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Generalized Bayesian Inference Study Based on Type-II Censored Data from the Class of Exponential Distributions

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ABSTRACT: Generalized Bayesian (GB) is a Bayesian approach based on the learning rate parameter (LRP) $(0 < \eta < 1)$ as a fraction of the power of the likelihood function. In this paper, we consider the GB method to perform inference studies for a class of exponential distributions. Generalized Bayesian estimators (GBE) and generalized empirical Bayesian estimators (GEBE) for the parameters of the considered distributions were obtained based on the censored type II samples. In addition, generalized Bayesian prediction (GBP) and generalized empirical Bayesian prediction (GEBP) were considered using a one-sample prediction scheme. Monte Carlo simulations were performed to compare the performance of the GBE and GEBE estimation results and the GBP and GEBP prediction results for different values of the LRP.

Keywords: generalized Bayesian estimators; generalized empirical Bayesian estimators; generalized Bayesian prediction; generalized empirical Bayesian prediction; learning rate parameter; prediction; type-II censored; Monte Carlo simulations

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1. Introduction

In the Bayesian inference techniques, GB analysis was introduced and studied based on the learning rate parameter $(0 < \eta < 1)$. The traditional Bayesian framework for $\eta = 1$ is a fraction of the power of the likelihood function $L(\theta) \equiv L(\theta;\, data)$ for the parameter $\theta \in \Theta$. This means that if the prior distribution of the parameter θ is $\pi(\theta)$, then the GB posterior distribution for θ is

$$\pi^*(\theta \mid data) \propto L^\eta(\theta)\pi(\theta), \quad \theta \in \Theta, \quad 0 < \eta < 1. \tag{1}$$

For more information on the GB approach and how to select the value for the rate parameter, see [1]- [13]. Specifically, the choice of the learning rate was studied in [3]- [6] using the Safe Bayes algorithm based on the minimization of a sequential risk measure. In [7] and [8], another learning rate selection method was proposed, which included two different information adaptation strategies. The authors in [12] investigated GBE based on a joint type-II censored sample from multiple exponential populations, using various values of the learning rate parameter. The same study was presented in [13] but was based on joint hybrid censoring.

A one-sample prediction scheme is a Bayesian prediction method that determines the point predictor or prediction interval for unknown future values in the same sample based on the currently available observations. A two-sample prediction scheme or a multiple-sample prediction scheme are two other ways in which Bayesian prediction can utilize currently available observations to predict one or more future samples. Numerous authors have addressed the prediction of future failures or samples using different censoring techniques in the context of different prediction methods. We

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highlight some points that are relevant to our research. For instance, [13] investigated the GBP using a combined type-II censored sample drawn from multiple exponential populations. A study using a joint type-II censored sample from two exponential populations for Bayes estimation and prediction was published in [14]. Based on a generalized order statistic and multiple type II censoring, a Bayesian prediction for the future values of distributions from the class of exponential distributions was constructed in [15, 16].

In the Bayesian study, the parameter of the distribution under investigation is a random variable, i.e. this unknown parameter is distributed according to the prior distribution. Empirical Bayes (EB) is a Bayesian study in which the parameters of the prior distribution (hyperparameters) are also unknown. By combining the density function of the distribution and the prior distributions, we obtain the marginal density function of the hyperparameters, which is used to estimate the hyperparameters. Therefore, the data of the original distribution are used to find the maximum likelihood estimators (MLEs) of these hyperparameters. EB has been introduced by many authors, for example, [17] studied the EBE of reliability performances with progressive type-II censoring of the Lomax model. The reliability and hazard function of the Kumaraswamy distribution were determined by [18] using progressive censored type II samples to estimate the EBE of the parameters. The Rayleigh distribution was studied in [19] to determine EBE and EBP.

In this study, the GBE, GEBE, GBP and GEBP distributions from the class of exponential distributions were examined using type II censored samples. Thus, the aim of this study is to examine all GB and GEB results for different LRP values, including $\eta = 1$, which discribe the traditional Bayes.

The rest of this article is organized as follows. Section 2 introduces the class of exponential models and then describes the problem of GB, GEB, GBP and GEBP for this class. Section 3 applies the investigation from Section 2 to the exponential and Rayleigh models, which are given as examples of the class. In addition, in that section, we present the simulation study for the exponential and Rayleigh models to obtain the GBE, GEBE, GBP and GEBP for different LRP values and compare the results. Finally, Section 4 discusses the results and concludes the paper.

