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Natalia Gorobey , Alexander Lukyanenko , Alexander V. Goltsev *

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Initial State in Quantum Cosmology and the Proper Mass of the Universe

Natalia Gorobey [1,†], Alexander Lukyanenko [1,†] and A. V. Goltsev [2,*,†]

[1] Peter the Great Saint Petersburg Polytechnic University, Polytekhnicheskaya 29, 195251, St. Petersburg, Russia; alex.lukyan@mail.ru
[2] Ioffe Physical- Technical Institute, Polytekhnicheskaya 26, 195251, St. Petersburg, Russia
* Correspondence: goltsev@ua.pt
† These authors contributed equally to this work.

Abstract: In the Euclidean form of the theory of gravity, where there is no dedicated time parameter, a generalized canonical form of the principle of least action is proposed. On its basis, the quantum principle of least action is formulated, in which the "dynamics" of the universe in the Origin is described by the eigenvector of the action operator - the wave functional on the space of 4D Riemannian geometries and configurations of matter fields in some compact region of 4D space. The corresponding eigenvalue of the action operator determines the initial state for the world history of the universe outside this region, where the metric signature is Lorentzian and thus the time parameter exists. The boundary of the Origin region is determined by the condition that the rate of change of the determinant of the 3D metric tensor is zero on it. A modification of quantum theory is proposed that eliminates the uncertainty of the sign of the Hilbert-Einstein action in the Euclidean region of the Origin. On this basis, we formulate a minimum principle for the eigenvalue of the action operator, similar to the principle of minimum energy for the ground state in quantum mechanics. It has been suggested that in the initial state the universe contains a certain distribution of its own mass, which is not directly related to the fields of matter.

Keywords: universe; time; own mass; quantum action; generalized Legendre transformation

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As the title of the article suggests, time is assumed to exist in quantum cosmology at the fundamental level. This issue remains controversial, and we will not substantiate any particular point of view here, but will consider a number of consequences of the assumption that time exists in quantum cosmology. Let us only recall that the formal solution of Einstein's equations, found in the works of Friedman [1] and Lemaitre [2], led to the idea of the evolution of the universe in time and the existence of its Beginning, and this was of epochal significance. The opposite idea, that time, as a physical reality, does not exist at all, also arose on formal grounds. We should start with the work of Dirac [3], which analyzes the canonical structure of dynamical systems possessing certain symmetries. Consequences of symmetries are additional conditions on dynamic variables - constraints. In quantum theory, constraints, according to Dirac, must be imposed on the wave function as additional conditions. In the theory of gravity, general covariance leads to the fact that the Hamilton function for a closed model of the universe is reduced to a linear combination of constraints [4], and, therefore, taking into account additional conditions, equals zero. In classical theory, this does not lead to the abandonment of time, since dynamic equations and constraints (as part of the equations of motion) do not contradict each other and are carried out simultaneously. The modern quantum theory of gravity (QTG) is based on constraints, i.e. the Wheeler-De Witt equation (WDW) [5,6], which is the source of the time problem and is called the 'frozen formalism' [7]. In his geometrydynamics (see [5]), Wheeler formulated an extreme point of view according to which time and 4D space-time are not fundamental concepts, but are emergent in nature.

An alternative to this would be to abandon the WDW equation and return to the formulation of dynamics in terms of the Schrödinger equation (SE). This is not attractive since the wave function of the universe is defined on 3D spatial sections obtained by an arbitrary 3+1 partition of spacetime (Arnowitt, Deser, Misner [8], and [7]), which is clearly not covariant. In this case, arbitrary parameters of the 3+1 partition – the lapse and shift functions of the ADM – will be included as coefficients in the SE. However, the 3+1 partition of the ADM does not violate the invariance of the action functional,

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provided that the lapse and shift functions have certain transformation properties. The works [9,10] proposed a covariant formulation of quantum dynamics in terms of a wave functional defined on the world histories of the universe (which assumes the presence of a time parameter) and equal to the product of all elements of the time sequence of wave functions. To find the wave functional, the quantum principle of least action was formulated in [9,10], which is equivalent to the SE for the wave function.

