Article

Not peer-reviewed version

A Zassenhaus Lemma for Digroups

Guy Biyogmam *

Posted Date: 26 August 2024

doi: 10.20944/preprints202408.1857.v1

Keywords: Digroup; Normal subdigroup; Zassenhaus Lemma.

Article

A Zassenhaus Lemma for Digroups

Guy Biyogmam *

Georgia College & State University, 231 W Hancock St, Milledgeville, GA 31061; guy.biyogmam@gcsu.edu

Abstract: In this paper, we construct a quotient structure on digroups. This construction yields a new functor from the category of digroups to the category of groups. We obtain a modular property for digroups, and use it to prove an analogue of the Zassenhaus Lemma in this framework.

Keywords: digroup; normal subdigroup; zassenhaus lemma

---

1. Introduction

Digroups are a generalization of groups which were independently introduced by J. L. Loday in his study of dialgebras [7], by K. Liu [5] and R. Felipe [3], and were further studied in [8]. A formal axiomatic definition was provided by M. Kinyon in [4], in which Kinyon used digroups to construct Lie racks in his attempt to solve the coquecigrue problem, which consists of finding an appropriate generalization of Lie's third theorem in the category of Leibniz algebras [6].

The notion of normal subgroups plays a fundamental role in defining quotient groups and obtaining the classical isomorphism theorems which are fundamental tools in the development of Group Theory (see [10]). Recently, Ongay, Velasquez and Wills-Toro defined normal subdigroups [9] and studied a construction of quotient digroups and the corresponding classical Isomorphism Theorems. In [2], It is defined the concept of trigroups as a generalization of digroups, following essentially Loday's axiomatic definition of associative trioids [7]. The authors in [2] also constructed quotient trigroups and proved classical isomorphism theorems in the category of trigroups [1]. In this paper we obtain the same results on digroups by considering that digroups have a trivial trigroup structure. This study produces a new functor from the category of digroups to the category of groups. More precisely, we use the notion of conjugation of digoups provided in [4] to define a congruence for which the quotient set has a group structure, i.e. a trivial digroup structure. In Section 2, we study the concept quotient digroups and state classical isomorphism theorems yielding from this concept. These theorems are independent of the theorems obtained in [9]. In Section 3, we establish an analogue of the Zassenhaus Lemma in the category of digroups.

2. Preliminaries

2.1. Normal Subdigroups

In this section, we provide a few results on normal subdigroups. Recall from [4, Definition 4.1] that a digroup $(D, \vdash, \dashv)$ is a set D equipped with two binary associative operations $\vdash$ and $\dashv$ respectively called left and right, and satisfying the following conditions:

$$\begin{cases} x \vdash (y \vdash z) = (x \dashv y) \vdash z & (p_1) \\ x \vdash (y \dashv z) = (x \vdash y) \dashv z & (p_2) \\ x \dashv (y \dashv z) = x \dashv (y \vdash z) & (p_3) \end{cases}$$

for all $x, y, z \in D$, and there exists an element $1 \in A$ satisfying

$$1 \vdash x = x = x \dashv 1 \ \text{ for all } \ x \in D \qquad (I)$$

and for all $x \in D$, there exists $x^{-1} \in D$ (called inverse of x) such that

$$x \vdash x^{-1} = 1 = x^{-1} \dashv x.$$

A subset S of a digroup D is said to be a subdigroup of D if $(S, \vdash, \dashv)$ is a digroup with distinguish bar-unit 1.

Note that the set $\mathfrak{U}_D := \{e \in D : e \vdash x = x = x \dashv e \text{ for all } x, y \in D\}$ of bar-units of D is a subdigroup of D.

Recall also that a morphism between two digroups is a map that preserves the three binary operations and is compatible with bar-units and inverses.

