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[Ghadah Alomani](#)^{*}, [Yahia Awad](#), [Mohamed Kayid](#)

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Article

Generalized Bayesian Inference Study Based on Type-II Censored Data from Pareto Distribution

Ghadah Alomani ^{1,*}, Yahia Abdel-Aty ² and Mohamed Kayid ³

¹ Department of Mathematical Sciences, College of Science, Princess Nourah Bint Abdulrahman University, Saudi Arabia

² Department of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt

³ Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* Correspondence: gaalomani@pnu.edu.sa

Abstract: This paper investigates generalized Bayesian (GB) inference in the context of a one-parameter Pareto distribution. The study focuses on the derivation of generalized Bayesian estimators (GBE) and generalized empirical Bayesian estimators (GEBE) for the parameters of the Pareto distribution using type II censored data. In addition, the study investigates a one-sample prediction approach to examine generalized Bayesian prediction (GBP) and generalized empirical Bayesian prediction (GEBP). To evaluate the performance of these estimation and prediction methods, Monte Carlo simulations were performed comparing GBE with GEBE and GBP with GEBP for different parameter values and learning rate parameters (LRP).

Keywords: Generalized Bayesian; learning rate parameter; Generalized Bayesian estimators; generalized empirical Bayesian estimators; generalized Bayesian prediction; generalized empirical Bayesian prediction; Simulation

1. Introduction

Generalized Bayesian is a Bayesian study based on the positive learning rate parameter (LRP) ($\eta > 0$) as a power on the likelihood function $L(\theta)$ for the parameter $\theta \in \Theta$, the traditional Bayesian framework is obtained for $\eta = 1$. In other words, if the parameter θ has a prior distribution $\pi(\theta)$, the GB posterior distribution $\pi^*(\theta | \text{data})$ is proportional to $L^\eta(\theta)\pi(\theta)$. This distribution is derived by raising $L(\theta)$ to the power η and then combining the result with the prioritized density $\pi(\theta)$. For more information on the GB method and how to select the value of the rate parameter, see [1–13]. In particular, the Safe Bayes algorithm based on the minimization of a sequential risk measure has been used in [3–6] to study learning rate selection. In [7,8], an alternative approach to learning rate selection was presented which includes two different information adaptation strategies. In [12], GBE was investigated using a common type-II censored sample from several exponential populations and different parameter values for the learning rate. A similar study was presented in [13], but focused on joint hybrid censoring.

Based on the currently available observations, Bayesian predictive research can be used to determine the point predictor or prediction interval for unknown future values in the same sample; this is called a one-sample prediction scheme. In addition to using the currently available observations, Bayesian prediction research can also be used to predict one or more future samples; these types of prediction schemes are referred to as two-sample and multi-sample prediction schemes. Many authors have dealt with the prediction of future failures or samples based on different censoring techniques and using different prediction methods. We list some of these that are relevant to our research. For example, [13] investigated the GBP using a combined type-II censored sample from several exponential populations. A study using a joint type-II censored sample from two exponential populations for Bayes estimation and prediction was published in [14]. Based on the

extended order statistics with multiple type II censoring, a Bayesian prediction for future values of distributions from the class of exponential distributions was created in [15,16].

Since the parameter of the analyzed distribution in the Bayesian analysis is random, its distribution is based on the previous distribution. In empirical Bayesian analysis (EB), the hyperparameters (i.e., the parameters of the prior distribution) are also unknown. The marginal density function of the hyperparameters, which is used to estimate the hyperparameters, is obtained by combining the density function of the distribution with prior distributions. Therefore, the maximum likelihood estimators (MLEs) of these hyperparameters were determined using the data from the original distribution. Several authors have used EB. For example, [17] used type II progressive censoring from the Lomax model to investigate the reliability of EBE. Based on progressively type II censored data, for the Kumaraswamy distribution [18] derived the EBE of the parameter, the reliability and the hazard function.