With the return of time to the quantum dynamics of the universe, the question arises about the initial conditions for this dynamics. Solving the WDW equation in the form of the Euclidean functional integral of Hartle and Hawking for the no boundary wave function of the universe [11] (see also [12]) will be useful for us. The functional integral is taken over all Riemannian metrics and configurations of matter fields in some compact region of 4D space with given boundary values of the metric and matter fields on the 3D boundary of the region. It also includes the time integral, which means that the latter is not available as a free parameter. In this case, in specific calculations [13], a representation is used in which coordinate time lines converge at one point—the pole. The natural smoothness conditions for the fundamental dynamic variables at this point do not completely determine the functional integral. The remaining arbitrariness in the choice of "polar" values of the scalar field is used in [13,14] to analyze various scenarios of cosmological evolution. In fact, this means the return of time to the polar region, which contradicts the WDW equation. In [10] it was proposed to do this explicitly and remove the integration over time. In this case, the possible non-zero value of the gravitational constraint turns into an additional parameter of evolution - the own mass of the universe.

The goal of this work is a consistent formulation of the quantum "dynamics" of the universe in the "subpolar" region using the formalism of the quantum principle of least action. The formalism is based on the action operator, obtained taking into account the equivalence of all measurements in the subpolar region. For this purpose, the generalized canonical De Donder-Weil (DDW) formalism is used [15,16]. We define the boundary of the subpolar region as formed by the cusp points (the rate of change equal to zero) of the scale factor of 3D geometry in the polar coordinate system, which is not singular near the boundary. The eigenvalue of the action operator determines the initial value of the wave function of the universe outside the polar region, where the space-time signature is Lorentzian. A clear representation of this is the picture of the tunneling of the universe from "nothing", proposed in the works of Vilenkin [17]. In the quantum action formalism, taking into account the equivalence of all spatial dimensions, there are no constraints and indefinite Lagrange multipliers, as is the case in the ADM formalism. Therefore, we should expect that in the initial state there is some distribution of the universe's own mass. There will also be no constraints corresponding to internal symmetries of matter fields, which raises the question of the existence of non-zero values of gauge charges in the initial state. Thus, the appearance of constraints is a consequence of the clearly non-covariant procedure of $3+1$ splitting of space-time. We believe that the proposed formalism of the Euclidean quantum theory of gravity, which does not use the $3+1$ splitting of 4D space-time, can serve as one of the options for the synthesis of relativistic and quantum principles.

The next section formulates the quantum principle of least action in non-relativistic quantum mechanics. The second section examines the generalized canonical representation of various classical fields and quantization. In the third section, a formulation of the Euclidean quantum theory of gravitational field is proposed, based on the generalized canonical form of the DDW. The fourth section formulates the quantum principle of the least action of the universe and the minimum principle for finding the numerical value of the action.

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1. Quantum Principle of Least Action

Let us formulate an alternative formalism of quantum theory, which we will use as the basis for our constructions in the polar region. Let us consider the simplest mechanical system described by the action in the canonical form

$$I[p,q] = \int_0^T dt\Big[\dot{q}(t)p(t) - H(t,p(t),q(t))\Big], \tag{1}$$

where

$$H(t,p,q) = \frac{p^2}{2} + V(t,q). \tag{2}$$

is the Hamilton function. Canonical quantization consists of replacing the main dynamic variables p, q with Hermitian operators

$$\widehat{q} \equiv q, \quad \widehat{p} \equiv \frac{\hbar}{i}\frac{\partial}{\partial q} \tag{3}$$

on the space of wave functions $\psi(t,q)$ with scalar product

$$(\psi_1, \psi_2) = \int dq \psi_1^* \psi_2 \tag{4}$$

The basic equation of motion in quantum theory is the SE:

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi, \tag{5}$$

where $\widehat{H}$ is the Hamilton operator, which is obtained by replacing the dynamic variables in Equation (2) with operators Equation (3). With appropriate normalization of the wave function, $|\psi(t,q)|^2$ determines the probability of detecting a particle in a small neighborhood of the point q at time t.

Let's move on to an alternative description of quantum dynamics. Let's divide the time interval T into small intervals $\epsilon = T/N$ and fix a certain sequence of coordinate values $q_n = q(\epsilon n), n = 0, 1, 2, \ldots N$. These points are the vertices of a piecewise linear particle trajectory in configuration space with start and end points (q_0, q_T). Obviously, the amplitude of the probability of particle motion in a small neighborhood of this broken line is determined by the product

$$\Psi[q_n] = \prod_{n=0}^{N} \psi(t_n, q_n). \tag{6}$$

As $\twoheadrightarrow \infty$, the broken line more and more accurately approximates the curvilinear trajectory $q = q(t)$ in the configuration space, and the function of many variables Equation (6) becomes the (wave) functional $\Psi[q(t)]$ in the space of world lines. With appropriate normalization of the wave function ψ, the quantity $|\Psi[q(t)]|^2$ has the meaning of the probability density of detecting a trajectory in a small neighborhood of $q = q(t)$. Now let us formulate a dynamic law that directly determines the wave functional. Let us introduce a new representation of dynamic variables by operators acting on the space of wave functionals:

