix} E(\varepsilon) & 0\\ 0 & E(\varepsilon)\end{bmatrix},\ \tilde{A}=\begin{bmatrix}\tilde{A}_1 & \tilde{A}_2\\ 0 & \tilde{A}_4\end{bmatrix},$$
$$\tilde{A}_1=\sum_{k=1}^{r}\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q,$$
$$\tilde{A}_2=-\lambda_i\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q,$$
$$\tilde{A}_4=\sum_{k=1}^{r}\eta_k(\xi)A_k-\lambda_i\sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)W_sC_k.$$

In order to analyze the stability of system (15), two independent SPSs are constructed as follows:

$$E(\varepsilon)D^\alpha X_{ei}(t)=\tilde{A}_1X_{ei}(t), \tag{16}$$

$$E(\varepsilon)D^\alpha\hat{X}_{ei}(t)=\tilde{A}_4\hat{X}_{ei}(t). \tag{17}$$

**Lemma 6.** *System (15) is stable iff both systems (16) and (17) are stable.*

**Proof.** According to Lemma 1, system (15) is stable iff $\left|\arg\left(\mathrm{spec}\left(\tilde{E}^{-1}(\varepsilon)\tilde{A}\right)\right)\right|>\frac{\pi}{2}\alpha$.

Factorize the characteristic determinant of the system (15) as

$$\det\left(s^\alpha\tilde{E}(\varepsilon)-\tilde{A}\right)=\det\left(s^\alpha E(\varepsilon)-\tilde{A}_1\right)\times\det\left(s^\alpha E(\varepsilon)-\tilde{A}_4\right).$$

It implies that the stability of system (15) is dependent on the stability of two subsidiary systems (16) and (17). Thus, system (15) is stable iff systems (16) and (17) are simultaneously stable. □

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**Definition 1.** *The consensus of T-S fuzzy FOSPMAS (6)-(7) is achieved via protocol (9) if*

$$\lim_{t\to\infty}\|x_i(t)-x_0(t)\|=0,\ i=1,2,\cdots,N.$$

From Definition 1 and Lemma 6, for achieving the consensus of fuzzy FOSPMAS (6)-(7), it is necessary for both systems (16) and (17) to be stable. Therefore, in order to derive the stability conditions of (16) and (17), the following equivalent transformation is presented.

The matrix $E(\varepsilon)$ is decomposed into

$$E(\varepsilon)=E_1+(\varepsilon-\beta)E_2,$$

where the scalar $\beta>0$, $E_1=\begin{bmatrix}I_{n_1} & 0\\ 0 & \beta I_{n_2}\end{bmatrix}$, $E_2=\begin{bmatrix}0 & 0\\ 0 & I_{n_2}\end{bmatrix}$.

Let $f_1(t)=D^\alpha X_{ei}(t)$, $x_z(t)=\left[X_{ei}^T(t),f_1^T(t)\right]^T$. (16) is derived as

$$ED^\alpha x_z(t)=\left(\sum_{k=1}^{r}\eta_k(\xi)A_{k1}+(\varepsilon-\beta)A_2+\lambda_i\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)\bar{B}_k\bar{K}_q\right)x_z(t), \tag{18}$$

where $E=\begin{bmatrix}I_n & 0\\ 0 & 0\end{bmatrix}$, $A_{k1}=\begin{bmatrix}0 & I_n\\ A_k & -E_1\end{bmatrix}$, $A_2=\begin{bmatrix}0 & 0\\ 0 & -E_2\end{bmatrix}$, $\bar{B}_k=\begin{bmatrix}0\\ B_k\end{bmatrix}$, $\bar{K}_q=\begin{bmatrix}K_q & 0\end{bmatrix}$.

Similarly, denote $f_2(t)=D^\alpha\hat{X}_{ei}(t)$, $\hat{x}_z(t)=\left[\hat{X}_{ei}^T(t),f_2^T(t)\right]^T$. (17) is reformulated as

$$ED^\alpha\hat{x}_z(t)=\left(\sum_{k=1}^{r}\eta_k(\xi)\bar{A}_{k1}+(\varepsilon-\beta)A_2-\lambda_i\sum_{s=1}^{r}\sum_{k=1}^{r}\eta_s(\xi)\eta_k(\xi)\bar{W}_s\bar{C}_k\right)\hat{x}_z(t), \tag{19}$$

where $\bar{A}_{k1}=\begin{bmatrix}0 & A_k\\ I_n & -E_1\end{bmatrix}$, $\bar{W}_q=\begin{bmatrix}W_q\\ 0\end{bmatrix}$, $\bar{C}_k=\begin{bmatrix}0 & C_k\end{bmatrix}$.