Based on type II censored samples, we examine the GBE, GEBE, GBP, and GEBP for the Pareto distribution ($Par(\theta)$). Thus, the main objective of this study is to compare all GB and GEB results for different LRP values, including the value $\eta = 1$, which characterizes classical Bayes.

The rest of the paper is structured as follows: Section 2 presents the Pareto distribution and reviews the research on GB and GEB, while Section 3 deals with GBP and GEBP. Section 4 presents a simulation study based on a censored Type II sample from $Par(\theta)$ to identify GBE, GEBE, GBP and GEBP for different LRP values and compares the results. Section 5 concludes the paper with a discussion and conclusions.

2. Estimation Study

In this section, we introduce our model, $Par(\theta)$, and study the GB and GEB problems for this model. The Pareto distribution can be applied to economic management, social geophysical phenomena, and science. More information about the distribution can be found in [19].

The Model

The probability density function (pdf), cumulative distribution function (cdf) and survival function of one parameter Pareto distribution respectively given by,

$$f(x; \theta) = \theta x^{-(\theta+1)} \quad (1)$$

$$F(x; \theta) = 1 - \exp[-\theta \ln x], \quad (2)$$

$$\bar{F}(x; \theta) = \exp[-\theta \ln x]. \quad (3)$$

2.1. Generalized Bayesian estimation

The likelihood function under type-II censored data from the class is given by,

$$L(\underline{x}; \theta) = c \bar{F}(x_r; \theta)^{n-r} \prod_{i=1}^r f(x_i; \theta) \\ \propto \theta^r \prod_{i=1}^r x_i^{-1} \exp[-\theta T(\underline{x})], \quad (4)$$

where, $c = \frac{n!}{(n-r)!}$, $T = \sum_{i=1}^r \ln x_i + (n-r) \ln x_r$, and $\underline{x} = (x_1, \dots, x_r)$.

Consider the prior distribution of θ is gamma distribution with pdf,

$$\pi(\theta; \delta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} \exp[-b\theta], \quad a > 0, b > 0. \quad (5)$$

where, $\delta = (a, b)$ is a vector of hyperparameters.

Raising (4) to the fractional power η , then combining it with the prior density (5) we get GB posterior distribution of θ given the data as follows,

$$\pi_{GB}^*(\theta; \delta, \underline{x}) = I_\delta^{*-1} L^\eta(\underline{x}; \theta) \pi(\theta; \delta) \\ = \frac{D_\delta^{C_\delta}}{\Gamma(C_\delta)} \theta^{C_\delta-1} \exp[-\theta D_\delta], \quad (6)$$

where $C_\delta = \eta r + a$, $D_\delta = \eta T + b$ and $I_\delta^{*-1} = \frac{D_\delta^{C_\delta}}{\Gamma(C_\delta)}$

Then GBE is given by,

$$\begin{aligned}\hat{\theta}_{GB} &= E(\theta) = \int_0^\infty \theta \pi_G^*(\theta; \delta, \underline{x}) d\theta. \\ &= \frac{\Gamma(C_\delta)}{D_\delta^{C_\delta}} \int_0^\infty \theta^{C_\delta} \exp[-\theta D_\delta] d\theta. \\ &= \frac{C_\delta}{D_\delta} \quad (7)\end{aligned}$$

2.2. Generalized Empirical Bayesian Estimation

We want to find the marginal pdf $f(x; \delta)$, by combining (1) and (5) then integrating with respect to θ as follows,

$$\begin{aligned}f(x; \delta) &= \int_0^\infty \pi(\theta; \delta) f(x; \theta) d\theta \\ &= \frac{b^a}{\Gamma(a)} \int_0^\infty \theta^a x^{-1} \exp[-\theta(\ln x + b)] d\theta. \\ &= \frac{ab^a}{x(\ln x + b)^{(a+1)}} \quad (8)\end{aligned}$$

