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Article

Projective Vector Fields on Semi-Riemannian Manifolds

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Abstract: This paper explores the properties of projective vector fields on semi-Riemannian manifolds. The main result establishes that if a projective vector field η on such a manifold is also a conformal vector field with potential function ψ, then η must either be homothetic, or the vector field ζ, which is dual to $d\psi$, is a light-like vector field. Additionally, it is shown that a complete Riemannian manifold admits a projective vector field that is also conformal and non-Killing if and only if it is locally Euclidean. The paper also presents other results related to the characterization of Killing and parallel vector fields using the Ricci curvature and the Hessian of the function given by the inner product of the vector field.

Keywords: semi-Riemannian manifolds; projective vector fields; conformal and killing vector fields; Ricci curvature; Hessian

MSC: 53A15; 53C40; 53C42; 53C50

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1. Introduction

Consider an n-dimensional semi-Riemannian manifold (N, h), $n \geq 2$, and denoted by $\mathfrak{X}(N)$ the collection of all smooth vector fields on N.

At a point p in N, the tangent vector X is called space-like (respectively, time-like or light-like) if $h_p(X, X) \geq 0$ (respectively, ≤ 0 or $= 0$). The zero vector is classified as space-like. The norm $|X|$ is defined as $|h(X, X)|^{\frac{1}{2}}$. Analogously, a vector field X on N is called space-like (respectively, time-like or light-like) if it is this at each point. The zero vector field is also considered space-like.

A vector field η is called a projective if its local flow preserves the geodesics of (N, h) in the set-theoretic sense. If the flow of η preserves geodesics in the mapping sense, then it is called affine. It is not difficult to see that a vector field η on N is projective if there exists a differential 1-form μ on N (called the associated differential 1-form to η) , such that

$$(\pounds_\eta \nabla)(U, V) = \mu(U)V + \mu(V)U, \tag{1}$$

for all $U, V \in \mathfrak{X}(N)$, where $\pounds_\eta$ is the Lie derivative along η, and ∇ is the Levi-Civita connection of (N, h). Here the Lie derivative $\pounds_\eta$ acts on ∇ as follows:

$$(\pounds_\eta \nabla)(U, V) = [\eta, \nabla_U V] - \nabla_{[\eta, U]} V - \nabla_U [\eta, V],$$

for all $U, V \in \mathfrak{X}(N)$.

Of course, when $\pounds_\eta \nabla = 0$, η is an affine vector field. It follows from (1) that

$$\mu(U) = \frac{1}{n+1} \nabla_U (div(\eta)), \tag{2}$$

for all $U \in \mathfrak{X}(N)$.

According to [8], a complete Riemannian manifold N with a parallel Ricci tensor, which admits a non-affine projective vector fields, has a positive constant curvature. In [16], it has been shown that if N is a compact Riemannian manifold with non-positive constant scalar curvature, any projective vector

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field on N is Killing. Furthermore, in [17], it is proven that if a compact simply connected Riemannian manifold with constant scalar curvature admits a projective vector field which is not Killing, then N must be isometric to sphere.

In [4], a set of integral inequalities within a compact, orientable Riemannian manifold with constant scalar curvature that allows for a projective vector field, subsequently deriving the necessary and sufficient conditions for such a Riemannian manifold to be isometric to a sphere.

In addition, Section 4 will explore conformal projective vector fields. Conformal vector fields are crucial. They are significant elements in the study of the geometry of various types of manifolds. A smooth vector field η on a semi-Riemannian manifold (N, h) is termed a conformal vector field if its flow results in conformal transformations or, equivalently, if the Lie derivative $\pounds_\eta h$ with respect to the metric h along the vector field η satisfies the condition [21] (see also [6]):

$$\pounds_\eta h = 2\psi h, \tag{3}$$

where ψ is a smooth function on N (called the potential function of η). In this case, it is straightforward to see that:

$$\psi = \frac{div(\eta)}{n}. \tag{4}$$

Examples of conformal vector fields include homothetic vector fields, where ψ remains constant, and Killing vector fields, where $\psi = 0$.