2. Estimation and Prediction

In this section, we introduce the exponential class of models and examine the problems of the GB, GEB, GBP, and GEBP for this class.

2.1. The Model

Let $\boldsymbol{\theta}$ be the vector of parameters, define a function $g(x;\boldsymbol{\theta}) \equiv g(x)$, and its derivative $g'(x)$ where $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to 0^+} g(x) = 0$. The probability density function (pdf), the cumulative probability density function (cdf) and the survival function (sf) of the exponential class are each given by:

$$f(x;\boldsymbol{\theta}) = g'(x)\exp[-g(x)], \quad x > 0,\ \boldsymbol{\theta} > 0; \tag{1}$$

$$F(x;\boldsymbol{\theta}) = 1 - \exp[-g(x)]; \tag{2}$$

and

$$\overline{F}(x;\boldsymbol{\theta}) = \exp[-g(x)]. \tag{3}$$

The likelihood function under type-II censored data from the class is given by,

$$L(\underline{x};\boldsymbol{\theta}) = c\,\overline{F}(x_r;\boldsymbol{\theta})^{n-r}\prod_{i=1}^{r} f(x_i;\boldsymbol{\theta})$$

$$\propto A(\underline{x};\boldsymbol{\theta})\exp[-B(\underline{x};\boldsymbol{\theta})], \tag{4}$$

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where, $c = \frac{n!}{(n-r)!}$, $A(\underline{x};\boldsymbol{\theta}) = \prod_{i=1}^{r} g'(x_i)$, $B(\underline{x};\boldsymbol{\theta}) = \sum_{i=1}^{r} g(x_i) + (n-r)g(x_r)$, $\underline{x} = (x_1, \dots, x_r)$.

Consider the prior distribution of $\boldsymbol{\theta}$ in the following general form,

$$\pi(\boldsymbol{\theta};\boldsymbol{\delta}) = I_{\boldsymbol{\delta}}^{-1} C(\boldsymbol{\theta};\boldsymbol{\delta}), \tag{5}$$

where, $I_{\boldsymbol{\delta}} = \int_{\boldsymbol{\theta}} C(\boldsymbol{\theta};\boldsymbol{\delta}) \exp[-D(\boldsymbol{\theta};\boldsymbol{\delta})]\, d\boldsymbol{\theta}$, $\boldsymbol{\delta}$ is a vector of hyperparameters.

Combining (4) and (5), after raising (4) to the fractional power η, the GB posterior distribution of $\boldsymbol{\theta}$ is given by,

$$\pi_G^*(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x}) = I_{\boldsymbol{\delta}}^{*-1} L^{\eta}(\underline{x};\boldsymbol{\theta})\pi(\boldsymbol{\theta};\boldsymbol{\delta})$$

$$= I_{\boldsymbol{\delta}}^{*-1}\, G(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x}) \exp[-H(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x})], \tag{6}$$

where $G_{\boldsymbol{\delta}} = A^{\eta}(\underline{x};\boldsymbol{\theta})C(\boldsymbol{\theta};\boldsymbol{\delta})$, $H_{\boldsymbol{\delta}} = \eta B(\underline{x};\boldsymbol{\theta}) + D(\boldsymbol{\theta};\boldsymbol{\delta})$, and $I_{\boldsymbol{\delta}}^* = \int_{\boldsymbol{\theta}} G_{\boldsymbol{\delta}} \exp[-H_{\boldsymbol{\delta}}]\, d\boldsymbol{\theta}$.

Then GBE is given by,

$$\widehat{\boldsymbol{\theta}}_{GB} = E(\boldsymbol{\theta}) = \int_{\boldsymbol{\theta}} \boldsymbol{\theta}\, \pi_G^*(\boldsymbol{\theta};\boldsymbol{\delta},\underline{x})\, d\boldsymbol{\theta}. \tag{7}$$

2.2. Generalized empirical Bayesian estimation

Combining (1) and (5), to get the marginal pdf $f(x;\boldsymbol{\delta})$ as follows

$$f(x;\boldsymbol{\delta}) = \int_{\boldsymbol{\theta}} \pi(\boldsymbol{\theta};\boldsymbol{\delta})\, f(x;\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$= I_{\boldsymbol{\delta}}^{-1} \int_{\boldsymbol{\theta}} C(\boldsymbol{\theta};\boldsymbol{\delta}) g'(x) \exp[-\{D(\boldsymbol{\theta};\boldsymbol{\delta}) + g(x)\}]\, d\boldsymbol{\theta}. \tag{8}$$