$$\widehat{q}(t) \equiv q(t), \quad \widehat{p}(t) \equiv \frac{\widetilde{\hbar}}{i}\frac{\delta}{\delta q(t)}, \tag{7}$$

where the first is the operator of multiplication by the coordinate value at time t, and the second is proportional to the variational derivative of the wave functional, which is determined by the following relation:

$$\delta\Psi = \int_0^T dt \frac{\delta\Psi}{\delta q(t)}\delta q(t). \tag{8}$$

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Let us note that the dimension of the variational derivative differs by the factor $\sec^{-1}$ from the ordinary partial derivative. Therefore, the variational derivative of the wave functional in the multiplicative representation Equation (6) has the form [18]

$$\frac{\delta\Psi}{\delta q(t_n)} = \frac{1}{\epsilon}\frac{\partial\Psi}{\partial q_n}, \tag{9}$$

and the modified Planck constant in Equation (7) is equal to

$$\widetilde{\hbar} = \hbar\epsilon. \tag{10}$$

Substituting operators Equation (7) into the canonical action functional Equation (1), we obtain the action operator

$$\widehat{I} = \int_0^T dt\left[\frac{\widetilde{\hbar}}{i}\dot{q}(t)\frac{\delta}{\delta q(t)} + \frac{\widetilde{\hbar}^2}{2}\frac{\delta^2}{\delta q^2(t)} - V(t, q(t))\right] \tag{11}$$

on the space of wave functionals. In specific calculations, the wave functional should be taken in the form of the product Equation (6) and the integral in Equation (11) should be replaced by the sum:

$$\int_0^T dt[...] \twoheadrightarrow \sum_{n=0}^{N}\epsilon[...]. \tag{12}$$

After all calculations, the passage to the limit $\epsilon \longrightarrow 0$ is assumed. However, we will continue to use continuous notation with an integral sign.

Let

$$\psi(t, q) = \exp\left[\frac{i}{\hbar}\chi(t, q)\right] \tag{13}$$

is a solution of the SE with a given initial value $\psi(0, q) = \psi_0(q)$. Then the wave functional in Equation (6) in the limit $\epsilon \longrightarrow 0$ can be written in the form

$$\Psi[q(t)] = \exp\left[\frac{i}{\widetilde{\hbar}}\int_0^T dt\chi(t, q(t))\right]. \tag{14}$$

Multiplicative functional (6) in this limit is singular, since $\widetilde{\hbar} \longrightarrow 0$. However, at the calculation stage we agreed to use a discrete approximation with a finite value of ϵ. It is easy to see that in this approximation, and then in the limit $\epsilon \longrightarrow 0$, the expression $\widehat{I}\Psi[q(t)]$ is finite. The second variational derivative in Equation (11) can cause difficulty, but it turns out that everything is in order [9]. The claim is that the expression

$$\Lambda = \frac{\widehat{I}\Psi}{\Psi} \tag{15}$$

does not depend on the internal points of an arbitrary trajectory $q(t)$ and is equal to

$$\Lambda = \frac{i}{\hbar}[\chi(T, q_T) - \chi(0, q_0)]. \tag{16}$$

Thus, the wave functional Equation (14) is an eigenvector of the action operator Equation (11) with eigenvalue Equation (16),

$$\widehat{I}\Psi = \Lambda\Psi, \tag{17}$$

if the wave function Equation (13) is a solution to the equation Equation (5). We call this statement as the quantum principle of least action.

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2. Covariant Quantum Field Theory in the Euclidean Space

Let's begin our consideration of quantum field theory in the polar region Ω with the simplest real scalar field $\varphi(x^\mu), \mu = 1,2,3,4$. Its Euclidean action has the form:

$$I_\varphi = -\int_\Omega \sqrt{g} d^4x \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi)\right]. \tag{18}$$

Taking into account the minus sign, action (18) is clearly negative definite. Since there is now no dedicated time parameter, we will use the generalized canonical representation of the DDW action [15,16]. Let us define generalized canonical momenta for each spatial coordinate,

$$p^\mu_\varphi \equiv -\sqrt{g} g^{\mu\nu} \partial_\nu \varphi, \tag{19}$$

and using the generalized Legendre transformation we write the action in the generalized canonical form:

$$I_\varphi = \int_\Omega d^4x \left[\partial_\mu \varphi p^\mu_\varphi - \frac{g_{\mu\nu} p^\mu_\varphi p^\nu_\varphi}{2\sqrt{g}} + V(\varphi)\right]. \tag{20}$$