A notable question in the study of Riemannian manifold geometry is identifying spheres within the category of compact connected Riemannian manifolds. Obata provided one such identification [9,10]. Many authors extensively studied Riemannian manifolds with constant scalar curvature allowing for non-isometric conformal vector fields. They aimed to prove a conjecture about the Euclidean sphere as the unique compact orientable Riemannian manifold admitting a metric of constant scalar curvature R carrying a conformal vector field X. Notable researchers include Goldberg and Kobayashi [2], Nagano [7], Obata [11], and Yano and Hagano [18]. Interested readers can find a summary of these results in Yano [19].

This paper examines the properties of projective vector fields in semi-Riemannian manifolds. Initially, we demonstrate that a projective field, which is also a conformal vector field within a semi-Riemannian manifold, is inherently homothetic. This paper is structured as follows. Section 2 provides some preliminaries and Section 3 focuses on validating various theorems related to projective vector fields within a semi-Riemannian manifold. This includes multiple characterization results and confirms certain theorems on projective vector fields in such manifolds. We demonstrate that any projective vector field η with a non-negative $\mu(\eta)$ on a Riemannian compact manifold must be a Killing vector field. In Theorem 7, we establish the impossibility of a non-parallel projective vector field η with a non-negative $\mu(\eta)$ on a Riemannian compact manifold with non-positive Ricci curvature. For non-compact manifolds where the metric h is not necessarily positive definite (i.e., (N, h) is semi-Riemannian), we show that a projective vector field η on N with constant length and fulfilling $Ric(\eta, \eta) \leq 0$ must be parallel. Furthermore, we prove that any projective vector field η with a non-negative $\mu(\eta)$ on a Riemannian manifold, where the Hessian of the function $h(\eta, \eta)$ is non-positive, is necessarily a geodesic vector field. We also identify several necessary and sufficient conditions for a projective vector field on a semi-Riemannian manifold to be Killing.

Additionally, in Theorems 11 and 13, we establish the necessary and sufficient conditions for a projective vector field on a semi-Riemannian manifold to be parallel.

In Section 4, we explore projective vector fields on semi-Riemannian manifolds that also serve as conformal vector fields. First, we show that if η is a projective vector field which is also a conformal vector field on a semi-Riemannian manifold, such that $\pounds_\eta h = 2\psi h$, then η is homothetic or the vector field ζ dual to $d\psi$ is a light-like vector field. Then, we prove that a complete Riemannian manifold has a non-Killing projective vector field that is also conformal if and only if it is locally Euclidean. We also

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generalize two results in [1,5] in two directions: We focus on semi-Riemannian manifolds rather than Riemannian manifolds, and we examine conformal vector fields instead of affine vector fields (referred to as Jacobi-type vector fields in [1]).

2. Perliminaries

On a semi-Riemannian manifold of dimension $n \geq 2$, denoted as (N, h), with a Levi-Civita connection ∇ and a local orthonormal frame $\{E_1, \ldots, E_n\}$. The Ricci curvature tensor is a symmetric tensor defined as follows:

$$Ric(U, V) = \sum_{i=1}^{n} \epsilon_i h(R(U, E_i)V, E_i), \tag{5}$$

where U and V are vector fields on N, and $\epsilon_i = h(E_i, E_i)$. Here, the curvature tensor of N is given by

$$R(U, V)W = \nabla_{[U,V]}W - \nabla_U \nabla_V W + \nabla_V \nabla_U W,$$

for all $U, V, W \in \mathfrak{X}(N)$. The divergence of a vector field U is defined by

$$div(U) = \sum_{i=1}^{n} \epsilon_i h(\nabla_{E_i} U, E_i). \tag{6}$$

where $\epsilon_i = h(E_i, E_i)$. The vector field U is called incomperssible if $div(U) = 0$. That means that the flow of U preserves the volume of (N, h). For a smooth function f on N, the Hessian, denoted $Hess(f)$, is a symmetric tensor of type $(0, 2)$. It is defined by the equation

$$Hess(f)(U, V) = h(\nabla_U \nabla f, V), \tag{7}$$

for all $U, V \in \mathfrak{X}(N)$, where the symbol ∇f represents the gradient of f.