**Lemma 7.** *With $\varepsilon>0$, system (16) is stable iff system (18) is admissible.*

**Proof.** Based on the aforementioned analysis, system (18) is reformulated as

$$ED^\alpha x_z(t)=\begin{bmatrix}0 & I_n\\ \sum_{k=1}^{r}\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q & -E(\varepsilon)\end{bmatrix}x_z(t). \tag{20}$$

According to Lemma 2, set $U=V=I$ in (3), then system (20) is admissible iff $E(\varepsilon)$ is nonsingular and

$$\left|\arg\left(\operatorname{spec}\left(E^{-1}(\varepsilon)\left(\sum_{k=1}^{r}\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q\right)\right)\right)\right|>\frac{\pi}{2}\alpha. \tag{21}$$

Rewrite system (16) as

$$D^\alpha X_{ei}(t)=E^{-1}(\varepsilon)\left(\sum_{k=1}^{r}\eta_k(\xi)A_k+\lambda_i\sum_{k=1}^{r}\sum_{q=1}^{r}\eta_k(\xi)\eta_q(\xi)B_kK_q\right)X_{ei}(t).$$

By Lemma 1, it is obtained from (21) that system (16) is stable iff system (18) is admissible. □

According to Lemma 7, the stability conditions of systems (16) and (17) are interpreted as the admissibility conditions for systems (18) and (19).

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### 3.3. Consensus Conditions of T-S Fuzzy FOSPMAS

In this section, the LMI criteria for consensus of fuzzy FOSPMAS (6)-(7) are proposed by studying the admissibility of systems (18) and (19).

**Theorem 1.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMAS (6)-(7) with $\alpha \in [1,2)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9), if there exist matrices $P_1$, $P_2$, $H_q$, $G_s$, and positive scalars $\mu_1$, $\mu_2$ such that*

$$EP_1 = P_1^T E^T \geqslant 0, \tag{22}$$

$$\Phi_{kk} < 0, k = 1, 2, \cdots, r, \tag{23}$$

$$\Phi_{kq} + \Phi_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{24}$$

$$E^T P_2 = P_2^T E \geqslant 0, \tag{25}$$

$$\Psi_{kk} < 0, k = 1, 2, \cdots, r, \tag{26}$$

$$\Psi_{ks} + \Psi_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{27}$$

*where*

$$\Phi_{kq} = \begin{bmatrix} \mathrm{sym}\{\Theta \otimes (A_{k1}P_1 + \lambda_i \bar{B}_k H_q)\} & I_2 \otimes P_1^T & \Theta \otimes \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_1 I_2 \end{bmatrix},$$

$$\Psi_{ks} = \begin{bmatrix} \mathrm{sym}\{\Theta \otimes (\bar{A}_{k1}^T P_2 - \lambda_i \bar{C}_k^T G_s^T)\} & I_2 \otimes P_2^T & \Theta \otimes \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_2 I_2 \end{bmatrix}.$$

*The gain matrices are derived as*

$$\bar{K}_q = H_q P_1^{-1}, \; \bar{W}_s = P_2^{-T} G_s.$$

**Proof.** Reformulate (18) with $\beta = \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}$ as

$$ED^\alpha x_z(t) = \left( \sum_{k=1}^{r} \eta_k(\xi) A_{k1} + \left( \varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2} \right) A_2 + \lambda_i \sum_{k=1}^{r} \sum_{q=1}^{r} \eta_k(\xi) \eta_q(\xi) \bar{B}_k \bar{K}_q \right) x_z(t). \tag{28}$$

From $\eta_k(\xi) \geqslant 0$, (23) and (24) ensure that the following inequality holds:

$$\begin{bmatrix} \Delta_1 & I_2 \otimes P_1^T & \mu_1(\Theta \otimes A_2) \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_1 I_2 \end{bmatrix} < 0, \tag{29}$$

where

$$\Delta_1 = \mathrm{sym}\left\{ \Theta \otimes \left( \sum_{k=1}^{r} \eta_k(\xi) A_{k1} P_1 + \lambda_i \sum_{k=1}^{r} \sum_{q=1}^{r} \eta_k(\xi) \eta_q(\xi) \bar{B}_k H_q \right) \right\}.$$

By pre- and post-multiplying (29) with $\mathrm{diag}\left(I, \frac{I}{\sqrt{\mu_1}}, \frac{I}{\sqrt{\mu_1}}\right)$, it is transformed as follows:

$$\begin{bmatrix} \Delta_1 & \frac{I_2 \otimes P_1^T}{\sqrt{\mu_1}} & \sqrt{\mu_1}(\Theta \otimes A_2) \\ * & -\frac{4}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -I_2 \end{bmatrix} < 0. \tag{30}$$

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According to the Schur complement, (30) is equivalent to

$$\Delta_1 + \begin{bmatrix} \frac{I_2 \otimes P_1^T}{\sqrt{\mu_1}} & \sqrt{\mu_1}(\Theta \otimes A_2) \end{bmatrix} \begin{bmatrix} \frac{(\bar{\varepsilon}-\underline{\varepsilon})^2}{4} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{I_2 \otimes P_1}{\sqrt{\mu_1}} \\ \sqrt{\mu_1}(\Theta \otimes A_2)^T \end{bmatrix} < 0. \tag{31}$$