Remark 1. *[4, Lemma 4.5].*

(a) *The set $J_D = \{x^{-1} : x \in D\}$ is a group in which $\vdash = \dashv$.*
(b) *The mapping $\phi : D \to J_D$ defined by $x \mapsto (x^{-1})^{-1}$ is an epimorphism of digroups that fixes J_D, and* $\operatorname{Ker} \phi = \mathfrak{U}_D$.
(c) $(x^{-1})^{-1} = x \vdash 1 = 1 \dashv x$ *for all $x \in D$.*
(d) $(x \vdash y)^{-1} = y^{-1} \vdash x^{-1} = y^{-1} \dashv x^{-1} = (x \dashv y)^{-1}$ *for all $x, y \in D$. Consequently, $((x^{-1})^{-1})^{-1} = x^{-1}$.*
(e) $x^{-1} \vdash 1 = x^{-1} = 1 \dashv x^{-1}$ *for all $x \in D$.*
(f) $x \vdash y = (x^{-1})^{-1} \vdash y$ *for all $x, y \in D$.*
(g) $x \dashv y = x \dashv (y^{-1})^{-1}$ *for all $x, y \in D$.*

Remark 2. *Let $(D, \vdash, \dashv)$ be a digroup. Then*

$$a^{-1} \dashv b = a^{-1} \dashv (b^{-1})^{-1} = a^{-1} \vdash (b^{-1})^{-1}$$

for all $a, b \in S$.

Proof. This is a consequence of Remark 1(d) and Remark 1(g). □

Definition 1. *[9, Definition 4]. A subdigroup S of a digroup $(D, \vdash, \dashv)$ is said to be normal if $x \vdash S \dashv x^{-1} \subseteq S$ for all $x \in D$.*

Remark 3. *By the assertions f) and g) of Remark 1, it follows that if S is normal in D then $x^{-1} \vdash S \dashv x \subseteq S$ for all $x \in D$.*

The following Lemma is the modular property for groups.

Lemma 1. *Let $(D, \vdash, \dashv)$ be a digroup, S and S' two subdigroups of D and R a subdigroup of S. Consider the set $\widehat{S} = \{x \in D \mid x^{-1} \in S\}$. Then*

$$\widehat{S} \cap S' \vdash R = S \cap (S' \vdash R) \quad \text{and} \quad R \dashv \widehat{S} \cap S' = S \cap (R \dashv S').$$

Proof. Let $x \in \widehat{S} \cap S'$ and $r \in R$. Clearly, $x \vdash r \in S' \vdash R$ and $x \vdash r = (x^{-1})^{-1} \vdash r \in S$ since $x^{-1} \in S$ and $R \subseteq S$. So, $\widehat{S} \cap S' \vdash R \subseteq S \cap (S' \vdash R)$. For the other inclusion, let $x \vdash r \in S \cap (S' \vdash R)$. It is enough to show that $x \in \widehat{S}$ i.e. $x^{-1} \in S$. Indeed, $(x^{-1})^{-1} = x \vdash 1 = x \vdash (r \vdash r^{-1}) = (x \vdash r) \vdash r^{-1} \in S$ since $x \vdash r \in S$ and $r^{-1} \in R \subseteq S$, and thus $x^{-1} = ((x^{-1})^{-1})^{-1} \in S$, thanks to Remark 1(d). This proves the first identity. The proof of the second identity is similar. □

Lemma 2. *Let $(D, \vdash, \dashv)$ be a digroup. If S and R are two normal subdigroups of D, then $S \vdash R$ is also a normal subdigroup of D.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 August 2024 doi:10.20944/preprints202408.1857.v1

Proof. First we show that $S \vdash R$ is closed under the digroup operations $\vdash$ and $\dashv$. Indeed, for all $x, y \in S$ and $r_1, r_2 \in R$, we have as R is normal in D and by Remark 3, $y^{-1} \vdash r_1 \dashv y \in R$. Similarly, $(y^{-1} \vdash r_2^{-1}) \vdash (r_2 \vdash r_1) \dashv (y \vdash r_2) \in R$. So,

$$\begin{aligned}(x \vdash r_1) \vdash (y \vdash r_2) &= (x \vdash ((y \vdash y^{-1}) \vdash r_1)) \vdash (y \vdash r_2)\\ &= ((x \vdash y) \vdash y^{-1}) \vdash r_1 \vdash (y \vdash r_2)\\ &= (x \vdash y) \vdash (y^{-1} \vdash r_1) \vdash (y \vdash r_2)\\ &\overset{p_1}{=} (x \vdash y) \vdash ((y^{-1} \vdash r_1) \dashv y) \vdash r_2)\\ &= (x \vdash y) \vdash ((y^{-1} \vdash r_1 \dashv y) \vdash r_2)\end{aligned}$$