The second covariant derivative of the vector field η in the direction of the vector fields U and V is defined by

$$\nabla^2_{U,V}\eta = \nabla_U \nabla_V \eta - \nabla_{\nabla_U V}\eta.$$

For operators A and B on N, the inner product between A and B is given by

$$< A, B >= tr(AB^t),$$

where tr denoted the trace. The norm of the operator A is determined as

$$||A|| = \sqrt{< A, A >}.$$

The following lemma characterizes projective vector fields in terms of the second covariant derivative and the curvature tensor.

Lemma 1. *Let η be a projective vector field on a semi-Riemannian manifold (N, h). Then η satisfies the following equation:*

$$\nabla^2_{U,V}\eta + R(U, \eta)V = \mu(U)V + \mu(V)U,$$

for all $U, V \in \mathfrak{X}(N)$, where μ the associated differential 1-form to η.

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Proof. For $U, V \in \mathfrak{X}(N)$, we have

$$\begin{aligned}(\pounds_\eta \nabla)(U,V) &= \pounds_\eta \nabla_U V - \nabla_{\pounds_\eta U} V - \nabla_U \pounds_\eta \\ &= [\eta, \nabla_U V] - \nabla_{[\eta,U]} V - \nabla_U [\eta, V] \\ &= \nabla_\eta \nabla_U V - \nabla_{\nabla_U V}\eta - \nabla_{[\eta,U]} V - \nabla_U \nabla_\eta V + \nabla_U \nabla_V \eta \\ &= \nabla^2_{U,V}\eta + [\nabla_\eta, \nabla_U]V - \nabla_{[\eta,U]} V \\ &= \nabla^2_{U,V}\eta + R(U,\eta)V.\end{aligned}$$

So, η is a projective vector field if and only if $\nabla^2_\eta(U,V) + R(\eta,U)V = \mu(U)V + \mu(V)U$ for all $U, V \in \mathfrak{X}(N)$.
□

For any vector field η on (N,h), let ω_η denote the 1- form dual to η, that is, $\omega_\eta(U) = h(X,\eta)$, for all $U \in \mathfrak{X}(N)$. We associate the (1, 1)-tensor A_η defined by

$$A_\eta(U) = \nabla_U \eta, \tag{8}$$

for all $U \in \mathfrak{X}(N)$.
We write

$$A_\eta = B + \theta, \tag{9}$$

where B and θ are the symmetric and anti-symmetric components of A_η, respectively.

The assertion presented here is an alternative form of Lemma 1 presented in terms of the operator of A_η.

Lemma 2. *A vector field η on a semi-Riemannian manifold (N,h) with an associated differential 1-form μ is projective if and only if it satisfies the following equation.*

$$\nabla_U A_\eta + R(U,\eta) = \mu(U)V + \mu(V)U,$$

for all $U \in \mathfrak{X}(N)$.

Lemma 3. *If η is a projective vector field on an n-dimensional semi-Riemannian manifold (N,h) with a associated differential 1-form μ, then*

$$\nabla_U(tr(A_\eta)) = (n+1)\mu(U).$$

Proof. Since $div(\eta) = tr(A_\eta)$, (2) yields

$$\nabla_U(tr(A_\eta)) = tr(\nabla_U(A_\eta)) = (n+1)\mu(U)$$

for all $U \in \mathfrak{X}(N)$. □

Now, we present a generalized formulation of the Bochner formula, which will be employed in the forthcoming sections. (cf. [13]).

Theorem 1. *Let (N,h) be a semi-Riemannian manifold. Then*

$$\nabla_U(div(U)) + Ric(U,U) - div(\nabla_U U) + tr(A_U^2) = 0, \tag{10}$$

for all $U \in \mathfrak{X}(N)$.

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Proof. Let $\{E_1, \ldots, E_n\}$ be a parallel orthonormal frame on N, where n is the dimension of N, and let $U \in \mathfrak{X}(N)$. It is straightforward to see that

$$(\pounds_U \nabla)(U, E_i) = (\pounds_U \nabla)(E_i, U),$$

for all $i = 1, ..., n$.