Considering $\left|\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right| \leqslant \frac{\bar{\varepsilon}-\underline{\varepsilon}}{2}$ in (31) and Lemma 5, it gives

$$\Delta_1 + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right)\mathrm{sym}\{\Theta \otimes A_2 P_1\} < 0. \tag{32}$$

Substituting $\sum_{q=1}^{r} \eta_q(\xi) H_q = \sum_{q=1}^{r} \eta_q(\xi) \bar{K}_q P_1$ into (32), it yields the subsequent expression as

$$\mathrm{sym}\left\{\Theta \otimes \left(\left(\sum_{k=1}^{r} \eta_k(\xi) A_{k1} + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) A_2 + \lambda_i \sum_{k=1}^{r}\sum_{q=1}^{r} \eta_k(\xi)\eta_q(\xi)\bar{B}_k \bar{K}_q\right) P_1\right)\right\} < 0. \tag{33}$$

According to Lemma 3, system (28) with $\alpha \in [1,2)$ is admissible from (22) and (33).

Similarly, substituting $\beta = \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}$ into system (19), it yields

$$ED^{\alpha}\hat{x}_z(t) = \left(\sum_{k=1}^{r} \eta_k(\xi)\bar{A}_{k1} + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right) A_2 - \lambda_i \sum_{s=1}^{r}\sum_{k=1}^{r} \eta_s(\xi)\eta_k(\xi)\bar{W}_s \bar{C}_k\right)\hat{x}_z(t). \tag{34}$$

Based on $\eta_k(\xi) \geqslant 0$, the following inequality is derived from (26) and (27):

$$\begin{bmatrix} \Delta_2 & I_2 \otimes P_2^T & \mu_2(\Theta \otimes A_2^T) \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_2 I_2 \end{bmatrix} < 0, \tag{35}$$

where

$$\Delta_2 = \mathrm{sym}\left\{\Theta \otimes \left(\sum_{k=1}^{r} \eta_k(\xi)\bar{A}_{k1}^T P - \lambda_i \sum_{s=1}^{r}\sum_{k=1}^{r} \eta_s(\xi)\eta_k(\xi)\bar{C}_k^T G_s^T\right)\right\}.$$

By pre- and post-multiplying (35) with $\mathrm{diag}\left(I, \frac{I}{\sqrt{\mu_2}}, \frac{I}{\sqrt{\mu_2}}\right)$, it is transformed as

$$\begin{bmatrix} \Delta_2 & \frac{I_2 \otimes P_2^T}{\sqrt{\mu_2}} & \sqrt{\mu_2}(\Theta \otimes A_2^T) \\ * & -\frac{4}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I_2 & 0 \\ * & * & -I_2 \end{bmatrix} < 0. \tag{36}$$

In the same way, (36) is equivalent to

$$\Delta_2 + \begin{bmatrix} \frac{I_2 \otimes P_2^T}{\sqrt{\mu_2}} & \sqrt{\mu_2}(\Theta \otimes A_2^T) \end{bmatrix} \begin{bmatrix} \frac{(\bar{\varepsilon}-\underline{\varepsilon})^2}{4} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{I_2 \otimes P_2}{\sqrt{\mu_2}} \\ \sqrt{\mu_2}(\Theta \otimes A_2^T)^T \end{bmatrix} < 0. \tag{37}$$

From Lemma 5 and $\left|\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right| \leqslant \frac{\bar{\varepsilon}-\underline{\varepsilon}}{2}$, it yields

$$\Delta_2 + \left(\varepsilon - \frac{\underline{\varepsilon}+\bar{\varepsilon}}{2}\right)\mathrm{sym}\left\{\Theta \otimes A_2^T P_2\right\} < 0. \tag{38}$$

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Substituting $\sum_{s=1}^{r} \eta_s(\xi) G_s = P_2^T \sum_{q=1}^{r} \eta_q(\xi) \bar{W}_s$ into (38), it gives

$$\text{sym}\left\{ \Theta \otimes \left( \left( \sum_{k=1}^{r} \eta_k(\xi) \bar{A}_{k1} + \left( \varepsilon - \frac{\underline{\varepsilon} + \bar{\varepsilon}}{2} \right) A_2 - \lambda_i \sum_{s=1}^{r} \sum_{k=1}^{r} \eta_s(\xi) \eta_k(\xi) \bar{C}_k \bar{W}_s \right)^T P_2 \right) \right\} < 0. \tag{39}$$

Given the equivalence between the admissibility of $(E, A, \alpha)$ and that of $(E^T, A^T, \alpha)$, from (25) and (39), it is deduced that system (34) with $\alpha \in [1,2)$ is admissible according to Lemma 3. To sum up, (6)-(7) achieves leader-following consensus via observer-based protocol (9). □

**Remark 1.** The conditions in Theorem 1 involve LMIs with equality constraints, rendering them fragile and potentially prone to computational difficulty. Consequently, the subsequent theorem presents strict LMI conditions to address these issues and enhance computational accuracy.