From pdf in (8) we get the cdf $F(x;\boldsymbol{\delta})$, then the likelihood function under type-II censored data is given by,

$$L_E(\underline{x};\boldsymbol{\delta}) = c\, \overline{F}(x_r;\boldsymbol{\delta})^{n-r} \prod_{i=1}^{r} f(x_i;\boldsymbol{\delta}), \tag{9}$$

Using the loglikelihood function $\mathcal{L}_E(\underline{x};\boldsymbol{\delta}) = \log L_E(\underline{x};\boldsymbol{\delta})$, to find the maximum likelihood estimator (MLE) $\widehat{\boldsymbol{\delta}}$ as follows

$$\widehat{\boldsymbol{\delta}} = \underset{\boldsymbol{\delta}}{\operatorname{argmax}}\, \mathcal{L}_E(\underline{x};\boldsymbol{\delta}). \tag{10}$$

By solving the following equation,

$$\frac{\partial \mathcal{L}_E(\underline{x};\delta)}{\partial \delta} = 0. \tag{11}$$

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Substituting by $\hat{\delta}$ in (6) we obtain the posterior GE as follows,

$$\pi^{*}_{GE}\left(\boldsymbol{\theta};\widehat{\boldsymbol{\delta}},\underline{x}\right)=I^{*^{-1}}_{\widehat{\boldsymbol{\delta}}}\;G_{\widehat{\boldsymbol{\delta}}}\exp[-H_{\widehat{\boldsymbol{\delta}}}]. \tag{12}$$

Then GEBE given by

$$\widehat{\boldsymbol{\theta}}_{GE}=E(\boldsymbol{\theta})=\int_{\boldsymbol{\theta}}\boldsymbol{\theta}\,\pi^{*}_{GE}\left(\boldsymbol{\theta};\widehat{\boldsymbol{\delta}},\underline{x}\right)d\boldsymbol{\theta}. \tag{13}$$

2.3. A one sample prediction scheme

To determine the GBP and GEBP intervals using a one-sample prediction scheme under the type-II censored sample from the class, the first r ordered statistics $\underline{x}$ are observed from a random sample of size n; $r<n$. A one sample prediction scheme is considered to predict the rest of unobserved values x_s, $s=r+1,\dots,n$. The conditional density function of x_s given $\underline{x}$ is given by

$$f\left(x_s|\underline{x}\right)=\frac{(n-r)!}{(s-r-1)!(n-s)!}\left[\overline{F}(x_r)-\overline{F}(x_s)\right]^{s-r-1}\overline{F}(x_s)^{n-s}\,\overline{F}(x_r)^{-(n-r)}f(x_s). \tag{14}$$

Substituting by (1), (3) in (14), the conditional density function of x_s given $\underline{x}$ is,

$$f\left(x_s|\underline{x}\right)=\sum_{j=0}^{s-r-1}c_j\,g'(x_s)\exp\left[-n_j\{g(x_s)-g(x_r)\}\right], \tag{15}$$

where, $c_j=\frac{(-1)^{s-r-j-1}\,(n-r)!}{j!(s-r-j-1)!(n-s)!}$, $n_j=n-r-j$.

Combining (6) and (15) then integrating with respect to $\boldsymbol{\theta}$, the GB predictive density function is given by,

$$f^{*}_{G}\left(x_s|\underline{x}\right)=I^{*^{-1}}_{\boldsymbol{\delta}}\sum_{j=0}^{s-r-1}c_j\int_{\boldsymbol{\theta}}g'(x_s)G_{\boldsymbol{\delta}}\exp\left[-\left\{n_j\left(g(x_s)-g(x_r)\right)+H_{\boldsymbol{\delta}}\right\}\right]d\boldsymbol{\theta}. \tag{16}$$

The predictive reliability function of x_s, $s=r+1,\dots,n$ is given by,

$$\left(t|\underline{x}\right)=I^{*^{-1}}_{\boldsymbol{\delta}}\sum_{j=0}^{s-r-1}\frac{c_j}{n_j}\int_{\boldsymbol{\theta}}G_{\boldsymbol{\delta}}\exp\left[-\left\{n_j\left(g(t)-g(x_r)\right)+H_{\boldsymbol{\delta}}\right\}\right]d\boldsymbol{\theta}. \tag{17}$$

Equating (17) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds $(L_{\boldsymbol{\delta}},U_{\boldsymbol{\delta}})$. GEBP bounds $(L_{\hat{\delta}},U_{\hat{\delta}})$, can be obtained by substituting by $\hat{\delta}$ in (17) then equating to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$.