and,

$$\begin{aligned}(x \vdash r_1) \dashv (y \vdash r_2) &= (x \dashv ((y \vdash y^{-1}) \vdash (r_2^{-1} \vdash r_2) \vdash r_1)) \dashv (y \vdash r_2)\\ &\overset{p_3}{=} ((x \dashv y) \dashv y^{-1}) \vdash (r_2^{-1} \vdash (r_2 \vdash r_1) \dashv (y \vdash r_2))\\ &\overset{p_1}{=} ((x \dashv y) \vdash (y^{-1} \vdash r_2^{-1}) \vdash (r_2 \vdash r_1) \dashv (y \vdash r_2)) \quad (eq.2)\end{aligned}$$

So $(x \vdash r_1) \vdash (y \vdash r_2), (x \vdash r_1) \dashv (y \vdash r_2) \in S \vdash R$.

Now for $x \in S$ and $r \in R$,

$$\begin{aligned}(x \vdash r)^{-1} = r^{-1} \vdash x^{-1} &= (r^{-1} \vdash x^{-1}) \dashv (r \vdash r^{-1}) \overset{p_3}{=} (r^{-1} \vdash x^{-1} \dashv r) \dashv r^{-1}\\ &= (r^{-1} \vdash x^{-1} \dashv (r^{-1})^{-1}) \dashv r^{-1} = (r^{-1} \vdash x \dashv r)^{-1} \dashv r^{-1}\\ &= (r^{-1} \vdash x \dashv r)^{-1} \vdash r^{-1} \in S \vdash R.\end{aligned}$$

Since $1 = 1 \vdash 1$, we conclude that $S \vdash R$ is a subdigroup of D. To show that $S \vdash R$ is normal, let $s \in S, r \in R$ and $x \in D$. Then

$$\begin{aligned}s \vdash r = (s \dashv 1) \vdash r &= (s \dashv (x^{-1} \dashv x)) \vdash r = ((s \dashv x^{-1}) \dashv x) \vdash r\\ &\overset{p_1}{=} (s \dashv x^{-1}) \vdash (x \vdash r).\end{aligned}$$

It follows that:

$$\begin{aligned}x \vdash (s \vdash r) \dashv x^{-1} &= x \vdash ((s \dashv x^{-1}) \vdash (x \vdash r)) \dashv x^{-1}\\ &= (x \vdash s \dashv x^{-1}) \vdash (x \vdash r \dashv x^{-1}) \in S \vdash R.\end{aligned}$$

□

Lemma 3. *Let D be a digroup, I_1, I_2 and J three subdigroups of D such that I_1 and I_2 are normal subdigroups of J. Then $I_1 \vdash I_2$ is a normal subdigroup of $J \vdash I_2$.*

Proof. Let $r_1 \in I_1$, $s, r_2 \in I_2$ and $r \in J$. We need to show that $(r \vdash s) \vdash (r_1 \vdash r_2) \dashv (r \vdash s)^{-1} \in I_1 \vdash I_2$. Set $c_1 := s \vdash r_1 \dashv s^{-1}$ and $c_2 := s \vdash r_2 \dashv s^{-1}$. Clearly $c_1 \in I_1$ and $c_2 \in I_2$ by the normality of I_1 and I_2 in J since $I_2 \subseteq J$. So $b_1 := r \vdash c_1 \dashv r^{-1} \in I_1$ and $b_2 := r \vdash c_2 \dashv r^{-1} \in I_2$ for the same reason. We claim that: $(r \vdash s) \vdash (r_1 \vdash r_2) \dashv (r \vdash s)^{-1} = b_1 \vdash b_2$. Indeed,