**Theorem 2.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMAS (6)-(7) with $\alpha \in [1,2)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9), if there exist matrices $X_1 > 0$, $X_2 > 0$, $Y_1$, $Y_2$, $H_q$, $G_s$, and positive scalars $\mu_1$, $\mu_2$ such that*

$$\Gamma_{kk} < 0, k = 1, 2, \cdots, r, \tag{40}$$

$$\Gamma_{kq} + \Gamma_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{41}$$

$$\Upsilon_{kk} < 0, k = 1, 2, \cdots, r, \tag{42}$$

$$\Upsilon_{ks} + \Upsilon_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{43}$$

*where*

$$\Gamma_{kq} = \begin{bmatrix} \text{sym}\{\Theta \otimes (A_{k0}(X_1 E^T + S_1 Y_1) + \lambda_i \bar{B}_k H_q)\} & I_2 \otimes (X_1 E^T + S_1 Y_1)^T & \Theta \otimes \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon} - \underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_1 I_2 \end{bmatrix},$$

$$\Upsilon_{ks} = \begin{bmatrix} \text{sym}\{\Theta \otimes (\tilde{A}_{k1}^T (X_2 E + S_2 Y_2) - \lambda_i \bar{C}_k^T G_s^T)\} & I_2 \otimes (X_2 E + S_2 Y_2)^T & \Theta \otimes \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon} - \underline{\varepsilon})^2} I_2 & 0 \\ * & * & -\mu_2 I_2 \end{bmatrix}.$$

*$S_1$, $S_2 \in \mathbb{R}^{n \times (n-m)}$ are arbitrary matrices with full column rank satisfying $ES_1 = 0$ and $E^T S_2 = 0$. The gain matrices are derived as*

$$\bar{K}_q = H_q \left( X_1 E^T + S_1 Y_1 \right)^{-1}, \ \bar{W}_s = (X_2 E + S_2 Y_2)^{-T} G_s.$$

**Proof.** Assume that there exist matrices $X_1 > 0$, $X_2 > 0$, $Y_1$, $Y_2$, $H_q$, $G_q$, and scalars $\mu_1 > 0$, $\mu_2 > 0$ such that (40)-(43). Let

$$P_1 = X_1 E^T + S_1 Y_1 \ , \ P_2 = X_2 E + S_2 Y_2.$$

Then by (40)-(43), it is easy to verify $P_1$, $P_2$, $H_q$, $G_q$ and scalars $\mu_1 > 0$, $\mu_2 > 0$ satisfying (22)-(27). Therefore, by Theorem 1, (6)-(7) achieves the consensus. □

**Theorem 3.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMAS (6)-(7) with $\alpha \in (0,1)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9), if there exist matrices $X_1, X_2, Y_1, Y_2, H_q, G_s$, and positive scalars $\mu_1$, $\mu_2$ such that*

$$\begin{bmatrix} EX_1 & EY_1 \\ -EY_1 & EX_1 \end{bmatrix} = \begin{bmatrix} X_1^T E^T & -Y_1^T E^T \\ Y_1^T E^T & X_1^T E^T \end{bmatrix} \geqslant 0, \tag{44}$$

$$\Pi_{kk} < 0, k = 1, 2, \cdots, r, \tag{45}$$

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$$\Pi_{kq} + \Pi_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{46}$$

$$\begin{bmatrix} E^T X_2 & E^T Y_2 \\ -E^T Y_2 & E^T X_2 \end{bmatrix} = \begin{bmatrix} X_2^T E & -Y_2^T E \\ Y_2^T E & X_2^T E \end{bmatrix} \geqslant 0, \tag{47}$$

$$\Omega_{kk} < 0, k = 1, 2, \cdots, r, \tag{48}$$

$$\Omega_{ks} + \Omega_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{49}$$

*where*

$$\Pi_{kq} = \begin{bmatrix} \mathrm{sym}\{A_{k1}(aX_1 - bY_1) + \lambda_i \bar{B}_k H_q\} & (aX_1 - bY_1)^T & \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I & 0 \\ * & * & -\mu_1 I \end{bmatrix},$$

$$\Omega_{ks} = \begin{bmatrix} \mathrm{sym}\{\bar{A}_{k1}^T(aX_2 - bY_2) - \lambda_i \bar{C}_k^T G_s^T\} & (aX_2 - bY_2)^T & \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I & 0 \\ * & * & -\mu_2 I \end{bmatrix}.$$