Action Equation (20), obviously, gives the same equation of motion of the scalar field as the original Equation (18). Quantization of this form of canonical theory can be carried out within the framework of the quantum principle of least action as follows. Let us introduce in $4D$ Euclidean space a rectangular lattice with constants ϵ^μ (edges of the unit cell). For the functional operator implementation of canonical momenta Equation (19) on the space of wave functionals $\Psi[\varphi(x)]$ we use four equivalent singular representations of the wave functional, similar to Equation (14). The significant difference is that in Euclidean quantum theory the wave functional is constructed to solve the diffusion equation, [19], and it is real,

$$\Psi[\varphi(x)] = \exp\left[\frac{1}{\widetilde{\hbar}^\mu} \int_0^{T^\mu} dx^\mu \chi^\mu \left(x^\mu, \left[\varphi\left(x^\mu, x^{\widetilde{\mu}}\right)\right]\right)\right]. \tag{21}$$

Here there is no summation over $\mu = 1,2,3,4$, and $\widetilde{\mu}$ means: all other values except μ. χ^μ is a function of the coordinate x^μ and the functional of the world line of the scalar field with respect to the coordinate time x^μ, provided that all other coordinates $x^{\widetilde{\mu}}$ have the meaning of numbering indices. Using representations Equation (21), we define the functional operator implementation of the canonical momenta Equation (19) for Euclidean quantum theory:

$$\widehat{p}^\mu_\varphi(x)\Psi \equiv \widetilde{\hbar}^\mu \frac{\delta\Psi}{\delta_\mu \varphi(x)} = \frac{\delta\chi^\mu}{\delta\varphi(x)}\Psi. \tag{22}$$

Here

$$\widetilde{\hbar}^\mu = \hbar\epsilon^\mu. \tag{23}$$

Let us note that in Euclidean quantum theory the Hilbert space is real [19], and the sign of the complex conjugation in the scalar product (4) has no meaning.

Now consider the covariant quantum theory of a more complex structure - the Yang-Mills field on Euclidean space with action [20],

$$I_A = \frac{1}{4} \int_\Omega \sqrt{g} d^4x F^a_{\mu\nu} F^{a\mu\nu}, \tag{24}$$

where

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \sigma f^{abc} A^b_\mu A^c_\nu \tag{25}$$

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f^{abc} are structural constants of the internal symmetry group. We denote the interaction constant by σ so as not to confuse it with the determinant of the metric tensor g. Let us introduce generalized canonical momenta,

$$p^{\mu}_{A^a_\nu}(x) \equiv \frac{\delta I_A}{\delta \partial_\mu A^a_\nu(x)} = \sqrt{g} F^{a\mu\nu}, \tag{26}$$

and write action Equation (24) in the generalized canonical form as follows:

$$\begin{aligned} I_A &= \frac{1}{2}\int_\Omega d^4x \Big[\Big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \sigma f^{abc} A^b_\mu A^c_\nu \Big) p^{\mu}_{A^a_\nu}(x) \\ &+ \frac{1}{2} \frac{g_{\mu\nu} g_{\alpha\beta} p^{\mu}_{A^a_\alpha} p^{\nu}_{A^a_\beta}}{\sqrt{g}} \Big]. \end{aligned} \tag{27}$$

This expression is clearly gauge invariant. When moving to quantum theory, one should also take into account the antisymmetry of generalized canonical momenta Equation (26) with respect to permutations of the indices μ, ν. This is achieved using the functional operator representation

$$\widehat{p}^{\mu}_{A^a_\nu}(x) \equiv \widetilde{\hbar}^{\mu} \frac{\delta}{\delta_\mu A^a_\nu(x)} - \widetilde{\hbar}^{\nu} \frac{\delta}{\delta_\nu A^a_\mu(x)}. \tag{28}$$

Let us note that in this covariant formulation of the gauge theory, all components of the $4D$ vector potential A^a_μ are equal dynamic variables and there is no Gaussian constraint. In the presence of a gauge multiplet of scalar fields φ^γ with minimal interaction with the Yang-Mills field, action Equation (20) should be written as:

$$\begin{aligned} I_\varphi &= \int_\Omega d^4x \Big[\Big(\partial\varphi^\gamma + \frac{i}{2}\sigma A^a_\mu (t^a)^\gamma_\delta \varphi^\delta \Big) p^{\mu}_{\varphi^\delta} \\ &- \frac{g_{\mu\nu} p^{\mu}_{\varphi^\gamma} p^{\nu}_{\varphi^\gamma}}{2\sqrt{g}} + V(\varphi) \Big], \end{aligned} \tag{29}$$

where t^a are generators of gauge transformations in the internal space of the multiplet.

