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The Energy-Momentum Tensor of Electromagnetism Revisited

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Abstract: Maxwell's energy-momentum tensor cannot be used as is in Einstein's equation for general relativity in the presence of sources. In this case, its four-divergence is non-zero, whereas the tensors in the equation must have this characteristic. However, the sun emits streams of ionized particles, gases are present within galaxies, material transfers in the form of ionized plasma can occur in binary star systems, and so on. These charges cannot be neglected in the study of space-time in these regions. Here, we present a modified version of this tensor that eliminates this flaw, provided that the potentials satisfy the Lorenz gauge. The distribution of sources will also be analyzed in light of fluid mechanics, allowing us to account for its influence in terms of generated pressure.

Keywords: electromagnetism; general relativity; cosmology; canonical tensor

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1. Introduction

In the context of Noether's theorem, an energy-momentum tensor is canonical if the four-divergence is zero. This corresponds to the conservation of energy and momentum. Unfortunately, the energy-momentum tensor of Maxwell commonly used satisfies this condition only in the absence of charges and currents and therefore cannot be used as is in presence of sources.

If, within the framework of general relativity, we wish to have a geometrically correct description of spacetime on scales of galaxies, star systems, etc., we cannot neglect the influence of the charged particles present and their interactions with electromagnetic fields. These phenomena must therefore be taken into account in the energy-momentum tensor that appears in Einstein's equations.

The problem of formulating a stress-energy-momentum tensor for electromagnetism dates back over a century, and has never really been solved in the case of the presence of sources (charges and/or currents). Some research has tackled the question, but generally from a specific angle [1–3]. In this article, we deal with the problem in all its generality.

The tensor presented in this article takes these effects into account and is thus likely to pave the way for better modeling and, in turn, improve the understanding and knowledge of the physics of these regions. If sources are present, they will be modeled by a fluid of charged particles. The rest space is considered to be comoving with the particles, and the values are the means of the total particles.

Symmetry comes at a cost: the potentials must satisfy the Lorenz gauge.

For the sake of simplicity in writing, the development is first carried out within the framework of special relativity in Minkowski spacetime and will then be extended to general relativity.

2. Notation

The Greek indices take the values 0, 1, 2, and 3, while the Latin indices range from 1 to 3. The main symbols used in this paper are summarized in the following table.

$\eta^{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1)$	Minkowski metric
$g_{\mu\nu}$	General relativity spacetime metric
$\boldsymbol{E}$	Electric field
$\boldsymbol{B}$	Magnetic induction
q	Particle charge
m	Particle mass

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P	Fluid pressure
n	Particle density (number of particles per m^3)
$\boldsymbol{x}$	Position in 3D
$x^\mu = (ct, \boldsymbol{x})$	Position contravariant 4-vector
$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$	3D Nabla operator
$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \boldsymbol{\nabla}\right)$	4D derivative operator
∇_μ	4D covariant derivative operator
$\boldsymbol{v}$	Speed in 3D
$x^\mu = (c, \boldsymbol{v})$	Speed contravariant 4-vector
$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$	Lorentz contraction factor
$u^\mu = \gamma(1, \frac{\boldsymbol{v}}{c})$	Normalized speed contravariant 4-vector
$\rho_m = nm$	Particle rest mass density
$\rho_e = nq$	Particle rest charge density
$\mathcal{E}$	Fluid rest total energy density
$\boldsymbol{J} = \rho_e \boldsymbol{v}$	Current density
Φ	Scalar potential
$\boldsymbol{A}$	Vector potential
$\phi^\mu = \left(\frac{\Phi}{c}, \boldsymbol{A}\right)$	Potential contravariant 4-vector
$F^{\mu\nu} = \partial^\mu \phi^\nu - \partial^\nu \phi^\mu$	Contravariant Faraday tensor
$j^\mu = (\rho_e \Phi, \rho_e \boldsymbol{v}) = \frac{1}{\mu_0} \partial_\alpha F^{\alpha\mu}$	Current density contravariant 4-vector
$f^\mu = \rho_e\left(\frac{\boldsymbol{E}.\boldsymbol{v}}{c}, \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) = j_\alpha F^{\mu\alpha}$	Lorentz force density contravariant 4-vector

3. Charged Particles Distribution

We will consider that distribution can be likened to an incompressible inviscid relativistic fluid. As a consequence, its rest mass densityρ_m is constant, and the speed divergence is zero $\boldsymbol{\nabla}.\boldsymbol{v} = 0$ and $\partial_\mu v^\mu = 0$.