From (8) we get the cdf $F(x; \delta)$ as follows,

$$\begin{aligned}F(x; \delta) &= \int_1^x \frac{ab^a}{u(\ln u + b)^{(a+1)}} du \\ &= 1 - \left(\frac{\ln x}{b} + 1\right)^{-a} \quad (9)\end{aligned}$$

From (8) and (9) the likelihood function $L_E(\underline{x}; \delta)$ under type-II censored data is given by,

$$\begin{aligned}L_E(\underline{x}; \delta) &= c \bar{F}(x_r; \delta)^{n-r} \prod_{i=1}^r f(x_i; \delta), \\ &\propto (ab^a)^r \left(\frac{\ln x_r}{b} + 1\right)^{-a(n-r)} \prod_{i=1}^r x_i^{-1} (\ln x_i + b)^{-(a+1)} \quad (10)\end{aligned}$$

Using the loglikelihood function $\mathcal{L}_E(\underline{x}; \delta) = \log L_E(\underline{x}; \delta)$, to find the maximum likelihood estimator (MLE) $\hat{\delta} = (\hat{a}, \hat{b})$, where $\hat{\delta} = \underset{\delta}{\operatorname{argmax}} \mathcal{L}_E(\underline{x}; \delta)$.

Differentiating $\mathcal{L}_E(\underline{x}; \delta)$ w. r. to $\delta = (a, b)$ then equating to zero as follows,

$$\frac{\partial \mathcal{L}_E(\underline{x}; \delta)}{\partial a} = 0, \quad \frac{\partial \mathcal{L}_E(\underline{x}; \delta)}{\partial b} = 0. \quad (11)$$

We get the MLE $\hat{\delta} = (\hat{a}, \hat{b})$ by solving the following two equations numerically,

$$\hat{a} = r \left((n-r) \log \left(\frac{\ln x_r}{b} + 1 \right) + \sum_{i=1}^r \log(\ln x_i + b) - r \log b \right)^{-1} \quad (12)$$

$$\frac{\hat{a}r}{b} + \frac{\hat{a}(n-r) \ln x_r}{b(\ln x_r + b)} - (\hat{a} + 1) \sum_{i=1}^r \frac{1}{(\ln x_i + b)} = 0 \quad (13)$$

$$\hat{a}r + \frac{\hat{a}(n-r) \ln x_r}{(\ln x_r + b)} - b(\hat{a} + 1) \sum_{i=1}^r (\ln x_i + b)^{-1} = 0$$

Substituting by $\hat{\delta} = (\hat{a}, \hat{b})$ in (7) we obtain the GE posterior as follows,

$$\pi_{GE}^*(\theta; \hat{\delta}, \underline{x}) = \frac{D_{\hat{\delta}}^{C_{\hat{\delta}}}}{\Gamma(C_{\hat{\delta}})} \theta^{C_{\hat{\delta}}-1} \exp[-\theta D_{\hat{\delta}}] \quad (14)$$

Then GEBE is given by,

$$\hat{\theta}_{GB} = \frac{C_{\hat{\delta}}}{D_{\hat{\delta}}} \quad (15)$$

3. One Sample Prediction

In this section we study GBP and GEBP intervals using one sample prediction scheme for Pareto distribution. Under type-II censored sample, the first r ordered statistics \underline{x} are observed from random sample of size n ; $r < n$. One sample prediction scheme is considered to predict the rest of unobserved values $x_s, r+1 \leq s \leq n$. The conditional density function of x_s given \underline{x} is given by,

$$f(x_s | \underline{x}) = \frac{(n-r)!}{(s-r-1)!(n-s)!} [\bar{F}(x_r) - \bar{F}(x_s)]^{s-r-1} \bar{F}(x_s)^{n-s} \bar{F}(x_r)^{-(n-r)} f(x_s). \quad (16)$$

Substituting by (1), (3) in (16), the conditional density function of x_s given \underline{x} is,

$$f(x_s | \underline{x}) = \sum_{j=0}^{s-r-1} c_j \frac{\theta}{x_s} \exp[-\theta n_j (\ln x_s - \ln x_r)], \quad (17)$$

where, $c_j = \frac{(-1)^{s-r-j-1} (n-r)!}{j!(s-r-j-1)!(n-s)!}$, $n_j = n - r - j$.