3. Covariant Euclidean Quantum Theory of Gravity

Let us now consider the theory of gravity in the polar region. Let us take the dimensionless Hilbert-Einstein action of the gravitational field in the form [7]

$$I_g = \frac{1}{l_P^2} \int_\Omega \sqrt{g} d^4x G, \tag{30}$$

where l_P is the Planck length,

$$G = g^{\mu\nu} \Big(\Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\alpha} - \Gamma^\alpha_{\mu\nu} \Gamma^\gamma_{\alpha\gamma} \Big), \tag{31}$$

and

$$\Gamma^\alpha_{\mu\gamma} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \tag{32}$$

are Christoffel symbols. The generalized canonical momenta of the DDW are determined by the formulas

$$p_g^{\alpha|\mu\nu}(x) \equiv \frac{\delta I_g}{\delta \partial_\alpha g_{\mu\nu}(x)}. \tag{33}$$

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As a result of cumbersome calculations we find:

$$\begin{aligned} p_g^{\alpha|\mu\nu} &= \frac{\sqrt{g}}{l_P^2}\Big[\Big(g^{\mu\gamma}g^{\nu\delta} - \frac{1}{2}g^{\mu\nu}g^{\gamma\delta}\Big)\Gamma^\alpha_{\gamma\delta} \\ &\quad -\Big(g^{\mu\delta}g^{\nu\alpha} + g^{\nu\delta}g^{\mu\alpha} - g^{\mu\nu}g^{\delta\alpha}\Big)\Gamma^\gamma_{\delta\gamma}\Big] \\ &\equiv \frac{\sqrt{g}}{l_P^2}\text{Ç}^{\alpha\mu\nu|\beta\gamma\delta}\partial_\beta g_{\gamma\delta}, \end{aligned} \tag{34}$$

where $\text{Ç}^{\alpha\mu\nu|\beta\gamma\delta}$ is a symmetric matrix formed by the components $g^{\mu\nu}$ of a contravariant metric tensor. This result differs from a similar expression in [9] in that the right-hand side is symmetric in the indices μ,ν, as it should be. In this regard, some elements of subsequent constructions will change. Let us note that some derivatives with respect to coordinates from the components of the $4D$ metric do not enter into action Equation (30), which means that the corresponding generalized momenta $p_g^{\alpha|\mu\nu}$ are equal to zero. A more precise indication is given by the $3+1$ splitting of the $4D$ metric $g_{\mu\nu}$ in the ADM representation. If we formally consider any spatial coordinate x^μ as a time parameter in the ADM formalism, then the derivatives ∂_μ from the components $g^{\mu\mu}, g^{\mu\tilde{\mu}}$ of the $4D$ metric in the original action Equation (30) are absent. This is reflected in the structure of the matrix $\text{Ç}^{\alpha\mu\nu|\beta\gamma\delta}$. From Equation (34) we find:

$$g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = -\frac{2}{3}\frac{\sqrt{g}}{l_P^2}g_{\gamma\delta}\Big(p_g^{\gamma|\delta\alpha} + p_g^{\alpha|\gamma\delta}\Big), \tag{35}$$

$$g^{\alpha\beta}\Gamma^\gamma_{\beta\gamma} = -\frac{2}{3}\frac{\sqrt{g}}{l_P^2}g_{\gamma\delta}\Big(p_g^{\gamma|\delta\alpha} - \frac{1}{2}p_g^{\alpha|\gamma\delta}\Big), \tag{36}$$

after which we express the Christoffel symbols (and therefore all derivatives of the metric tensor) in terms of generalized canonical momenta:

$$\begin{aligned} g^{\mu\gamma}g^{\nu\delta}\Gamma^\alpha_{\gamma\delta} &= \frac{l_P^2}{\sqrt{g}}\Big\{p_g^{\alpha|\mu\nu} - \frac{1}{2}g^{\mu\gamma}g_{\gamma\delta}p_g^{\alpha|\gamma\delta} \\ &\quad -\frac{1}{3}\Big[g^{\alpha\nu}g_{\gamma\delta}\Big(p_g^{\gamma|\delta\mu} - \frac{1}{2}p_g^{\mu|\gamma\delta}\Big) \\ &\quad +g^{\alpha\mu}g_{\gamma\delta}\Big(p_g^{\gamma|\delta\nu} - \frac{1}{2}p_g^{\nu|\gamma\delta}\Big)\Big]\Big\} \end{aligned} \tag{37}$$