This gives us two alternatives for $\boldsymbol{v}$: either it does not depend explicitly on spatial coordinates, or it derives from the curl of a potential vector. For each kind of particles, the quantities m and q are constants, yielding the ratio $\frac{\rho_e}{\rho_m} = \frac{q}{m}$.

The expression for the canonical relativistic energy-momentum tensor can be found in [4–7] and is

$$T_{fl}^{\mu\nu} = (\mathcal{E} + P)u^\mu u^\nu - \eta^{\mu\nu} P \tag{1}$$

with the symbols defined previously. In a comoving rest frame, this tensor is diagonal: $diag(\mathcal{E}, P, P, P)$.

The conservation laws of energy and momentum imply that $\partial_\mu T_{fl}^{\mu\nu} = 0$.

The evidence that this relationship is verified is available in [4–7].

4. Construction of a Canonical Tensor, Neither Symmetric nor Anti-Symmetric

Let us start with the Lagrangian density of electromagnetism in the presence of charges, augmented by the Lagrangian density of incompressible relativistic fluid $\mathcal{L}_{fl}$, as we consider it to represent the particle distribution. This provides

$$\mathcal{L}(\phi_\alpha, \partial_\mu \phi_\alpha) = -\frac{1}{4\mu_0} F_{\mu\alpha} F^{\mu\alpha} - \phi_\alpha j^\alpha + \mathcal{L}_{fl}$$

with $F_{\mu\alpha} = \partial_\mu \phi_\alpha - \partial_\alpha \phi_\mu$.

Given that we already know the canonical tensor of the fluid by (1), we will not further develop $\mathcal{L}_{fl}$; this approach is beyond the scope of this publication.

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However, for the electromagnetic part, the contravariant canonical tensor is obtained from the relation $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)}\partial^\nu \phi_\alpha - \eta^{\mu\nu}\mathcal{L}$, which allows us to write an initial tensor

$$\Theta^{\mu\nu}(\phi_\alpha, \partial_\mu\phi_\alpha, u_\alpha) = -\frac{1}{\mu_0}F^\mu{}_\alpha \partial^\nu \phi^\alpha + \frac{\eta^{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} + \eta^{\mu\nu}\phi_\alpha j^\alpha$$
$$+(\mathcal{E}+P)u^\mu u^\nu - \eta^{\mu\nu}P \tag{2}$$

This tensor will be made symmetric in the next section.

5. Angular Momentum Conservation

If the canonical tensor $\Theta^{\mu\nu}$ is not symmetric, the angular momentum tensor is written as $M^{\alpha\mu\nu} = x^\mu \Theta^{\alpha\nu} - x^\nu \Theta^{\alpha\mu} + S^{\alpha\mu\nu}$, where $S^{\alpha\mu\nu}$ is an intrinsic spin tensor. By construction, $S^{\alpha\mu\nu}$ is anti-symmetric in (μ, ν).

The angular momentum conservation implies $\partial_\alpha M^{\alpha\mu\nu} = 0$, and thus, taking into account that $\partial_\alpha \Theta^{\alpha\mu} = \partial_\alpha \Theta^{\alpha\nu} = 0$, the relation must be satisfied $\Theta^{\mu\nu} - \Theta^{\nu\mu} = \partial_\alpha S^{\alpha\nu\mu}$.

The anti-symmetric part of the tensor to be made symmetric must therefore be expressible as the four-divergence of a rank-3 tensor if one wishes to make it symmetric.

The calculation of the anti-symmetric part yields

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = \frac{1}{\mu_0}(\partial^\nu \phi_\alpha \partial^\mu \phi^\alpha - \partial_\alpha \phi^\nu \partial^\mu \phi^\alpha - \partial^\mu \phi_\alpha \partial^\nu \phi^\alpha + \partial_\alpha \phi^\mu \partial^\nu \phi^\alpha)$$

Examination of this relationship shows that the only way to express it solely in the form of a four-divergence involves $\partial_\alpha \phi^\alpha = 0$, the Lorenz gauge.

Under this condition, we finally obtain $S^{\alpha\mu\nu} = \frac{1}{\mu_0}(\phi^\mu \partial^\nu \phi^\alpha - \phi^\nu \partial^\mu \phi^\alpha)$.