$$\begin{aligned}b_1 \vdash b_2 = (r \vdash c_1 \dashv r^{-1}) \vdash (r \vdash c_2 \dashv r^{-1}) &\overset{p_1}{=} (r \vdash c_1 \dashv (r^{-1} \dashv r)) \vdash (c_2 \dashv r^{-1})\\ &= (r \vdash c_1 \dashv 1) \vdash (c_2 \dashv r^{-1}) = r \vdash c_1 \vdash c_2 \dashv r^{-1}\\ &= r \vdash (s \vdash r_1 \dashv s^{-1}) \vdash (s \vdash r_2 \dashv s^{-1}) \dashv r^{-1}\\ &\overset{p_1}{=} r \vdash (s \vdash r_1 \dashv (s^{-1} \dashv s)) \vdash r_2 \dashv s^{-1} \dashv r^{-1}\\ &= r \vdash (s \vdash r_1 \dashv 1) \vdash r_2 \dashv s^{-1} \dashv r^{-1} = r \vdash (s \vdash r_1) \vdash r_2 \dashv s^{-1} \dashv r^{-1}\\ &= r \vdash s \vdash (r_1 \vdash r_2) \dashv s^{-1} \dashv r^{-1} = (r \vdash s) \vdash (r_1 \vdash r_2) \dashv (r \vdash s)^{-1}.\end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 August 2024 doi:10.20944/preprints202408.1857.v1

□

Lemma 4. *Let $(D, \vdash, \dashv)$ be a digroup. If S, S', R and R' are subdigroups of D such that R a normal subdigroup of S and R' a normal subdigroup of S', then*

(a) $(S \cap R') \vdash R$ *is a normal subdigroup of* $(S \cap S') \vdash R$
(b) $(S' \cap R) \vdash R'$ *is a normal subdigroup of* $(S' \cap S) \vdash R'$.

Proof. Since R and R' are respectively normal subdigroups of S and S', one easily verify that they are, along with $S \cap R'$ and $S' \cap R$, normal subgroups of $S \cap S'$. The results a) and b) now follow from Lemma 3 □

2.2. Quotient Digroups

This section proposes a new notion of quotient of a given digroup by a normal subdigroup. We construct an equivalence relation for which the equivalence classes are the cosets of the normal subdigroup, and the equivalence class of the identity element is the normal subdigroup. This construction is identical to the work presented in [2] on trigroups by considering their underlying digroup structure. Consequently, the proofs of all results in this section follow by their corresponding results in [2].

Lemma 5. *[2, Lemma 4.1] Let $(D, \vdash, \dashv)$ be a digroup, and S a subdigroup of D. Then the following assertions are true:*

(a) $g \vdash S = S \iff g^{-1} \in S \iff S \dashv g = S$ *for all* $g \in D$.
(b) $g \vdash S = h \vdash S \iff g^{-1} \dashv h \in S$.
(c) $S \dashv g = S \dashv h, \iff g \vdash h^{-1} \in S$

Proposition 1. *[2, Proposition 4.2] Let $(D, \vdash, \dashv)$ be a digroup and S a subdigroup of D. Define the relation: For $x, y \in D$,*

$$x \sim y \iff x^{-1} \dashv y \ \in \ S.$$

Then $\sim$ is an equivalence relation and the equivalence classes are the left cosets $x \vdash S$, $x \in D$ (orbits of the action of S on D).

By the fundamental theorem of equivalence relations, the relation $\sim$ partitions D into the left cosets $x \vdash S$, $x \in D$. Let D/S be the set of left cosets. Define the following binary operations $\Vdash, \dashv\!\!|: D/S \times D/S \to D/S$ by:

$$(g \vdash S) \Vdash (h \vdash S) = (g \vdash h) \vdash S$$

$$(g \vdash S) \dashv\!\!| \ (h \vdash S) = (g \dashv h) \vdash S.$$

The following Proposition provides a functor from the category of digroups to the category of groups.

Proposition 2. *[2, Proposition 4.4] Let $(D, \vdash, \dashv)$ be a digroup and S a normal subdigroup of D. Then the binary operations $\Vdash, \dashv\!\!|$ are well-defined and equip D/S with a structure of a group with identity $\{S\}$ and the inverse of the class $g \vdash S$ is the class $g^{-1} \vdash S$.*

The following results are isomorphism theorems on digroups. They are obtained from isomorphism theorems proven in [2] on trigroups by using the trivial trigroup structure of digroups.

