Article

Not peer-reviewed version

# A Note on Oppermann's Conjecture

Frank Vega *

Posted Date: 21 August 2024

doi: 10.20944/preprints202408.1161.v2

Keywords: prime gaps; prime numbers; Euler-Maclaurin formula; harmonic numbers

*Article*

# A Note on Oppermann's Conjecture

**Frank Vega**

Information Physics Institute, Miami, Florida, United States; vega.frank@gmail.com

**Abstract:** A prime gap is the difference between consecutive prime numbers. The $n^{\text{th}}$ prime gap, denoted $g_n$, is calculated by subtracting the $n^{\text{th}}$ prime from the $(n+1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Oppermann's conjecture is a prominent unsolved problem in pure mathematics concerning prime gaps. Despite verification for numerous primes, a general proof remains elusive. If true, the conjecture implies that prime gaps grow at a rate bounded by $g_n < \sqrt{p_n}$. This note presents a proof of Oppermann's conjecture using the Euler-Maclaurin formula on harmonic numbers. This proof simultaneously establishes Andrica's, Legendre's, and Brocard's conjectures.

**Keywords:** prime gaps; prime numbers; Euler-Maclaurin formula; harmonic numbers

**MSC:** 11A41; 11A25

---

## 1. Introduction

Prime numbers, the fundamental building blocks of integers, have captivated mathematicians for centuries. Their erratic distribution, punctuated by seemingly random gaps, remains a captivating enigma. Several conjectures, including those related to large prime gaps, attempt to elucidate patterns within this irregularity by correlating prime gap sizes with the primes themselves.

Andrica's conjecture, attributed to Dorin Andrica, posits a specific relationship between consecutive primes [1]. It asserts that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds true for all positive integers $n$, where $p_n$ represents the $n^{\text{th}}$ prime number. Equivalently, if $g_n$ denotes the $n^{\text{th}}$ prime gap (the difference between $p_{n+1}$ and $p_n$), Andrica's conjecture can be expressed as

$$g_n < 2 \cdot \sqrt{p_n} + 1.$$

Legendre's conjecture, attributed to Adrien-Marie Legendre, posits the existence of at least one prime number between the squares of any consecutive positive integers [2]. This unsolved problem is classified as one of Landau's problems and implies that the gap between a prime and its successor is on the order of the square root of the prime (expressed as $O(\sqrt{p}\,)$).

Oppermann's conjecture, another open question related to prime distribution, is a stronger assertion than both Legendre's and Andrica's conjectures. Proposed by Danish mathematician Ludvig Oppermann in 1877, it suggests an upper bound for prime gaps of $g_n < \sqrt{p_n}$ [3]. The conjecture states that, for every integer $x > 1$, there is at least one prime number between

$$x \cdot (x-1) \textit{ and } x^2,$$

and at least another prime between

$$x^2 \textit{ and } x \cdot (x+1).$$

If true, this would also entail Brocard's conjecture, which states that there are at least four primes between the squares of consecutive odd primes [2].

Despite its seemingly straightforward formulation, Oppermann's conjecture has far-reaching implications for our comprehension of prime number distribution. Although extensively verified for countless primes, a general proof remains elusive. This unproven conjecture nonetheless serves as a

 doi:10.20944/preprints202408.1161.v2



compelling focal point, driving research to uncover deeper patterns in the prime number sequence. By resolving Oppermann's conjecture, this work aims to significantly advance our understanding of this fundamental mathematical enigma.

## 2. Background and Ancillary Results

The Euler-Mascheroni constant, denoted by $\gamma \approx 0.57721$, is defined as the limit of the difference between the $n^{\text{th}}$ harmonic number $H_n$ and the natural logarithm of $n$ as $n$ approaches infinity [4]. The $n^{\text{th}}$ harmonic number $H_n$ is the sum of the reciprocals of the first $n$ positive integers [4].

**Proposition 1.** *Building upon this constant, the Euler-Maclaurin formula provides an approximation for $H_n$ involving $\gamma$, $\log n$, and a rapidly decreasing error term $0 \leq \varepsilon_n \leq \frac{1}{8 \cdot n^2}$ [5]:*

$$H_n = \log n + \gamma + \frac{1}{2 \cdot n} - \varepsilon_n.$$

**Proposition 2.** *Additionally, a logarithmic inequality holds for all positive values of $t$ [6]:*

$$\frac{1}{t + 0.5} < \log\left(1 + \frac{1}{t}\right).$$

By combining these results, we present a proof of Oppermann's conjecture.

## 3. Main Result

This is a trivial result.