*Choose the gain matrices as*

$$\bar{K}_q = H_q(aX_1 - bY_1)^{-1}, \ \bar{W}_s = (aX_2 - bY_2)^{-T} G_s.$$

**Proof.** The proof parallels that of Theorem 1 and bases on Lemma 4, which is omitted here for brevity. □

**Theorem 4.** *Given $0 < \underline{\varepsilon} < \bar{\varepsilon}$, the consensus of fuzzy FOSPMAS (6)-(7) with $\alpha \in (0,1)$ and any $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ is achieved via protocol (9), if there exist matrices $X_1$, $X_2$, $Y_1$, $Y_2$, $Q_1$, $Q_2$, $H_q$, $G_s$, and positive scalars $\mu_1$, $\mu_2$ such that*

$$\begin{bmatrix} X_1 & Y_1 \\ -Y_1 & X_1 \end{bmatrix} > 0, \tag{50}$$

$$\Xi_{kk} < 0, k = 1, 2, \cdots, r, \tag{51}$$

$$\Xi_{kq} + \Xi_{qk} < 0, 1 \leqslant k < q \leqslant r, \tag{52}$$

$$\begin{bmatrix} X_2 & Y_2 \\ -Y_2 & X_2 \end{bmatrix} > 0, \tag{53}$$

$$\Lambda_{kk} < 0, k = 1, 2, \cdots, r, \tag{54}$$

$$\Lambda_{ks} + \Lambda_{sk} < 0, 1 \leqslant k < s \leqslant r, \tag{55}$$

*where*

$$\Xi_{kq} = \begin{bmatrix} \mathrm{sym}\{A_{k1}((aX_1 - bY_1)E^T + S_1 Q_1) + \lambda_i \bar{B}_k H_q\} & ((aX_1 - bY_1)E^T + S_1 Q_1)^T & \mu_1 A_2 \\ * & -\frac{4\mu_1}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I & 0 \\ * & * & -\mu_1 I \end{bmatrix},$$

$$\Lambda_{ks} = \begin{bmatrix} \mathrm{sym}\{\bar{A}_{k1}^T((aX_2 - bY_2)E + S_2 Q_2) - \lambda_i \bar{C}_k^T G_s^T\} & ((aX_2 - bY_2)E + S_2 Q_2)^T & \mu_2 A_2^T \\ * & -\frac{4\mu_2}{(\bar{\varepsilon}-\underline{\varepsilon})^2} I & 0 \\ * & * & -\mu_2 I \end{bmatrix}.$$

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*$S_1$, $S_2$ satisfy conditions in Theorem 2. The gain matrices are chosen as*

$$\bar{K}_q = H_q\left((aX_1 - bY_1)E^T + S_1Q_1\right)^{-1}, \; \bar{W}_s = ((aX_2 - bY_2)E + S_2Q_2)^{-T}G_s.$$

**Proof.** Assume that there exist matrices $X_1, X_2, Y_1, Y_2, Q_1, Q_2, H_q, G_q$, and scalars $\mu_1 > 0$, $\mu_2 > 0$ such that (50)-(55). Let

$$\bar{X}_1 = X_1E^T + a^{-1}S_1Q_1, \; \bar{Y}_1 = Y_1E^T, \; \bar{X}_2 = X_2E + a^{-1}S_2Q_2, \; \bar{Y}_2 = Y_2E.$$

By (50)-(55), it is easy to verify $\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2, H_q, G_q$ and scalars $\mu_1 > 0$, $\mu_2 > 0$ satisfying (44)-(49). Therefore, by Theorem 3, the consensus of (6)-(7) is achieved. □

## 4. Numerical Examples

This section presents two demonstrative instances that highlight the effectiveness of control protocol in achieving the consensus of fuzzy FOSPMASs with order $\alpha$ in $(0,1)$ and $[1,2)$, respectively.

**Example 1.** Consider a T-S fuzzy FOSPMAS composed of one leader and four followers, and the behavior of each agent is described by the fractional order RLC circuit model as shown in Figure 1.

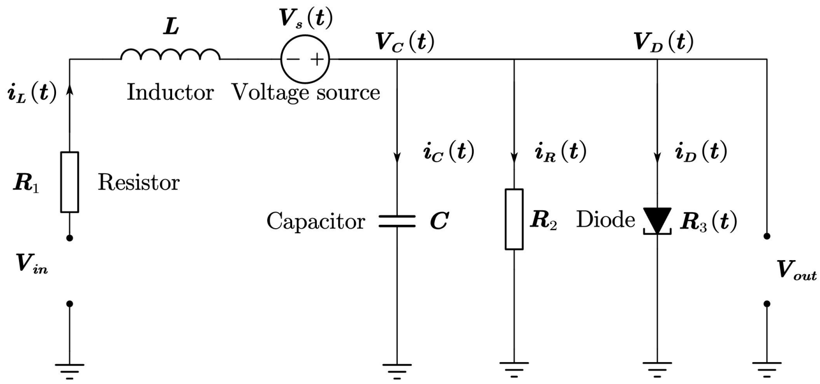


**Figure 1.** The RLC circuit with the diode.

The capacitor and inductor have fractional characteristics with order $\alpha$. $L$ represents a very small parasitic inductance. $R_1$, $R_2$ denote the resistance of corresponding resistors. $R_3$ is a diode and its characteristic function is $R_3 = 1/0.4 + 0.15V_D^2(t)$. It is known that the relationships between voltages are $V_D(t) = V_C(t)$ and $V_S(t) = -V_C(t)$. The dynamic of each agent is subsequently described by

$$\begin{cases} CD^\alpha V_C(t) = -\frac{V_C(t)}{R_2} - \frac{V_C(t)}{R_3} + i_L(t) \\ LD^\alpha i_L(t) = V_S(t) - V_C(t) + R_1 i_L(t) + u(t) \end{cases} \tag{56}$$