Proposition 3. *[2, Proposition 4.8] Let D and D' be two digroups and S a normal subdigroup of D. Let $\phi : D \to D'$ be a morphism of digroups such that $\mathrm{Ker}(\phi) \subseteq S$. Then there is an isomorphism of groups $\hat{\phi} : D/S \cong Im\phi/\phi(S)$. In particular, if $S = ker(\phi)$ then this isomorphism becomes $\hat{\phi} : D/ker(\phi) \cong Im\phi/\{1\}$.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 August 2024 doi:10.20944/preprints202408.1857.v1

Proposition 4. *[2, Corollary 4.3] Let D be a digroup, and S and R two subdigroups of D such that $s \vdash R = R \dashv s$ for all $s \in S$. Then there is a group isomorphism*

$$(S \vdash R)/R \cong S/(S \cap \widehat{R}).$$

Proposition 5. *[2, Proposition 4.17] Let D be a digroup, and S and R two normal subdigroups of D such that S is a normal subgroup of R. Then there is a group isomorphism*

$$(D/S)/(\hat{R}/S) \cong D/R.$$

3. The Zassenhaus Lemma for Digroups

In this section, we prove the Zassenhaus Lemma for digroups. We use the following Lemmas.

Lemma 6. *Let $(D, \vdash, \dashv)$ be a digroup. If S_1, S_2 are normal subdigroups of D, then there is a group isomorphism*

$$\frac{D}{S_1 \vdash S_2} \cong \frac{D}{S_2 \vdash S_1}.$$

Proof. Consider the map

$$\Gamma : \frac{D}{S_1 \vdash S_2} \longrightarrow \frac{D}{S_2 \vdash S_1} \quad \text{defined by} \quad a \vdash (S_1 \vdash S_2) \longmapsto a \vdash (S_2 \vdash S_1).$$

Notice that for all $a, b \in D$, we have by Remark 2,

$$a \vdash (S_1 \vdash S_2) = b \vdash (S_1 \vdash S_2) \iff a^{-1} \vdash (b^{-1})^{-1} \in (S_1 \vdash S_2)$$

which implies $a^{-1} \vdash (b^{-1})^{-1} = s_1 \vdash s_2$ for some $s_1 \in S_1$ and $s_2 \in S_2$, and thus $b^{-1} \vdash (a^{-1})^{-1} = s_2^{-1} \vdash s_1^{-1} \in S_2 \vdash S_1$, hence $a \vdash (S_2 \vdash S_1) = b \vdash (S_2 \vdash S_1)$. So Γ is well-defined. Γ is clearly a digroup homomorphism by definition of the operations $\Vdash, \dashV$ on left cosets. By interchanging the positions of S_1 and S_2 in the proof of the well-definition of Γ, one proves injection. That Γ is surjective is trivial. □

The following is an analogue of the Zassenhaus Lemma (also known as the Butterfly Lemma on groups) [10] for digroups.

Corollary 1. *Let $(D, \vdash, \dashv)$ be a digroup. If S, S', R and R' are subdigroups of D such that R is a normal subdigroup of S and R' is a normal subdigroup of S', then*

$$\frac{S \cap (S' \vdash R)}{S \cap (R' \vdash R)} \cong \frac{S \cap (S' \vdash R')}{S' \cap (R \vdash R')}$$

Proof. By the modular property of digroups (Lemma 1), it suffices to show that

$$\frac{(\widehat{S} \cap S') \vdash R}{(\widehat{S} \cap R') \vdash R} \cong \frac{(\widehat{S'} \cap S) \vdash R'}{(\widehat{S'} \cap R) \vdash R'}.$$

Set $\mathfrak{N} := (\widehat{S} \cap R') \vdash (\widehat{S'} \cap R)$. Clearly, $\widehat{S} \cap R'$ and $\widehat{S'} \cap R$ are normal subdigroup of $\widehat{S} \cap \widehat{S'}$. It follows by Lemma 2 that $\mathfrak{N}$ is a normal subdigroup of $\widehat{S} \cap \widehat{S'}$. Now consider the map

$$\Phi : (\widehat{S} \cap S') \vdash R \longrightarrow \frac{(\widehat{S} \cap \widehat{S'})}{\mathfrak{N}} \quad \text{defined by} \quad x \vdash r \longmapsto x \vdash \mathfrak{N}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 August 2024 doi:10.20944/preprints202408.1857.v1