Combining (6) and (17) then integrating with respect to θ , the GB predictive density function is given by,

$$f_G^*(x_s|\underline{x}) = \frac{D_\delta^{C_\delta}}{\Gamma(C_\delta)} \sum_{j=0}^{s-r-1} c_j \int_0^\infty \frac{\theta^{C_\delta}}{x_s} \exp[-\theta\{n_j(\ln x_s - \ln x_r) + D_\delta\}] d\theta$$

$$= \frac{C_\delta}{D_\delta} \sum_{j=0}^{s-r-1} \frac{c_j}{x_s} \left[\frac{n_j(\ln x_s - \ln x_r)}{D_\delta} + 1 \right]^{-(C_\delta+1)} \quad (18)$$

The predictive reliability function of x_s is given by,

$$\bar{F}_\delta(t|\underline{x}) = \frac{C_\delta}{D_\delta} \sum_{j=0}^{s-r-1} \frac{c_j}{x_s} \int_t^\infty \left[\frac{n_j(\ln x_s - \ln x_r)}{D_\delta} + 1 \right]^{-(C_\delta+1)} dx_s$$

$$= \sum_{j=0}^{s-r-1} \frac{c_j}{n_j} \left[\frac{n_j(\ln t - \ln x_r)}{D_\delta} + 1 \right]^{-C_\delta} \quad (19)$$

Equating (19) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds (L_δ, U_δ) .

GEGB bounds $(L_{\hat{\delta}}, U_{\hat{\delta}})$, can be obtained by substituting by $\hat{\delta}$ in (19) then equating to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$.

4. Numerical Study

In this section, the results of the Monte Carlo simulation are presented to evaluate the performance of the inference methods derived in the previous sections.

The simulation study about $Par(\theta)$ distribution is designed and carried out as follows:

- We consider two values for $\theta = 0.5, 3$ based on the hyperparameters $(a, b) = (5, 10)$ and $(a, b) = (9, 3)$, respectively, where θ is obtained as the mean of gamma distribution (5).
- Generate one sample from $Par(0.5)$ and $Par(3)$ with size $n = 100$, and choosing $r = 50, 75, 100$.
- For EB, we use MLE (\hat{a}, \hat{b}) to compute $\hat{\theta}_{GE}$, where the results MLE (\hat{a}, \hat{b}) based on $Par(0.5)$ and $Par(3)$ are shown in Table 1.
- For the Monte Carlo simulations we use $M = 10,000$ replicates, therefore the estimator $\hat{\theta} = \frac{\sum_{i=1}^M \hat{\theta}_i}{M}$, and the estimated risk, $ER = \sqrt{\frac{\sum_{i=1}^M (\theta_i - \hat{\theta}_i)^2}{M}}$.
- Using (7) and (15), the estimation results are obtained and expressed by the estimator $\hat{\theta}$ and ER for different values of LRP, where $\eta = 0.1, 0.5, 1, 2$.
- The results of GBE and GEGB for $Par(0.5)$ and $Par(3)$ are shown in Table 2 and Table 4.
- Prediction results are based on one sample from $Par(0.5)$ and $Par(3)$ with size $n = 25$, the number of observations is $r = 20$, we then compute the GBP, GEGB bounds and its lengths at $\alpha = 0.05$, for the future values with $s = 21, 23, 25$ using (19).
- The results of GBP and GEGB for $Par(0.5)$ and $Par(3)$ are shown in Table 3 and Table 5.

Table 1. The MLEs of the hyperparameters (\hat{a}, \hat{b}) under different censored samples.