Then, by (5) and (6), we get

$$\begin{aligned}\nabla_U(div(U)) + Ric(U,U) - div(\nabla_U U) + tr(A_U^2) &= \sum_{i=1}^{n} \epsilon_i h(\nabla_U \nabla_{E_i} U + R(E_i, U)U \\ &- \nabla_{E_i} \nabla_U U + \nabla_{\nabla_{E_i} U} U, E_i) \\ &= \sum_{i=1}^{n} \epsilon_i h((\pounds_U \nabla)(U, E_i), E_i)) \\ &- h((\pounds_U \nabla)(E_i, U), E_i) = 0,\end{aligned}$$

where $\epsilon_i = h(E_i, E_i)$. □

3. Characterizations of Projective Vector Fields on Semi-Riemannian Manifolds

In this section, we will investigate projective vector fields on semi-Riemannian manifolds. We obtain the following theorem for a projective vector field.

Theorem 2. *Let (N, h) be a semi-Riemannian manifold. For a projective vector field η on N, the equation below holds for η*

$$\frac{1}{2}\Delta h(\eta, \eta) = -Ric(\eta, \eta) + 2\mu(\eta) + ||A_\eta||^2. \tag{11}$$

Proof. Let η be a projective vector field on a semi-Riemannian manifold (N, h). By (1), and the proof of Lemma 1, it follows that

$$\nabla^2_{U,V}\eta + R(U, \eta)V = \mu(U)V + \mu(V)U, \tag{12}$$

for all $U, V \in \mathfrak{X}(N)$. Furthermore, by defining a function f on M by $f = \frac{1}{2}h(\eta, \eta)$, we obtain

$$h(\nabla f, V) = \frac{1}{2}\nabla_V f = h(A_\eta(V), \eta), \tag{13}$$

for all $V \in \mathfrak{X}(N)$. It follows that for any $U \in \mathfrak{X}(N)$, we have

$$\nabla_U h(\nabla f, V) = \nabla_U h(A_\eta(V), \eta),$$

which implies that

$$h(\nabla_U \nabla f, V) + h(\nabla f, \nabla_U V) = h(\nabla_U \nabla_V \eta, \eta) + h(\nabla_V \eta, \nabla_U \eta).$$

Thus, according to (7) and (13), we conclude that

$$Hessf(U, V) = h(\nabla^2_{U,V}\eta, \eta) + h(\nabla_V \eta, \nabla_U \eta), \tag{14}$$

By substituting (12) into (14), we get

$$Hessf(U, V) = -h(R(U, \eta)\eta, V) + \mu(U)h(V, \eta) + \mu(V)h(U, \eta) + h(\nabla_V \eta, \nabla_U \eta) \tag{15}$$

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By computing the trace of equation (15) with respect to an orthonormal frame $\{E_1, \ldots, E_n\}$, and considering both the symmetry of B and the anti-symmetry of θ, together with the fact that $\Delta f = tr(Hess f)$, we can obtain

$$\begin{aligned}
\Delta f &= -\sum_{i=1}^{n} \epsilon_i h(R(\eta, E_i)\eta, E_i) + 2\sum_{i=1}^{n} \epsilon_{E_i} \mu(E_i) h(\eta, E_i) + \sum_{i=1}^{n} \epsilon_i h(\nabla_{E_i}\eta, \nabla_{E_i}\eta) \\
&= -Ric(\eta,\eta) + 2\mu\left(\sum_{i=1}^{n} \epsilon_i h(\eta, E_i) E_i\right) + \sum_{i=1}^{n} \epsilon_i h(A_\eta(E_i), A_\eta(E_i)) \\
&= -Ric(\eta,\eta) + 2\mu(\eta) + \sum_{i=1}^{n} \epsilon_i h((B+\theta)(E_i), (B+\theta)(E_i)) \\
&= -Ric(\eta,\eta) + 2\mu(\eta) + \sum_{i=1}^{n} \epsilon_i h\left(B^2(E_i) - \theta^2(E_i), E_i\right) \\
&= -Ric(\eta,\eta) + 2\mu(\eta) + tr(B^2) - tr(\theta^2) \\
&= -Ric(\eta,\eta) + 2\mu(\eta) + tr(B^t B) + tr(\theta^t \theta) \\
&= -Ric(\eta,\eta) + 2\mu(\eta) + ||B||^2 + ||\theta||^2 \\
&= -Ric(\eta,\eta) + 2\mu(\eta) + ||A_\eta||^2.
\end{aligned}$$