Let $x_{i1}(t) = V_C(t)$, $x_{i2}(t) = i_L(t)$, $\varepsilon = L$, then the circuit model (56) is reformulated as follows:

$$\begin{bmatrix} D^\alpha x_{i1}(t) \\ \varepsilon D^\alpha x_{i2}(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_2C} - \frac{1}{C}(0.4 + 0.15x_{i1}^2(t)) & \frac{1}{C} \\ -2 & R_1 \end{bmatrix} \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i(t), \tag{57}$$

where $i = 1,2,3,4$.

The parameters are chosen as $C = 0.3$F, $\varepsilon = 0.02$H, $R_1 = 0.1\Omega$, $R_2 = 0.5\Omega$, $\alpha = 0.5$. It is assumed that $x_{11}(t)$ belongs to $[-2,2]$. Subsequently, the fuzzy rules are set as follows:

Rule 1: If the value of $x_{11}(t)$ is approximately 0, then

$$\begin{cases} E(\varepsilon)D^\alpha x_i(t) = A_1x_i(t) + B_1u_i(t) \\ y_i(t) = C_1x_i(t) \end{cases} \quad \text{and} \quad \begin{cases} E(\varepsilon)D^\alpha x_0(t) = A_1x_0(t) \\ y_0(t) = C_1x_0(t) \end{cases}$$

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Rule 2: If the value of $x_{11}(t)$ is approximately $\pm 2$, then

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_2x_i(t) + B_2u_i(t) \\ y_i(t) = C_2x_i(t) \end{cases} \text{ and } \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_2x_0(t) \\ y_0(t) = C_2x_0(t) \end{cases}$$

where

$$E(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, A_1 = \begin{bmatrix} -8 & 3 \\ -2 & 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} -10 & 3 \\ -2 & 0.1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, C_1 = C_2 = \begin{bmatrix} 1 & 0.5 \end{bmatrix}.$$

The fuzzy weighting function is selected as $\eta_1(\xi(t)) = 1 - 0.25x_{e11}^2(t)$, $\eta_2(\xi(t)) = 1 - \eta_1(\xi(t))$. The communication network between these agents is depicted in Figure 2.

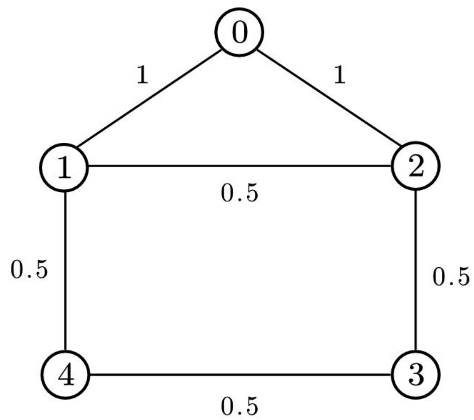


**Figure 2.** The weighted undirected graph.

With $\underline{\varepsilon} = 0.001, \bar{\varepsilon} = 0.04$, by solving LMIs (50)-(55) in Theorem 4, the feasible solutions are obtained as

$$\bar{K}_1 = \begin{bmatrix} 0.4367 & -0.0916 & 0 & 0 \end{bmatrix},$$

$$\bar{K}_2 = \begin{bmatrix} 0.4371 & -0.0916 & 0 & 0 \end{bmatrix},$$

$$\bar{W}_1 = \begin{bmatrix} 1.4511 & 0.1933 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{W}_2 = \begin{bmatrix} 1.4562 & 0.1938 & 0 & 0 \end{bmatrix}^T.$$

Let $\varepsilon = 0.02$, then the tracking errors and estimation errors are depicted in Figures 3 and 4. The consensus errors to zero means that each follower converges toward the leader, which demonstrates that system achieves the consensus by using the observer-based protocol (9). It indicates the practical applicability and efficacy of the proposed method in $\alpha \in [1, 2)$.

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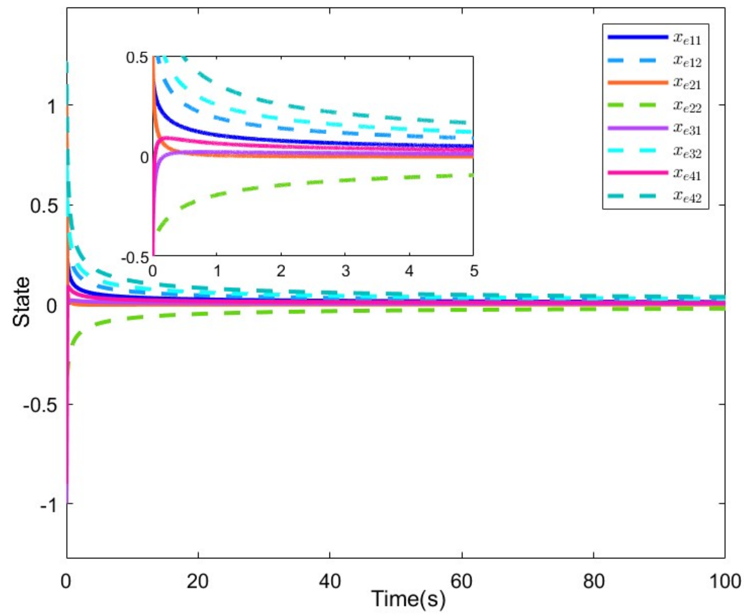