(\hat{a}, \hat{b})		
(n, r)	$\theta = 0.5$	$\theta = 3$
(25, 20)	(4.56, 8.41)	(10.883, 3.756)
(100, 50)	(4.253, 8.25)	(10.3955, 3.657)
(100, 75)	(4.642, 9.134)	(11.6707, 3.81)
(100, 100)	(5.153, 10.2)	(11.9567, 4)

Table 2. GBE and GEBE for the parameter $\theta = 0.5$.

r	η	$\hat{\theta}_{GB}$	ER_{GB}	$\hat{\theta}_{GE}$	ER_{GE}
50	0.1	0.5028	0.0005	0.5096	0.0036
75		0.5024	0.0004	0.5058	0.0030
100		0.5022	0.0002	0.5037	0.0016
50	0.5	0.5072	0.0029	0.5097	0.0060
75		0.5056	0.0013	0.5059	0.0036
100		0.5041	0.0006	0.5046	0.0025
50	1	0.5083	0.0035	0.5099	0.0076
75		0.5061	0.0018	0.5064	0.0039
100		0.5045	0.0013	0.5049	0.0026
50	2	0.5096	0.0043	0.5103	0.0099
75		0.5062	0.0021	0.5066	0.0048
100		0.5050	0.0016	0.5051	0.0032

Table 3. GBP and GEBP bound for Pareto future values at $\theta = 0.5$.

s	η	$(L, U)_{GB}$	$length$	$(L, U)_{GE}$	$length$
21	0.1	(3.0731,5.0055)	1.9324	(3.0726, 4.9361)	1.8635
23		(3.3535,8.6084)	5.2549	(3.3363, 8.4412)	5.1049
25		(4.2030,18.4935)	14.2905	(4.1332,18.0386)	13.9054
21	0.5	(3.0731,4.7355)	1.6624	(3.0729, 4.7010)	1.6281
23		(3.3667,7.6165)	4.2498	(3.3589, 7.5285)	4.1696
25		(4.2789,15.6562)	11.3773	(4.2475,15.4145)	11.1670
21	1	(3.0731,4.6526)	1.5795	(3.0730, 4.6318)	1.5588
23		(3.3718,7.3151)	3.9433	(3.3672, 7.2613)	3.8941
25		(4.3107,14.8007)	10.4900	(4.2918,14.6528)	10.3610
21	2	(3.0731,4.6000)	1.5269	(3.0730, 4.5887)	1.5157
23		(3.3754,7.1258)	3.7504	(3.3728, 7.0957)	3.7229
25		(4.3332,14.2663)	9.9331	(4.3229,14.1833)	9.8604

Table 4. GBE and GEBE for the parameter $\theta = 3$.

r	η	$\hat{\theta}_{GB}$	ER_{GB}	$\hat{\theta}_{GE}$	ER_{GE}
50	0.1	3.0079	0.0048	2.8975	0.0072
75		3.0078	0.0024	3.0443	0.0060
100		3.0076	0.0016	3.0014	0.0046
50	0.5	3.0317	0.0416	2.9815	0.0218
75		3.0260	0.0220	3.0383	0.0167
100		3.0198	0.0054	3.0172	0.0132
50	1	3.0427	0.0256	3.0115	0.0397
75		3.0316	0.0209	3.0387	0.0229
100		3.0257	0.0137	3.0230	0.0207
50	2	3.0529	0.0434	3.0356	0.0743
75		3.0353	0.0313	3.0393	0.0551
100		3.0287	0.0253	3.0254	0.0378

Table 5. GBP and GEBP bound for Pareto future values at $\theta = 3$.