□

Now, for a projective vector field η on semi-Riemannian manfold (N, h), we deduce from (10) and (11) a very useful formula.

Theorem 3. *Let (N, h) be a semi-Riemannian manifold. For a projective vector field η on N, the equation below holds*

$$div(\nabla_\eta \eta) + \frac{1}{2}\Delta h(\eta, \eta) = 2||B||^2 + (n+3)\mu(\eta), \tag{16}$$

where B is the symmetric part of A_η.

We can derive several consequences from (16). The first one is a characterization of Killing vector fields on compact Riemannian manifolds among projective ones.

Theorem 4. *Let (N, h) be an n-dimensional compact Riemannian manifold, and let η be a projective vector field on N. If $\mu(\eta) \geq 0$, then η is a Killing vector field.*

Proof. Given that η is a projective vector field on the compact Riemannian manifold (N, h), by integrating equation (16), we obtain

$$\int_N \left(2||B||^2 + (n+3)\mu(\eta)\right) dV = 0. \tag{17}$$

This leads to the deduction that $B = 0$, as $\mu(\eta) \geq 0$, which implies that A_η is anti-symmetric, and meaning that η is a Killing vector field. □

When considering a semi-Riemannian manifold (N, h) which may not be compact, an interesting problem arises: What conditions need to be satisfied for a projective vector field to become a Killing vector field? The following two theorems can be derived directly from (11) and the important formulae (16).

Theorem 5. *Let (N, h) be an n-dimensional semi-Riemannian manifold, with a projective geodesic vector field η where $\mu(\eta) \geq 0$. Then, η has a constant length if and only if it is a Killing vector field. In this case, $\mu(\eta) = 0$.*

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Theorem 6. *Let (N,h) be an n-dimensional semi-Riemannian manifold, with a projective vector field η of constant length and $\mu(\eta) \geq 0$. Then, $\nabla_\eta \eta$ is an incompressible vector field if and only if η is a Killing vector field. In this case, $\mu(\eta) = 0$.*

The result below guarantees that a non-parallel projective vector field cannot exist on a compact Riemannian manifold with non-positive Ricci curvature. This is a consequence of formula (12).

Theorem 7. *Let η be a projective vector field on a compact Riemannian manifold (N,h), with $\mu(\eta) \geq 0$. If $Ric(\eta,\eta) \leq 0$, then η is a parallel vector field.*

Proof. By integrating both sides of (11), we obtain

$$\int_N \Big(||A_\eta||^2 + 2\mu(\eta) \Big) dV = \int_N Ric(\eta,\eta) dV.$$

Considering that $\mu(\eta) \geq 0$ and $Ric(\eta,\eta) \leq 0$, we deduce that $||A_\eta||^2$ is zero. Thus, $A_\eta = 0$ and η must be a parallel vector field. □

When N is not necessarily compact, the following holds. This is also a consequence of formula (12).

Theorem 8. *Let η be a projective vector field of constant length on the semi-Riemannian manifold (N,h) such that $\mu(\eta) \geq 0$. If $Ric(\eta,\eta) \leq 0$, then η is a parallel vector field.*

Proof. Given that $h(\eta,\eta)$ is constant, (11) reduces to

$$||A_\eta||^2 + 2\mu(\eta) = Ric(\eta,\eta).$$

Since $\mu(\eta) \geq 0$ and $Ric(\eta,\eta) \leq 0$, it follows that $A_\eta = 0$, which means that η is parallel. □

Corollary 1. *If the Ricci curvature of a semi-Riemannian manifold (N,h) is non-positive, then (N,h) admits no non-zero parallel projective vector field η with $\mu(\eta) \geq 0$.*

The subsequent result characterizes projective vector fields on a Riemannian manifold in terms of the Hessian of the length of these vector fields.