3. Applications

In this section, we apply the results in the previous section to one parameter models, therefore the models that discussed here are exponential $\mathrm{Exp}(\theta)$ and Rayliegh $\mathrm{Ray}(\theta)$. For these two distributions, the parameter θ assumed to be unknown, we may consider the conjugate prior distribution of θ as gamma prior distribution, $\theta\sim Gam(\delta_1,\delta_2)$, hence,

$$C(\theta;\boldsymbol{\delta})=\theta^{\delta_1-1},\;D(\theta;\boldsymbol{\delta})=\delta_2\theta;\;I^{-1}_{\boldsymbol{\delta}}=\frac{{\delta_2}^{\delta_1}}{\Gamma(\delta_1)},\quad\delta_1,\delta_2>0 \tag{18}$$

3.1. Exponential model

Here we give the essential functions and important forms derived in Section 2 for the exponential model as follows:

$$g(x)=\theta x,\;\;x>0. \tag{19}$$

For the likelihood function we have

$$A\left(\underline{x};\theta\right)=\theta^{r},\quad T_E=\sum_{i=1}^{r}x_i+(n-r)x_r\;\;\text{and}\quad B\left(\underline{x};\theta\right)=\theta T_E.$$

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Generalized posterior function, can be formed from the following,

$$G_{\boldsymbol{\delta}} = \theta^{\eta r+\delta_1-1},\ H_{\boldsymbol{\delta}} = \theta(\eta T_E + \delta_2) \quad \text{and} \quad I_{\boldsymbol{\delta}}^* = \frac{\Gamma(\eta r+\delta_1)}{(\eta T_E+\delta_2)^{\eta r+\delta_1}}.$$

The GBE of the parameter θ is given by,

$$\hat{\theta}_{GB} = \frac{\eta r+\delta_1}{\eta T_E+\delta_2} \ . \qquad (20)$$

The predictive reliability function of x_s is given by,

$$\overline{F}_{\boldsymbol{\delta}}(t|\underline{x}) = \sum_{j=0}^{s-r-1} \frac{c_j}{n_j}[1 + \frac{n_j(t-x_r)}{\eta T_E+\delta_2}]^{-(\eta r+\delta_1)}. \qquad (21)$$

Equating (21) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds $(L_{\boldsymbol{\delta}}, U_{\boldsymbol{\delta}})$.
The functions and forms under GEB study can be illustrated as follows:
The marginal pdf $f(x;\boldsymbol{\delta})$ is given in the following form,

$$f(x;\boldsymbol{\delta}) = \frac{\delta_1 {\delta_2}^{\delta_1}}{(x+\delta_2)^{(\delta_1+1)}}. \qquad (22)$$

Using pdf $f(x;\boldsymbol{\delta})$ and cdf $F(x;\boldsymbol{\delta})$ we get the likelihood function based on type-II censored data, as follows,

$$L_{\boldsymbol{E}}(\underline{x};\boldsymbol{\delta}) \propto \frac{\left(\delta_1 {\delta_2}^{\delta_1}\right)^r \prod_{i=1}^{r}(x_i+\delta_2)^{-(\delta_1+1)}}{\left(1+\frac{x_r}{\delta_2}\right)^{(n-r)\delta_1}}. (23)$$

By differentiating the loglikelihood function $\mathcal{L}_E(\underline{x};\boldsymbol{\delta})$ w. r. to δ_1 and δ_2 and equating each equation to zero, then solving them numerically we get the estimators $\hat{\delta}_1$ and $\hat{\delta}_2$.
The GEBE of the parameter θ is given by,

$$\hat{\theta}_{GE} = \frac{\eta r+\hat{\delta}_1}{\eta T_E+\hat{\delta}_2} \ . \qquad (24)$$

The predictive reliability function of x_s is given by,

$$\overline{F}_{\hat{\boldsymbol{\delta}}}(t|\underline{x}) = \sum_{j=0}^{s-r-1} \frac{c_j}{n_j}\left[1 + \frac{n_j(t-x_r)}{\eta T_E+\hat{\delta}_2}\right]^{-(\eta r+\hat{\delta}_1)} . \quad (25)$$

Equating (25) to $(1+\alpha)/2$ and $(1-\alpha)/2$, respectively, we obtain $(1-\alpha)\%$ GEBP bounds $(L_{\hat{\boldsymbol{\delta}}}, U_{\hat{\boldsymbol{\delta}}})$.