Φ is well-defined since for all $x, y \in \widehat{S} \cap S'$ and $r_1, r_2 \in R$ such that $x \vdash r_1 = y \vdash r_2$, we have as $r_1 \vdash r_1^{-1} = 1$ and $y^{-1} \vdash y \in \mathfrak{U}_D$,

$$
\begin{aligned}
y^{-1} \dashv x = y^{-1} \dashv (x^{-1})^{-1} = y^{-1} \vdash (x^{-1})^{-1} = y^{-1} \vdash (x \vdash 1) \\
= y^{-1} \vdash (x \vdash (r_1 \vdash r_1^{-1})) = y^{-1} \vdash (x \vdash r_1) \vdash r_1^{-1} \\
= y^{-1} \vdash (y \vdash r_2) \vdash r_1^{-1} = (y^{-1} \vdash y) \vdash (r_2 \vdash r_1^{-1}) = r_2 \vdash r_1^{-1}.
\end{aligned}
$$

$y^{-1} \dashv x = r_2 \vdash r_1^{-1} \in S' \cap R = 1 \vdash S' \cap R \subseteq 1 \vdash \widehat{S}' \cap R \subseteq \mathfrak{N}$. So $x \sim y$, i.e. $x \vdash \mathfrak{N} = y \vdash \mathfrak{N}$. To show that Φ is a digroup homomorphism, notice that as R is normal in S, for all $x, y \in S \cap S'$ and $r_1, r_2 \in R$, we have $y^{-1} \vdash r_1 \dashv y \in R$ and $(y^{-1} \vdash r_2^{-1}) \vdash (r_2 \vdash r_1) \dashv (y \vdash r_2) \in R$, thanks to Remark 3. So we have from the proof of Lemma 2

$$(x \vdash r_1) \vdash (y \vdash r_2) = (x \vdash y) \vdash \big((y^{-1} \vdash r_1 \dashv y) \vdash r_2\big) \in (S \cap S') \vdash R,$$

and

$$(x \vdash r_1) \dashv (y \vdash r_2) = (x \dashv y) \vdash \big((y^{-1} \vdash r_2^{-1}) \vdash (r_2 \vdash r_1) \dashv (y \vdash r_2)\big) \in (S \cap S') \vdash R.$$

Therefore,

$$\Phi((x \vdash r_1) \vdash (y \vdash r_2)) = (x \vdash y) \vdash \mathfrak{N} = (x \vdash \mathfrak{N}) \Vdash (y \vdash \mathfrak{N}) = \Phi(x \vdash r_1) \Vdash \Phi(y \vdash r_2)$$

and

$$\Phi((x \vdash r_1) \dashv (y \vdash r_2)) = (x \vdash y) \vdash \mathfrak{N} = (x \vdash \mathfrak{N}) \dashv\!\!\mid (y \vdash \mathfrak{N}) = \Phi(x \vdash r_1) \dashv\!\!\mid \Phi(y \vdash r_2).$$

Φ is surjective since if $x \in \widehat{S} \cap \widehat{S}'$, then $(x^{-1})^{-1} \in \widehat{S} \cap S'$, and we have by Remark 1(f) that $\Phi((x^{-1})^{-1} \vdash R) = (x^{-1})^{-1} \vdash \mathfrak{N} = x \vdash \mathfrak{N}$. It remains to show that Φ is injective. Indeed, $x \vdash a \in \mathrm{Ker}\,(\Phi) \iff x \vdash \mathfrak{N} = \mathfrak{N} \iff x^{-1} \in \mathfrak{N} \iff (x^{-1})^{-1} \in \mathfrak{N}$, thanks to Lemma 5(a). So $(x^{-1})^{-1} = y_1 \vdash y_2$ for some $y_1 \in \widehat{S} \cap R'$ and $y_2 \in \widehat{S}' \cap R$. It follows that

$$x \vdash a = (x^{-1})^{-1} \vdash a = (y_1 \vdash y_2) \vdash a = y_1 \vdash (y_2 \vdash a) \in \widehat{S} \cap R' \vdash R.$$