Theorem 9. *Let η be a projective vector field on a Riemannian manifold (N,h) with $\mu(\eta) \geq 0$, and let $f = \frac{1}{2}|\eta|^2$. If $Hessf(\eta,\eta) \leq 0$, then η is a geodesic vector field.*

Proof. Taking $U = V = \eta$ into (15), it follows that

$$Hessf(\eta,\eta) = 2\mu(\eta)|\eta|^2 + |\nabla_\eta \eta|^2. \tag{18}$$

Since $\mu(\eta) \geq 0$ and $Hessf(\eta,\eta) \leq 0$, it follows that $|\nabla_\eta \eta|^2 = 0$. Thus, η is a geodesic vector field. □

From this result, we obtain an important consequence.

Corollary 2. *Consider a Riemannian manifold (N,h). There does not exist any nonzero geodesic projective vector field η such that $\mu(\eta) \geq 0$ and $Hessf \leq 0$, where $f = \frac{1}{2}|\eta|^2$.*

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Now, we return to the decomposition (9), from which we deduce that

$$tr(A_\eta^2) = ||B||^2 - ||\theta||^2,$$

and

$$||A_\eta||^2 = ||B||^2 + ||\theta||^2.$$

Thus, $tr(A_\eta^2) = ||A_\eta||^2$ if A_η is symmetric, and $tr(A_\eta^2) = -||A_\eta||^2$ if A_η is anti-symmetric (that is, η is a Killing vector field). Also, from (9) and (13), we get

$$\frac{1}{2}\Delta h(\eta,\eta) = B(\eta) - \theta(\eta). \tag{19}$$

Consequently, we generalize Theorem 2 in [1] to projective vector fields on semi-Riemannian manifolds.

Theorem 10. *Let η be a projective vector field on a semi-Riemannian manifold. Then, η is a Killing vector field if and only if the following holds*

$$\frac{1}{2}\Delta h(\eta,\eta) \leq ||\theta||^2 + 2\mu(\eta) - Ric(\eta,\eta), \tag{20}$$

where θ is the anti-symmetric part of A_η.

Proof. Assuming (20) holds, then by (11), we have

$$||A_\eta||^2 \leq ||\theta||^2.$$

Since $||A_\eta||^2 = ||B||^2 + ||\theta||^2$, it follows that $B = 0$, and η is a Killing vector field. The converse is trivial. □

In the following result, we prove that a simple condition in terms of Ricci curvature, a geodesic projective vector field must be parallel.

Theorem 11. *Let (N, h) be an n-dimensional connected semi-Riemannian manifold, admitting a geodesic projective vector field η with $\mu(\eta) \geq 0$. Then, η is parallel field if and only if the following holds*

$$Ric(\eta,\eta) + ||B||^2 + (n+1)\mu(\eta) \leq 0. \tag{21}$$

In particular, if η is a geodesic vector field, then $Ric(\eta,\eta) \leq 0$.

Proof. Let η be a geodesic projective vector field. Then, by applying the generalized Bochner formula (10) and referring to (2), we obtain

$$Ric(\eta,\eta) = -tr(A_\eta^2) - \eta div(\eta) = ||\theta||^2 - ||B||^2 - (n+1)\mu(\eta). \tag{22}$$

Assuming that $Ric(\eta,\eta) + ||B||^2 + (n+1)\mu(\eta) \leq 0$, we deduce from (22) that $\theta = 0$. Since η is geodsic, it follows from (9) that $B(\eta) = 0$.

By substituting these quantities into (19), we deduce that $h(\eta,\eta)$ is constant. Substituting this into (11), we obtain

$$Ric(\eta,\eta) = ||B||^2 + 2\mu(\eta).$$

Given that $Ric(\eta,\eta) + ||B||^2 + (n+1)\mu(\eta) \leq 0$ and $\mu(\eta) \geq 0$, it follows that $B = 0$. This, with the fact $\theta = 0$ implies that $A_\eta = 0$, which means that η is parallel. □

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Next, we generalize Theorem 4 in [1] to semi-Riemannian manifolds admitting a projective vector field.