We will now apply the Belinfante method [8].

Let us define $\theta^{\alpha\mu\nu} = \frac{1}{2}(S^{\alpha\nu\mu} + S^{\nu\alpha\mu} - S^{\mu\nu\alpha})$; its computation yields $\theta^{\alpha\mu\nu} = \frac{1}{2\mu_0}(\phi^\mu \partial^\nu \phi^\alpha - \phi^\nu \partial^\mu \phi^\alpha + \phi^\mu \partial^\alpha \phi^\nu - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu + \phi^\nu \partial^\alpha \phi^\mu)$.

The symmetrical tensor $T^{\mu\nu}$ is obtained by subtracting

$$\partial_\alpha \theta^{\alpha\mu\nu} = \frac{1}{2\mu_0}\partial_\alpha(\phi^\mu \partial^\nu \phi^\alpha + \phi^\nu \partial^\mu \phi^\alpha - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu) - \frac{1}{\mu_0}F^\mu{}_\alpha \partial^\alpha \phi^\nu + \frac{1}{2}(\phi^\mu j^\nu + \phi^\nu j^\mu)$$

from egation (2), yielding

$$T^{\mu\nu}(\phi_\alpha, \partial_\mu\phi_\alpha, u_\alpha) = \frac{1}{\mu_0}F^\mu{}_\alpha F^{\alpha\nu} + \frac{\eta^{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} + (\mathcal{E}+P)u^\mu u^\nu$$
$$-\frac{1}{2}\phi^\mu j^\nu - \frac{1}{2}\phi^\nu j^\mu + \eta^{\mu\nu}(\phi_\alpha j^\alpha - P)$$
$$-\frac{1}{2\mu_0}\partial_\alpha(\phi^\mu \partial^\nu \phi^\alpha + \phi^\nu \partial^\mu \phi^\alpha - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu) \tag{3}$$

Even though the term $\partial_\alpha(\phi^\mu \partial^\nu \phi^\alpha + \phi^\nu \partial^\mu \phi^\alpha - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu)$ is a four-divergence term, it cannot be eliminated in any way. If this is the case, $\partial_\mu T^{\mu\nu} = 0$ would no longer be satisfied.

We also see from (3) that the energy-momentum tensor of Maxwell appears spontaneously in the first two terms of the right-hand side.

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6. Hamiltonian Density

In our canonical tensor, the Hamiltonian density is provided by the term T^{00}. Calculating this element leads to the following expression:

$$T^{oo} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \frac{\mathcal{E} + P\frac{V^2}{c^2}}{1 - \frac{V^2}{c^2}}$$

which corresponds well to the sum of the electromagnetic energy and the fluid energy densities.

To find the nonrelativistic limit ($v << c$), it is convenient to express the total energy density $\mathcal{E}$ as the sum of the mass energy $\rho_m c^2$ and the internal energy $\rho_m \varpi$ densities.

In doing so, it brings us to

$$T^{oo} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \rho_m c^2 + \rho_m \varpi + \frac{1}{2}\rho_m v^2$$

where the contribution of $P\frac{V^2}{c^2}$ is not taken into account as negligible with respect to the other terms.

In a comoving frame, as $\boldsymbol{v} = 0$, $\boldsymbol{A}.\boldsymbol{J} = \rho_e \boldsymbol{A}.\boldsymbol{v} = 0$ and the Hamiltonian density reduces to $T^{00} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) + \mathcal{E}$.

7. $T^{\mu}_{\ \mu}$ in Special Relativity

In Minkowski spacetime, $T^{\mu}_{\ \mu}$ is readily calculated, which leads to

$$T^{\mu}_{\ \mu} = \mathcal{E} - 3(P - \phi_\alpha j^\alpha)$$

where $\phi_\alpha j^\alpha$, which is the dot product of two 4-vectors, is a relativistic invariant.

In a comoving frame, as $\boldsymbol{v} = 0$, $\phi_\alpha j^\alpha$ reduces to $\rho_e \Phi$ and $T^{\mu}_{\ \mu}$ reduces to $\mathcal{E} - 3(P - \rho_e \Phi)$.

In astrophysical scenarios, relativistic fluids are often encountered in extreme environments, such as close to massive objects like black holes or in high-energy events like supernovae. In these situations, the speeds of particles or matter involved are comparable to the speed of light.