**Figure 3.** Errors between leader and followers in Example 1.

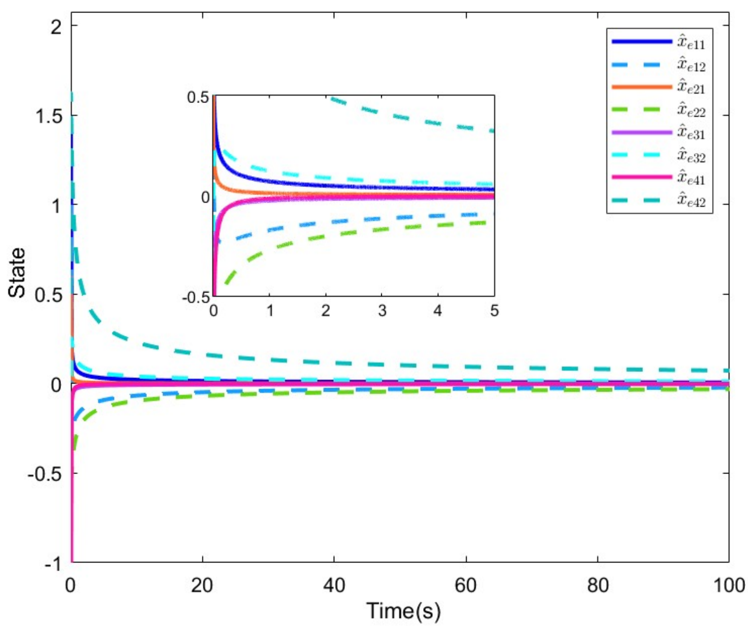


**Figure 4.** Estimation errors of followers in Example 1.

**Example 2.** Considering the T-S fuzzy FOSPMAS (6)-(7) within the topology in Figure 2, the fuzzy rules of the system are established in the manner outlined below:

Rule 1: If $x_{e11}(t)$ is $\Pi_{k1}$, then

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_1x_i(t) + B_1u_i(t) \\ y_i(t) = C_1x_i(t) \end{cases} \quad \text{and} \quad \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_1x_0(t) \\ y_0(t) = C_1x_0(t) \end{cases}$$

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Rule 2: If $x_{e11}(t)$ is $\Pi_{k2}$, then

$$\begin{cases} E(\varepsilon)D^{\alpha}x_i(t) = A_2x_i(t) + B_2u_i(t) \\ y_i(t) = C_2x_i(t) \end{cases} \text{ and } \begin{cases} E(\varepsilon)D^{\alpha}x_0(t) = A_2x_0(t) \\ y_0(t) = C_2x_0(t) \end{cases}$$

The remaining parameters are proposed as follows:

$$\alpha = 1.2,\ E(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix},\ A_1 = \begin{bmatrix} -5 & 9 & 5 \\ 12 & -5 & 13 \\ 5 & -10 & -5 \end{bmatrix},\ A_2 = \begin{bmatrix} -4 & 5 & 5 \\ 5 & -18 & 15 \\ 5 & -10 & -5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 & 0.5 \end{bmatrix}^T,\ B_2 = \begin{bmatrix} 0.5 & 0.5 & 1 \end{bmatrix}^T,$$

$$C_1 = \begin{bmatrix} 1 & 0.5 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.5 & 1 & 0.5 \end{bmatrix}.$$

Assuming that the state $x_{e11}(t)$ belongs to $[-1,1]$, the fuzzy weighting function is selected as $\eta_1(\xi(t)) = |x_{e11}|, \eta_2(\xi(t)) = 1 - \eta_1(\xi(t))$.

Considering $\underline{\varepsilon} = 0.001, \bar{\varepsilon} = 0.03$, the feasible solutions are presented based on the LMIs (40)-(43) in Theorem 2:

$$\bar{K}_1 = \begin{bmatrix} 0.1193 & -0.2976 & -0.1530 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{K}_2 = \begin{bmatrix} -0.0783 & 0.1285 & 0.0181 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{W}_1 = \begin{bmatrix} -0.0009 & -0.5099 & 0.0147 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\bar{W}_2 = \begin{bmatrix} 0.0150 & 0.4288 & 0.0108 & 0 & 0 & 0 \end{bmatrix}^T.$$

Figures 5 and 6 present the simulation results of the error system (14) with $\varepsilon = 0.02$. As depicted in Figure 5, the state of follower agents exhibits a successful tracking of the state of leader agent, indicating that the consensus issue of fuzzy FOSPMAS (6)-(7) with $\alpha \in [1,2)$ is solved by the criteria in Theorem 2.