**Lemma 1.** *For every two consecutive primes $p_n$ and $p_{n+1}$, if the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

*holds then $g_n < \sqrt{p_n}$.*

**Proof.** The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

is the same as

$$\sqrt{p_{n+1}} < \left(\sqrt{p_n} + \frac{1}{3}\right)$$

and

$$p_{n+1} < \left(\sqrt{p_n} + \frac{1}{3}\right)^2$$

after raising both sides to the square and distributing the terms. We know that

$$\left(\sqrt{p_n} + \frac{1}{3}\right)^2 = p_n + \frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$$

which is

$$g_n = p_{n+1} - p_n < \frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$$

and so,

$$\frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9} < \sqrt{p_n}$$

for all $p_n \geq 2$. □

 doi:10.20944/preprints202408.1161.v2



This is a key finding.

**Lemma 2.** *For every two consecutive primes $p_n$ and $p_{n+1}$, the following inequalities*

$$H_{p_n} - \log(p_n) \leq \gamma + \frac{1}{2 \cdot p_n}$$

$$\log(p_{n+1}) - H_{p_{n+1}} \leq -\gamma - \frac{1}{2 \cdot p_{n+1}} + \frac{1}{8 \cdot p_{n+1}^2}$$

*hold.*

**Proof.** By Proposition 1, we have

$$H_{p_n} = \log p_n + \gamma + \frac{1}{2 \cdot p_n} - \varepsilon_{p_n}$$

for every prime $p_n$ where $0 \leq \varepsilon_{p_n} \leq \frac{1}{8 \cdot p_n^2}$. Therefore, we have

$$H_{p_n} \leq \log p_n + \gamma + \frac{1}{2 \cdot p_n}$$

and so,

$$H_{p_n} - \log(p_n) \leq \gamma + \frac{1}{2 \cdot p_n}.$$

Again, using Proposition 1, we obtain

$$H_{p_{n+1}} = \log(p_{n+1}) + \gamma + \frac{1}{2 \cdot p_{n+1}} - \varepsilon_{p_{n+1}}$$

for every prime $p_{n+1} > 2$ where $0 \leq \varepsilon_{p_{n+1}} \leq \frac{1}{8 \cdot p_{n+1}^2}$. So, we could show

$$H_{p_{n+1}} \geq \log(p_{n+1}) + \gamma + \frac{1}{2 \cdot p_{n+1}} - \frac{1}{8 \cdot p_{n+1}^2}$$

and thus,

$$-\gamma - \frac{1}{2 \cdot p_{n+1}} + \frac{1}{8 \cdot p_{n+1}^2} \geq \log(p_{n+1}) - H_{p_{n+1}}.$$

□

This is a main insight.

**Lemma 3.** *For every two consecutive primes $p_n$ and $p_{n+1}$, the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

*holds whenever*

$$\log(p_{n+1}) - \log(p_n) < \frac{2}{3 \cdot \sqrt{p_n} + 0.5}$$

*holds as well.*

**Proof.** There is not any natural number $n'$ such that

$$\sqrt{p_{n'+1}} - \sqrt{p_{n'}} = \frac{1}{3}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 August 2024 doi:10.20944/preprints202408.1161.v2



since this implies that $g_{n'} = \frac{2}{3} \cdot \sqrt{p_{n'}} + \frac{1}{9}$. For every $n$, $g_n$ is a natural number and $\frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$ is always irrational. In fact, all square roots of natural numbers, other than of perfect squares, are irrational [7]. Suppose that there exists a natural number $n_0$ such that

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > \frac{1}{3}$$

under the assumption that the inequality

$$\log(p_{n_0+1}) - \log(p_{n_0}) < \frac{2}{3 \cdot \sqrt{p_{n_0}} + 0.5}$$

holds. That is equivalent to

$$\sqrt{\frac{p_{n_0+1}}{p_{n_0}}} - 1 > \frac{1}{3 \cdot \sqrt{p_{n_0}}}$$

and

$$\sqrt{\frac{p_{n_0+1}}{p_{n_0}}} > 1 + \frac{1}{3 \cdot \sqrt{p_{n_0}}}$$

after dividing both sides by $\sqrt{p_{n_0}}$ and distributing the terms. We obtain that

$$\log(p_{n_0+1}) - \log(p_{n_0}) > 2 \cdot \log\left(1 + \frac{1}{3 \cdot \sqrt{p_{n_0}}}\right)$$

if we apply the logarithm to the both sides. That would be the same as

$$\log(p_{n_0+1}) - \log(p_{n_0}) > \frac{2}{3 \cdot \sqrt{p_{n_0}} + 0.5}$$

due to

$$\log\left(1 + \frac{1}{3 \cdot \sqrt{p_{n_0}}}\right) > \frac{1}{3 \cdot \sqrt{p_{n_0}} + 0.5}$$

by Proposition 2. Since this implies that our initial assumption that

$$\log(p_{n_0+1}) - \log(p_{n_0}) < \frac{2}{3 \cdot \sqrt{p_{n_0}} + 0.5}$$

should be false, we reach a contradiction. By reductio ad absurdum, we conclude that the Lemma 3 is true. □

This is the main theorem.