For a relativistic perfect fluid, one can demonstrate [6] that the trace of its energy-momentum tensor equals $\gamma^{-1}\rho_m c^2$.

As for ultra-relativistic fluids $\gamma^{-1} \approx 0$ and hence $\mathcal{E} \approx 3P$. It means that $T^{\mu}_{\ \mu}$ will then depends almost only on the electromagnetism through $3\phi_\alpha j^\alpha$ or $3\rho_e \Phi$ in a co-moving rest frame.

For baryonic matter, $T^{\mu}_{\ \mu}$ must be positive or null which implies $\mathcal{E} \geq 3(P - \phi_\alpha j^\alpha)$. Violation of this condition would indicate that we are in the presence of exotic matter.

As it's always possible to find a region of space small enough to use the usual Minkowski spacetime coordinates, it would be interesting to check whether $3\phi_\alpha j^\alpha$ does not become (transiently) negative enough for $T^{\mu}_{\ \mu}$ to become negative too, especially in regions where ultrarelativistic fluids are present.

8. General Relativity

In the presence of charges, the Maxwell stress-energy momentum tensor is not suitable for use, as is the case for Einstein's equation, because its 4-divergence is not null. On the other hand, the tensor that we developed in (3) does not suffer from this defect and can be introduced directly, as is Einstein's equation.

Its covariant form, adapted to general relativity, is given by

$$\begin{aligned} T_{\mu\nu}(\phi_\alpha, \partial_\mu\phi_\alpha, u_\alpha) &= \frac{1}{\mu_0}F_{\mu\alpha}F^{\alpha}_{\ \nu} + \frac{g_{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} + (\mathcal{E}+P)u_\mu u_\nu \\ &-\frac{1}{2}\phi_\mu j_\nu - \frac{1}{2}\phi_\nu j_\mu + g_{\mu\nu}(\phi_\alpha j^\alpha - P) \\ &-\frac{1}{2\mu_0}\nabla_\alpha(\phi_\mu\nabla_\nu\phi^\alpha + \phi_\nu\nabla_\mu\phi^\alpha - \phi^\alpha\nabla_\mu\phi_\nu - \phi^\alpha\nabla_\nu\phi_\mu) \end{aligned} \tag{4}$$

that can be split into the electromagnetic part

$$\begin{aligned} T_{EM\,\mu\nu}(\phi_\alpha, \partial_\mu\phi_\alpha) &= \frac{1}{\mu_0}F_{\mu\alpha}F^{\alpha}_{\ \nu} + \frac{g_{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\phi_\mu j_\nu - \frac{1}{2}\phi_\nu j_\mu + g_{\mu\nu}\phi_\alpha j^\alpha \\ &-\frac{1}{2\mu_0}\nabla_\alpha(\phi_\mu\nabla_\nu\phi^\alpha + \phi_\nu\nabla_\mu\phi^\alpha - \phi^\alpha\nabla_\mu\phi_\nu - \phi^\alpha\nabla_\nu\phi_\mu) \end{aligned} \tag{5}$$

and the fluid part $T_{fl\,\mu\nu}(u_\alpha) = (\mathcal{E}+P)u_\mu u_\nu - g_{\mu\nu}P$.

9. Conclusion

Starting from the "noncanonical" Maxwell stress-energy-momentum tensor in the presence of sources, we have augmented it to make it canonical, meaning that its four-divergence is zero in the presence of charged or uncharged massive particles.

The most outstanding result is given by (5), which provides a new SEM tensor for electromagnetism that is suitable for direct use in Einstein's equation, and which we have completed by modeling charges limited to the form of an incompressible, inviscid fluid to give (4). We also open some paths to detect exotic matter in vicinity of black holes and supernovae. Clearly, this modeling is only a basic example, and further modeling can be envisaged by adding additional tensors, provided that they agree with Einstein's equation of general relativity.

By treating these particles as incompressible fluids, we observe that these sources modify the fluid pressure, as can be calculated, for instance, from the ideal gas law.

We hope that this article will contribute to better modeling ionized gas behavior within the interstellar medium and, linked to metrics such as the Schwarzschild or Friedmann–Lemaître–Robertson–Walker metrics, improve cosmology understanding.

The price to pay for achieving tensor symmetry is the adherence to the Lorenz gauge. Therefore, it is tempting to consider the Lorenz gauge as a fifth equation of electromagnetism that would complement the four Maxwell's equations.

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