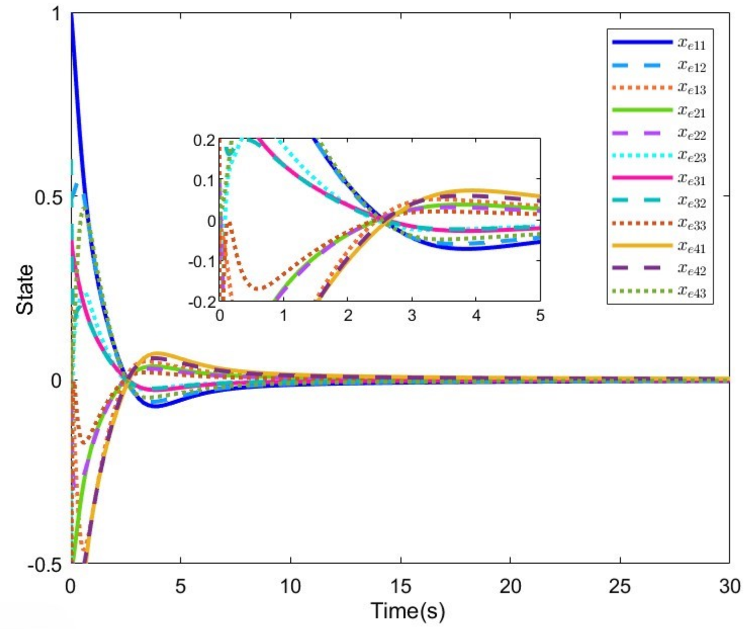


**Figure 5.** Errors between leader and followers in Example 2.

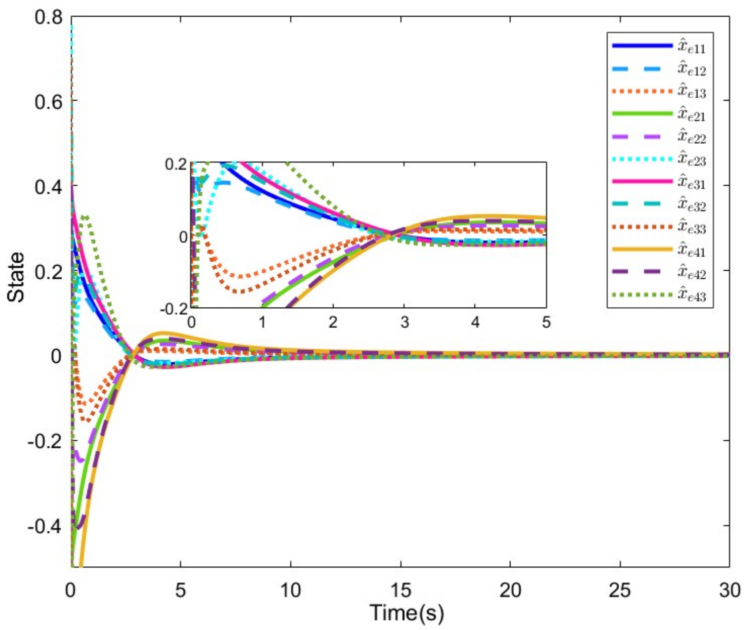


**Figure 6.** Estimation errors of followers in Example 2.

## 5. Conclusions

In this paper, the T-S fuzzy FOSPMAS with $\alpha \in (0,2)$ has been modeled and studied for more accurately describing actual complex systems. The consensus problem of T-S fuzzy FOSPMAS (6)-(7) is transformed into admissibility assessment of fuzzy SFOSs (18) and (19). In distinction to the methodologies in previous literature, the proposed method not only overcomes the pathological problem arising from multiple time-scale, but also is applicable to both standard and non-standard SPMASs. Theorems 1 and 3 provide sufficient conditions for achieving the consensus of (6)-(7) with $\alpha \in (0,1)$ and $[1,2)$. Furthermore, strict LMI criteria are given in Theorems 2 and 4, which are solved

 doi:10.20944/preprints202407.2480.v1



easily with LMI toolbox. Future research is certainly required to overcome the challenges in consensus and $H_\infty$ control of uncertain FOSPMASs.

**Author Contributions:** Conceptualization, methodology, validation, X.W. and X.Z.; writing—original draft preparation, X.W.; writing—review and editing, X.W. and X.Z.; visualization, X.W. and S.-H.Y.; supervision, W.P. and D.B.; project administration, X.Z.; funding acquisition, X.Z. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported in part by the National Natural Science Foundation of China under Grant 62103093, the National Key Research and Development Program of China under Grant 2022YFB3305905, fundamental research funds for the central universities N2224005-3 and national key research and development program topic 2020YFB1710003.

**Data Availability Statement:** Data are contained within the article.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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