So $\mathrm{Ker}\,(\Phi) \subseteq \widehat{S} \cap R' \vdash R$. But $\widehat{S} \cap R' \vdash R \subseteq \mathrm{Ker}\,(\Phi)$ since for all $x \in \widehat{S} \cap R'$ and $a \in R$, $(x^{-1})^{-1} = x \vdash 1 \in \mathfrak{N}$ implying that $\Phi(x \vdash a) = \Phi((x^{-1})^{-1} \vdash a) = (x^{-1})^{-1} \vdash \mathfrak{N} = \mathfrak{N}$. Hence $\mathrm{Ker}\,(\Phi) = (\widehat{S} \cap R') \vdash R$. Therefore we have by Proposition 3 that

$$\frac{(\widehat{S} \cap S') \vdash R}{(\widehat{S} \cap R') \vdash R} \cong \frac{(\widehat{S} \cap \widehat{S}')}{\mathfrak{N}} / \{\mathfrak{N}\} \cong \frac{(\widehat{S} \cap \widehat{S}')}{\mathfrak{N}}.$$

The last isomorphism holds since $\{\mathfrak{N}\}$ is the identity element in the group $\frac{\widehat{S} \cap \widehat{S}'}{\mathfrak{N}}$. Similarly, setting $\mathfrak{N}' := (\widehat{S}' \cap R) \vdash (\widehat{S} \cap R')$ and considering the map

$$\Phi : (\widehat{S}' \cap S) \vdash R' \longrightarrow \frac{(\widehat{S} \cap \widehat{S}')}{\mathfrak{N}'} \quad \text{defined by} \quad x \vdash r \longmapsto x \vdash \mathfrak{N}'.$$

we show by that

$$\frac{(\widehat{S}' \cap S) \vdash R'}{(\widehat{S}' \cap R) \vdash R'} \cong \frac{(\widehat{S} \cap \widehat{S}')}{\mathfrak{N}'}.$$

 doi:10.20944/preprints202408.1857.v1

Since by Lemma 6, $\frac{\widehat{S}\cap\widehat{S'}}{\mathfrak{N}} \cong \frac{\widehat{S}\cap\widehat{S'}}{\mathfrak{N}'}$, It follows that

$$\frac{(\widehat{S} \cap S') \vdash R}{(\widehat{S} \cap R') \vdash R} \cong \frac{(\widehat{S'} \cap S) \vdash R'}{(\widehat{S'} \cap R) \vdash R'}.$$

This completes the proof. □

References

1. Biyogmam, G. R.; Tcheka, C. From trigroups to Leibniz 3-algebras. *arXiv:1904.12030 [math.RA]*.
2. Biyogmam, G. R. Generalized Loday algebras and digroups. *Quasigroups and Related systems*, **28**(1) (2020), 29-41.
3. Felipe, R.; Tcheka, C.; Tchamna, S. From quotient trigroups to groups. *preprint, available at www.cimat.mx/reportes/enlinea/I-04-01.pdf* .
4. Kinyon, M. K. Leibniz algebra, Lie racks and digroups. em J. Lie Theory, **17** (2007) 99-11.
5. Liu, K. A class of grouplike objects. *www.arXiv.org/math.RA/0311396.*
6. Loday, J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *L'Enseignement Mathématique* **39** (1993), 269–292.
7. Loday, J.-L.; Ronco, M.O. Trialgebras and families of polytopes. *In "Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory" Contemporary Mathematics* **346** (2004), 369-398.
8. Phillips, J. D. A short Basis for the Variety of Digroups. *Semigroup Forum* OF1-OF5, 2004.
9. Ongay, F., Velasquez, R., Wills-Toro, L. A. Normal subdigroups and the isomorphism theorems for digroups. *Algebra and Discrete Mathematics*, **22** (2016), 262-283.
10. Robinson , D.J.S. *A Course in the Theory of Groups*, 2nd Edition; Graduate Texts in Mathematics, 80. Springer-Verlag,1995.