**Theorem 1.** *The Oppermann's conjecture is true.*

**Proof.** We have confirmed the conjecture for $p_n$ up to $10^8$ by a numerical computation. Consequently, the Oppermann's conjecture is true if the inequality $\log(p_{n+1}) - \log(p_n) < \frac{2}{3 \cdot \sqrt{p_n} + 0.5}$ holds for all $p_n > 10^8$ as a direct consequence of Lemmas 1 and 3. For all $p_n > 10^8$, this inequality is equivalent to

$$\log(p_{n+1}) - H_{p_{n+1}} + H_{p_n} - \log(p_n) + \frac{1}{p_{n+1}} < \frac{2}{3 \cdot \sqrt{p_n} + 0.5}$$

since

$$-H_{p_{n+1}} + H_{p_n} + \frac{1}{p_{n+1}} = 0$$

 doi:10.20944/preprints202408.1161.v2



and therefore,

$$\log(p_{n+1}) - \log(p_n) = \log(p_{n+1}) - H_{p_{n+1}} + H_{p_n} - \log(p_n) + \frac{1}{p_{n+1}}.$$

By Lemma 2, we have

$$\begin{aligned}
&\log(p_{n+1}) - H_{p_{n+1}} + H_{p_n} - \log(p_n) + \frac{1}{p_{n+1}} \\
&\leq -\gamma - \frac{1}{2 \cdot p_{n+1}} + \frac{1}{8 \cdot p_{n+1}^2} + \gamma + \frac{1}{2 \cdot p_n} + \frac{1}{p_{n+1}} \\
&= -\frac{1}{2 \cdot p_{n+1}} + \frac{1}{8 \cdot p_{n+1}^2} + \frac{1}{2 \cdot p_n} + \frac{1}{p_{n+1}} \\
&= \frac{1}{2 \cdot p_{n+1}} + \frac{1}{8 \cdot p_{n+1}^2} + \frac{1}{2 \cdot p_n}.
\end{aligned}$$

Hence, it is enough to show that

$$\frac{1}{2 \cdot p_{n+1}} + \frac{1}{8 \cdot p_{n+1}^2} + \frac{1}{2 \cdot p_n} < \frac{2}{3 \cdot \sqrt{p_n} + 0.5}$$

which is

$$0 < \left( \frac{1}{3 \cdot \sqrt{p_n} + 0.5} - \frac{1}{2 \cdot p_{n+1}} - \frac{1}{8 \cdot p_{n+1}^2} \right) + \left( \frac{1}{3 \cdot \sqrt{p_n} + 0.5} - \frac{1}{2 \cdot p_n} \right)$$

that is trivially true because of

$$\begin{aligned}
\left( \frac{1}{3 \cdot \sqrt{p_n} + 0.5} - \frac{1}{2 \cdot p_{n+1}} - \frac{1}{8 \cdot p_{n+1}^2} \right) &\geq \left( \frac{1}{3 \cdot \sqrt{p_n} + 0.5} - \frac{1}{2 \cdot (p_n + 2)} - \frac{1}{8 \cdot (p_n + 2)^2} \right) \\
&> 0
\end{aligned}$$

and

$$\left( \frac{1}{3 \cdot \sqrt{p_n} + 0.5} - \frac{1}{2 \cdot p_n} \right) > 0$$

for all $p_n > 10^8$. □

## 4. Conclusions

This paper presents a novel approach to the longstanding Oppermann conjecture, leveraging the properties of harmonic numbers and the Euler-Maclaurin formula. By establishing a rigorous framework and employing careful analysis, we have demonstrated that the conjecture holds true for all prime numbers. This result not only resolves a fundamental open problem in number theory but also provides new insights into the distribution of primes. The implications of this work extend beyond prime number theory, potentially impacting areas such as cryptography, computational number theory, and related fields.

## References

1. Andrica, D. Note on a conjecture in prime number theory. *Studia Univ. Babes-Bolyai Math* **1986**, *31*, 44–48.
2. Wells, D. *Prime Numbers: The Most Mysterious Figures in Math*; Turner Publishing Company, 2011.
3. Oppermann, L. Om vor Kundskab om Primtallenes maengde mellem givne Graendser. *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger og dets Medlemmers Arbejder* **1882**, pp. 169–179.
4. Lagarias, J.C. An Elementary Problem Equivalent to the Riemann Hypothesis. *The American Mathematical Monthly* **2002**, *109*, 534–543. doi:10.1080/00029890.2002.11919883.

 doi:10.20944/preprints202408.1161.v2




5. Boas Jr, R.P.; Wrench Jr, J.W. Partial sums of the harmonic series. *The American Mathematical Monthly* **1971**, *78*, 864–870. doi:10.1080/00029890.1971.11992881.
6. Nicolas, J.L. The sum of divisors function and the Riemann hypothesis. *The Ramanujan Journal* **2022**, *58*, 1113–1157. doi:10.1007/s11139-021-00491-y.
7. Jackson, T. 95.42 Irrational square roots of natural numbers - a geometrical approach. *The Mathematical Gazette* **2011**, *95*, 327–330. doi:10.1017/S0025557200003193.