Substituting the Christoffel symbols into Equation (31) and then into Equation (30), we write the Hilbert-Einstein action in the generalized canonical DDW form:

$$\begin{aligned} I_g &= \int_\Omega d^4x\Big[\partial_\alpha g_{\mu\nu}p_g^{\alpha|\mu\nu} \\ &\quad -\frac{l_P^2}{2\sqrt{g}}p_g^{\alpha|\mu\nu}\text{Ç}_{\alpha|\mu\nu|\beta|\gamma\delta}\,p_g^{\beta|\gamma\delta}\Big], \end{aligned} \tag{38}$$

in which the symmetric matrix $\text{Ç}_{\alpha|\mu\nu|\beta|\gamma\delta}$ is formed by the components $g_{\mu\nu}$ of the covariant metric tensor and is the inverse of $\text{Ç}^{\alpha\mu\nu|\beta\gamma\delta}$. Note that in this form of action there are no additional conditions on dynamic variables (constraints). The transition to quantum theory is carried out by the following implementation of the momentum operator

$$\widehat{p}_g^{\alpha|\mu\nu}(x) \equiv \widetilde{\hbar}^\alpha\frac{\delta}{\delta_\alpha g_{\mu\nu}(x)} \tag{39}$$

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on the space of wave functionals $\Psi[g,\phi]$. Substituting Equation (39) into the generalized canonical action Equation (38), we obtain (with proper ordering) the Hermitian operator of the action of the gravitational field

$$\begin{aligned}\widehat{I}_g \;=\; & \frac{1}{2}\int_\Omega d^4x\Bigg[\Bigg(\partial_\alpha g_{\mu\nu}\widetilde{\hbar}^\alpha\frac{\overrightarrow{\delta}}{\delta_\alpha g_{\mu\nu}(x)}\\ & +\widetilde{\hbar}^\alpha\frac{\overleftarrow{\delta}}{\delta_\alpha g_{\mu\nu}(x)}\partial_\alpha g_{\mu\nu}\Bigg)\\ & +\frac{l_P^2}{2\sqrt{g}}\widetilde{\hbar}^\alpha\frac{\overleftarrow{\delta}}{\delta_\alpha g_{\mu\nu}(x)}\varsigma_{\alpha|\mu\nu|\beta|\gamma\delta}\,\widetilde{\hbar}^\beta\frac{\overrightarrow{\delta}}{\delta_\beta g_{\gamma\delta}(x)}\Bigg]\end{aligned} \tag{40}$$

on the space of wave functionals. The arrows above the variational derivatives indicate the direction of their action in the quadratic form of the action operator (see below Equation (54)). The Hermitian scalar product in the space of wave functionals is defined as follows:

$$\begin{aligned}&(\Psi_1,\Psi_2)\\ \equiv\;&\int\prod_{x^\mu}Jd^{12}g\,d^M\psi\,\Psi_1^*\Psi_2.\end{aligned} \tag{41}$$

Recall that the sign of complex conjugation in Euclidean quantum theory does not matter. Here, J is a multiplier containing gauge conditions and the corresponding Faddeev-Popov determinant in the measure of functional integration. Let us note here that below we will need to regularize the action Equation (40), which we will achieve by changing the operator implementation of the momenta Equation (39).

An alternative representation of the gravitational field action operator is possible, based on its self-dual (and anti-self-dual) representation (see [21]):

$$I_{eA} = i\int_\Omega d^4x e^{AA'}\wedge e_{BA'}\wedge F_A^B\;, \tag{42}$$

where

$$F_A^B \equiv dA_A^B + A_A^P\wedge A_P^B \tag{43}$$

and $A_{\nu A}{}^B$ is the complex left $SL(2,C)$ connection and Hermitian field of the tetrad $e_\mu{}^{AA'}$, which form differential 1-forms. Action Equation (42) is also the starting point for the complex canonical formulation of Ashtekar's theory of gravity [22]. Within the framework of the approach developed in this work, we introduce a generalized canonical momentum for the connection $A_{\mu A}{}^B$

$$\begin{aligned}P^{\mu\nu A}{}_B(x) \;\equiv\;& \frac{\delta I_{eA}}{\delta\partial_\mu A_{\nu A}{}^B(x)}\\ =\;& iE^{\mu\nu\alpha\beta}e_\alpha{}^{AA'}e_{\beta\,a'B'}\end{aligned} \tag{44}$$

where $E^{\mu\nu\alpha\beta}$ is a completely antisymmetric unit tensor density of weight 1 [23]. Thus, the generalized canonical form of action Equation (42) has the simple form:

$$I_{eA} = \int_\Omega d^4x\,F_{\mu\nu A}{}^B\,P^{\mu\nu A}{}_B. \tag{45}$$

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As in the case of the Yang-Mills field, we introduce an antisymmetric functional operator implementation of momenta Equation (44) on the space of wave functionals $\Psi[A]$:

$$\begin{aligned} &\widehat{P}^{\mu\nu A}{}_B(x) \\ \equiv\; &\widetilde{\hbar}^{\mu} \frac{\delta}{\delta_{\mu}\, A_{\nu A}{}^{B}(x)} - \widetilde{\hbar}^{\nu} \frac{\delta}{\delta_{\nu}\, A_{\mu A}{}^{B}(x)}. \end{aligned} \tag{46}$$