Theorem 12. *Let η a projective vector field on a connected semi- Riemannian manifold with $\mu(\eta) \geq 0$. Assume that $\nabla_\eta \eta$ is space-like, and define $f = \frac{1}{2}h(\eta, \eta)$. Then η is a Killing vector field with constant length if and only if $Hessf(\eta, \eta) \leq 4\mu(\eta)f$ and $\Delta f \leq 0$. In this case, $Ric(\eta, \eta) + (n+1)\mu(\eta) \geq 0$, where the equality is valid if and only if eta is a parallel vector field.*

Proof. Assume that η is a Killing vector field. This means that $B = 0$. Since f is constant, by (19) η is a geodesic vector field. Referring to (18), we observe $Hessf(\eta, \eta) = 4\mu(\eta)f$. Also, since f is constant, $\Delta f = 0$. Moreover, according to (10), we have $Ric(\eta, \eta) + (n+1)\mu(\eta) = ||\theta||^2 \geq 0$, where equality is valid if and only if $\theta = 0$. Thus, η is parallel.

Conversely, if $Hessf(\eta, \eta) \leq 4\mu(\eta)f$. By (19), it follows that η is a geodesic. From (16), we deduce that $\mu(\eta) = 0$ and $B = 0$. Hence, A_η is anti-symmetric. Consequently, η is a Killing vector field. Since η is geodesic and $B = 0$, f is constant. Hence, relating to equation (10), we determine that $Ric(\eta, \eta) + (n+1)\mu(\eta) \geq 0$, where the equality occurs precisely when η is a parallel vector. □

When the projective vector field *eta* is a light-like, we drive the following conseqience.

Corollary 3. *Let η be a ligth-like projective vector field on a connected semi-Riemannian manifold with $\mu(\eta) \geq 0$. Assume that $\nabla_\eta \eta$ is space-like, and define $f = \frac{1}{2}h(\eta, \eta)$. Then η is a Killing vector field if and only if $Hessf(\eta, \eta) \leq 0$.*

Next, we give a characterization of Killing vector fields on semi-Riemannian manifolds in terms of the Ricci curvature and the Hessian of the length of such a vector field.

Theorem 13. *Let η be a projective vector field on a connected semi-Riemannian manifold with $\mu(\eta) \geq 0$. Assume that $\nabla_\eta \eta$ is space-like, and define $f = \frac{1}{2}h(\eta, \eta)$. Then, η is a Killing vector field with constant length if and only if $Hessf(\eta, \eta) \leq 4\mu(\eta)f$ and $Ric(\eta, \eta) + (n+1)\mu(\eta) \geq 0$.*

Proof. Assume that η is a Killing vector field. This means that $B = 0$. Since η has a constant length, by (19), we get $\theta(\eta) = 0$. It follows that η is geodesic. Referring to equation (18), we observe $Hessf(\eta, \eta) = 4\mu(\eta)f$. Thus, equation (22) shows $Ric(\eta, \eta) + (n+1)\mu(\eta) = ||\theta||^2 \geq 0$.

Conversely, assume that $Hessf(\eta, \eta) - 4\mu(\eta)f \leq 0$. Since $\nabla_\eta \eta$ is space-like, then (18) implies that η is geodesic. Since $Ric(\eta, \eta) + (n+1)\mu(\eta) \geq 0$, and by (16), we see that $B = 0$. Hence, η is a Killing vector field. Using equation (13), we see that $\theta(\eta) = 0$. Thus, f is constant. □

4. Conformal Projective Vector Fields Are Homothtic

The main objective of this section is to investigate whether a complete semi-Riemannian manifold, which admits a projective vector field that is also a conformal vector field, can be characterized as a Euclidean space. Initially, we show that a projective vector field, which is also a conformal vector field on a semi-Riemannian manifold is homothetic.