s	η	$(L, U)_{GB}$	$length$	$(L, U)_{GE}$	$length$
21	0.1	(0.5127,0.8032)	0.2905	(0.5127,0.8043)	0.2916
23		(0.5609,1.3189)	0.7580	(0.5628,1.3151)	0.7523
25		(0.7094,2.7496)	2.0402	(0.7178,2.7366)	2.0188
21	0.5	(0.5127,0.7824)	0.2697	(0.5127,0.7849)	0.2722
23		(0.5621,1.2431)	0.6810	(0.5632,1.2476)	0.6844
25		(0.7164,2.534)	1.8176	(0.7212,2.5455)	1.8243
21	1	(0.5127,0.7732)	0.2605	(0.5127,0.7754)	0.2627
23		(0.5627,1.2098)	0.6471	(0.5634,1.2148)	0.6514
25		(0.7201,2.4395)	1.7194	(0.7232,2.4527)	1.7295
21	2	(0.5127,0.7664)	0.2537	(0.5127,0.7680)	0.2553
23		(0.5631,1.1851)	0.6220	(0.5636,1.1889)	0.6253

25	(0.7231,2.3697)	1.6466	(0.7249,2.3801)	1.6552
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From Table 2. According to $\hat{\theta}$ and ER , GBE becomes better for small value of LRP but for the large value of r , that means getting the best result at $\eta = 0.1$ and $r = 100$ (complete sample). GEBE also becomes better for small value of LRP but for the large value of r , that means getting the best result at $\eta = 0.1$ and $r = 100$. Generally, for $Par(0.5)$ the result of GBE is better than that of GEBE. The values of (L, U) in Table 3 and Table 5 are presented as the logarithmic value of the lower and upper bounds of $x_i, i = r + 1, \dots, n$.

According to the length of the interval, GBP and GEBP becomes better for large value of LRP, but for the small value of s , that means getting the best result at $\eta = 2$ and $s = 21$. In general, the result of GEBP is better than that of GBP for $Par(0.5)$.

From Table 4. GBE becomes better for small value of LRP but for the large value of r , that means getting the best result at $\eta = 0.1$ and $r = 100$ (complete sample). GEBE becomes better for small value of LRP but for the large value of r , except for $r = 50$, the results are better for $\eta = 0.5, 1$; we can conclude that the best result at $\eta = 0.1$ and $r = 100$. For $Par(3)$ the result of GBE is better than that of GEBE.

From Table 5. GBP and GEBP becomes better for large value of LRP but for the small value of s , that means getting the best result as the smallest length at $\eta = 2$ and $s = 21$. In general, the result of GBP is better than that of GEBP for $Par(3)$.

5. Discussion and Conclusion

In this study, one parameter Pareto distribution is considered and studied based on type-II censored sample. GB, GEB, GBP and GEBP are discussed for the distribution with different values of LRP η . According to the results in Table 2. to Table 5. We can conclude the results of the Pareto distribution as follows:

5.1. The Result of **Par(0.5)** Distribution

- Both of GBE and GEBE become better for small value of LRP but for the large value of r , that means getting the best result at $\eta = 0.1$ and $r = 100$ (complete sample).
- GBP and GEBP becomes better for large value of LRP, that means getting the best result at $\eta = 2$ and $s = 21$.
- The result of GBE is better than that of GEBE but the result of GEBP is better than that of GBP.
- Small values of LRP give the best result for GBE but vice versa for GEBP.

5.2. The Result of **Par(3)** Distribution

- GBE becomes better for small value of LRP but for the large value of r , that means getting the best result at $\eta = 0.1$ and $r = 100$ (complete sample).
- GEBE becomes better for small value of LRP but for the large value of r , except for $r = 50$, the results are better for $\eta = 0.5, 1$.
- The best result of GEBE at $\eta = 0.1$ and $r = 100$.
- The result of GBE is better than that of GEBE.
- GBP and GEBP becomes better for large value of LRP, that means getting the best result at $\eta = 2$ and $s = 21$.
- The result of GBE is better than that of GEBE and the result of GBP is better than that of GEBP.
- Small values of LRP give the best result for GBE but vice versa for GEBP.

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