Substituting Equation (46) into Equation (45), we obtain the quantum principle of least action of the theory of gravity in self-dual form:

$$\begin{aligned} &\int_{\Omega} d^4x\, F_{\mu\nu A}{}^{B} \\ &\times \left(\widetilde{\hbar}^{\mu} \frac{\delta\Psi}{\delta_{\mu}\, A_{\nu A}{}^{B}(x)} - \widetilde{\hbar}^{\nu} \frac{\delta\Psi}{\delta_{\nu}\, A_{\mu A}{}^{B}(x)} \right) \\ = \;& \Lambda\Psi. \end{aligned} \tag{47}$$

In the theory of gravity, we are interested in the real part of the eigenvalue Λ and its minimum.

4. Quantum Principle of Least Action in Quantum Cosmology

We are ready to formulate the quantum principle of least action in cosmology. We will use the real form of the theory of gravity. Let us add to the operator of the action of the gravitational field Equation (40) the operator of the action of the fields of matter $\widehat{I}_{\phi}$ and obtain the operator of the action of the universe $\widehat{I} = \widehat{I}_g + \widehat{I}_{\phi}$. For the complete action operator, we write a secular equation in the region Ω of the Origin of the Universe:

$$\widehat{I}\Psi[g, \phi] = -\Lambda\Psi[g, \phi], \tag{48}$$

which is the quantum principle of least action for universe in original form. In this problem, we are primarily interested not in the eigenfunctional $\Psi[g, \phi]$ inside the domain Ω, but in its value on the boundary of the domain, which, according to Equation (16), is determined by the eigenvalue Λ of the action operator. The size and shape of the polar region remain undetermined in our problem. From the very beginning we abandoned polar coordinates, which are singular at the pole. But near the boundary, polar coordinates $(r, \theta^1, \theta^2, \theta^3)$ can be introduced, and are uniquely related to the internal "Cartesian" coordinates x^{μ}. We will further assume that the components of the $4D$ geometry on the $3D$ boundary $\partial\Omega$ are specified in polar coordinates. We also include here the lapse and shift functions, which arise in the $3+1$ splitting of the $4D$ metric,

$$ds^2 = (Ndr)^2 + g_{kl}\left(dx^k + N^k dr\right)\left(dx^l + N^l dr\right). \tag{49}$$

The transition to polar coordinates at the boundary of the polar region serves as a bridge between the Euclidean "dynamics" inside Ω and the real history of the universe outside Ω, in which the role of time will be played by the radial coordinate r (after the Wick rotation $N \longrightarrow iN$).

The initial state of the universe, which is also the boundary state for the internal "dynamics" in the polar region, in accordance with Equation (16), is defined as follows:

$$\psi = \exp\left(\frac{i}{\hbar}\Lambda\right). \tag{50}$$

Without moving on to real time yet, let us impose an additional condition on the eigenvalue Λ, and therefore on state Equation (50):

$$\left(2N^k_{|k} - \frac{N}{\sqrt{\det(g_{kl})}}\widehat{\pi}\right)\Lambda = 0, \tag{51}$$

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where $\widehat{\pi} = g_{kl}\widehat{\pi}^{kl}$ is the trace of the canonical momentum conjugate to the $3D$ metric in the generally accepted canonical formulation of ADM, which in the Euclidean form we implement as follows:

$$\widehat{\pi}^{kl}(x) \equiv \hbar\frac{\delta}{\delta g_{kl}(x)}. \tag{52}$$

According to the equations of motion of the theory of gravitation [7], equation Equation (51) is equivalent to the requirement

$$\frac{\partial}{\partial t}\det(g_{kl}) = 0, \tag{53}$$

which means that the boundary $\partial\Omega$ is formed by cusp points for the dynamics of the scale factor $\det(g_{kl})$ inside the polar region. This determines the size and shape of the polar region Ω.