If η is a conformal vector field on a semi-Riemannian manifold (N, h) such that $£_\eta h = 2\psi h$, then the following equation holds

$$(£_\eta \nabla)(U, V) = d\psi(U)V + d\psi(V)U - h(U, V)\zeta, \tag{23}$$

where ζ is the vector field associated to the 1-form $d\psi$, i.e. $d\psi(U) = h(\zeta, U)$, for all $U \in \mathfrak{X}(N)$. See, for example [19].

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Theorem 14. *Let η be a projective vector field on a n-dimensional semi-Riemannian manifold (N, h), $n \geq 2$. If η is conformal, such that $\pounds_\eta h = 2\psi h$, then η is homothetic or the vector field ζ dual to $d\psi$ is a light-like vector field.*

Proof. Using (1) and (23), we have

$$h(U,V)\zeta = (d\psi - \mu)(U)V + (d\psi - \mu)(V)U, \tag{24}$$

for all $U, V \in \mathfrak{X}(N)$.

On the other hand, by (2) and (4), we have

$$\mu(U) = \frac{n}{n+1} d\psi(U) = \frac{n}{n+1} h(U,\zeta),$$

for all $U \in \mathfrak{X}(N)$.

Substituting this into (24), it becomes

$$h(U,V)\zeta = \frac{1}{n}(\mu(U)V + \mu(V)U),$$

for all $U, V \in \mathfrak{X}(N)$.

By setting $U = V = \zeta$ in the above equation, we get

$$h(\zeta,\zeta)\zeta = \frac{2}{n+1} h(\zeta,\zeta)\zeta,$$

which implies

$$\left(\frac{n-1}{n+1}\right) h(\zeta,\zeta)\zeta = 0.$$

Given that $n \geq 2$, we deduce that $\zeta = 0$ or ζ is ligth-like. However, if $\zeta = 0$, then $\mu = 0$ and ψ is a constant. Thus, η is homothetic.
□

In the next theorem, we show that a complete Riemannian manifold possesses a non-Killing projective vector field which is also conformal if and only if it is locally Euclidean.

Theorem 15. *If (N, h) is an n-dimensional complete Riemannian manifold, $n \geq 2$, that admits a non-Killing projective vector field that is also conformal, then (N, h) is locally Euclidean.*

Proof. According to Theorem 14, such a vector field must be homothetic. By Lemma 2, page 242, in [5], and by [15], (N, h) is necessarily a locally Euclidean space. □

Remark 1. *In [5], Lemma 2, page 244 (see also [1], Theorem 6), it has been proved that if (N, h) is a complete Riemannian manifold that admits an affine vector field that is also a non-Killing gradient conformal vector field, then (N, h) is isometric to a Euclidean space. Furthermore, it was proved in [1], that a complete Riemannian manifold (N, h) admits an affine vector field η that is also a non-Killing conformal vector field that annihilates the operator ϕ (the anti-symmetric part of A_η) if and only if (N, h) is locally Euclidean. It is clear that these results evidently represent particular cases of our Theorem 15.*

5. Conclusions

This paper has presented a detailed investigation of projective vector fields on semi-Riemannian manifolds, focusing on their geometric properties and interactions with the curvature tensor. We derived essential results that relate the curvature tensor and the second covariant derivative of a projective vector field. Our results show that on a semi-Riemannian manifold with non-positive Ricci

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curvature, there is no non-zero parallel projective vector field η with the corresponding differential 1-form μ satisfies the condition $\mu(\eta) \geq 0$. Furthermore, we showed that on a compact Riemannian manifold, a projective vector field with constant length and non-positive Ricci curvature must be parallel. Additionally, we explored the conditions under which a projective vector field becomes a Killing vector field, specifically in the case of light-like vector fields. We also showed that a non-Killing, conformal projective vector field exists on a complete Riemannian manifold if and only if the manifold is locally Euclidean.

These results contribute to the broader understanding of the interplay between projective, conformal, and Killing vector fields, with potential implications for future studies in differential geometry and general relativity.

This section is not mandatory, but may be added if there are patents resulting from the work reported in this manuscript.

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