When we talk about the principle of least action in classical mechanics, we mean, first of all, the extremum conditions, the consequences of which are the equations of motion. For now, we also consider Equation (54) as a functional-differential equation on the space of wave functionals. In order to turn Equation (54) into a real minimum principle,

$$\Lambda = -\min\frac{\left(\Psi, \widehat{I}\Psi\right)}{(\Psi, \Psi)}, \tag{54}$$

we must know that this minimum really exists. The problem is that the original Euclidean Hilbert-Einstein action does not have a specific sign [24]. The minus sign in front of the second term in Equation (31) can be leveled out by harmonic coordinate conditions [25]:

$$\Gamma^{\alpha} \equiv g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0. \tag{55}$$

These coordinate conditions can be taken into account before quantization by adding them to the original action with Lagrange multipliers. In this case, the factor J in Equation (41) is not needed, but the minimum Equation (54) should be sought by also varying the Lagrange multipliers. However, there is still uncertainty in the sign of the first term in Equation (31). It is easy to see that this uncertainty is associated with the incomplete symmetry of the expression

$$\Gamma_{\beta,\mu\nu} = \frac{1}{2}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \tag{56}$$

relative to permutations of all indices and cannot be eliminated by coordinate conditions. The obstacle to complete symmetry is the expression

$$\partial_{[\mu}g_{\beta]\nu} \equiv \frac{1}{2}(\partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}) \neq 0. \tag{57}$$

Let us represent the derivatives of the metric tensor as a sum

$$\partial_\mu g_{\beta\nu} = \partial_{(\mu}g_{\beta)\nu} + \partial_{[\mu}g_{\beta]\nu}, \tag{58}$$

in which the first term is completely symmetrical (taking into account the symmetry $g_{\mu\nu}$). The "unfavorable" second term requires special consideration. We have the opportunity to change the sign of the corresponding contribution to the quadratic form of the generalized canonical action

Equation (38) by modifying the quantization rules. Namely, instead of Equation (39), we define the generalized canonical momentum operator as a sum:

$$\begin{aligned} &\widehat{p}_g^{\alpha|\mu\nu}(x) \\ \equiv\ &\frac{1}{2}\left(\widetilde{\hbar}^{\alpha}\frac{\delta}{\delta_{\alpha}g_{\mu\nu}(x)}+\widetilde{\hbar}^{\mu}\frac{\delta}{\delta_{\mu}g_{\alpha\nu}(x)}\right) \\ &+\frac{1}{2}\left(\frac{\widetilde{\hbar}^{\alpha}}{i}\frac{\delta}{\delta_{\alpha}g_{\mu\nu}(x)}-\frac{\widetilde{\hbar}^{\mu}}{i}\frac{\delta}{\delta_{\mu}g_{\alpha\nu}(x)}\right). \end{aligned} \tag{59}$$

where the second term is formed by Hermitian functional differentiation operators. With this modification, the Hilbert space of wave functional becomes complex, and the sign of complex conjugation in the Hermitian scalar product (41) becomes significant. For matter fields, we will leave the quantization rules unchanged, since the Euclidean action of matter fields has a certain sign. The gravitational field action operator modified in this way becomes bounded from below, and the minimum problem Equation (54) becomes meaningful. This opens up the possibility of formulating the quantum principle of least action in the polar region as a computational problem.

5. Conclusions

The purpose of this work is to determine the initial state of the universe on the border with the polar region, where there is no dedicated time parameter. The last step we must take is the Wick rotation in the complex plane of the succession function: $N \longrightarrow iN$ in expression Equation (49). After this, the history of the universe is described within the framework of the generally accepted QTG formalism with the evolution parameter $t = r$. The structure of quantum "dynamics" in the polar region is dictated by the equality of all dimensions of $4D$ space and the internal symmetries of the multiplet of matter fields. A convenient formalism here turns out to be the quantum principle of least action, which allows the formulation of the corresponding computational problem for a minimum Equation (53). This principle is similar to the principle of minimum energy in quantum mechanics, which determines the ground state (vacuum) of the system. Therefore, the initial state Equation (49) can also be called a vacuum in quantum cosmology. Time is implicitly present at all stages of our consideration. It is present in $4D$ space in the polar region, being completely equal with other spatial dimensions. It then appears as a radial coordinate in the polar coordinate grid at the exit from the polar region. In this case, in the minimum problem Equation (53) no constraints arise, except for the boundary condition Equation (57). This means that the initial wave function Equation (49), generally speaking, does not obey the constraint equations of the generally accepted QTG. If this is so, we can talk about corresponding deviations from the classical Einstein equations, which we associate in [10] with the distribution and movement of the universe's own mass. Since Gaussian constraints corresponding to internal symmetries in the polar region are also absent, one should expect the presence of uncompensated gauge charges in the initial state Equation (49).

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