3.2. Rayleigh model

The essential functions and important forms derived in Section. 2 and Section. 3 for Rayliegh distribution are derived as follows:

$$g(x) = \frac{\theta x^2}{2}, \ \ x > 0. \qquad (26)$$

For the likelihood function we have

$$A(\underline{x};\theta) = \theta^r \prod_{i=1}^{r} x_i,\ T_R = \sum_{i=1}^{r} x_i^2/2 + (n-r)x_r^2/2 \ \text{ and } \ B(\underline{x};\theta) = \theta T_R.$$

Generalized posterior function, can be formed from the following,

$$G_{\boldsymbol{\delta}} = \theta^{\eta r+\delta_1-1}(\prod_{i=1}^{r} x_i)^{\eta},\ H_{\boldsymbol{\delta}} = \theta(\eta T_R + \delta_2) \quad \text{and} \quad I_{\boldsymbol{\delta}}^* = \frac{(\prod_{i=1}^{r} x_i)^{\eta}\,\Gamma(\eta r+\delta_1)}{(\eta T_R+\delta_2)^{\eta r+\delta_1}}.$$

The GBE of the parameter θ is given by,

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$$\hat{\theta}_{GB} = \frac{\eta r+\delta_1}{\eta T_R+\delta_2}. \qquad (27)$$

The predictive reliability function of x_s is given by,

$$\bar{F}_{\boldsymbol{\delta}}(t|\underline{x}) = \sum_{j=0}^{s-r-1} \frac{c_j}{n_j} [1 + \frac{n_j(t^2-x_r^2)/2}{\eta T_R+\delta_2}]^{-(\eta r+\delta_1)}. \qquad (28)$$

Equating (28) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GBP bounds $(L_{\boldsymbol{\delta}}, U_{\boldsymbol{\delta}})$.
The marginal pdf $f(x;\boldsymbol{\delta})$ is given in the following form,

$$f(x;\boldsymbol{\delta}) = \frac{\delta_1 \delta_2^{\delta_1}}{(x+\delta_2)^{(\delta_1+1)}}. \qquad (29)$$

Using pdf $f(x;\boldsymbol{\delta})$ and cdf $F(x;\boldsymbol{\delta})$ we get the likelihood function based on type-II censored data, as follows:

$$L_{\boldsymbol{E}}(\underline{\boldsymbol{x}};\boldsymbol{\delta}) \propto \frac{(\delta_1\delta_2^{\delta_1})^r \prod_{i=1}^{r}(x_i+\delta_2)^{-(\delta_1+1)}}{(1+x_r/\delta_2)^{(n-r)\delta_1}}. \qquad (30)$$

By differentiating the loglikelihood function $\mathcal{L}_E(\underline{x};\boldsymbol{\delta})$ with respect to δ_1 and δ_2 and equating each equation to zero, then solving them numerically we get the estimators $\hat{\delta}_1$ and $\hat{\delta}_2$.
The GEBE of the parameter θ is given by,

$$\hat{\theta}_{GE} = \frac{\eta r+\hat{\delta}_1}{\eta T_E+\hat{\delta}_2}. \qquad (31)$$

The predictive reliability function of x_s is given by,

$$\bar{F}_{\hat{\boldsymbol{\delta}}}(t|\underline{x}) = \sum_{j=0}^{s-r-1} \frac{c_j}{n_j} [1 + \frac{n_j(t^2-x_r^2)/2}{\eta T_R+\hat{\delta}_2}]^{-(\eta r+\hat{\delta}_1)}. \qquad (32)$$

Equating (31) to $\frac{1+\alpha}{2}$ and $\frac{1-\alpha}{2}$, respectively, we obtain $(1-\alpha)\%$ GEBP bounds $(L_{\hat{\delta}}, U_{\hat{\delta}})$.

3.3. Numerical analysis

In this subsection, the results of the Monte Carlo simulation are presented to evaluate the performance of the inference methods derived in the previous sections.

The simulation study is designed and carried out for the two models as follows:

- Generate one sample from each distribution with size $n = 50$, and choosing $r = 30, 40, 50$.
- Based on the chosen values of the hyperparameters $(\delta_1, \delta_2) = (4, 2)$, the suggested value for the parameter is $\theta = 2$, where θ is obtained as the mean of gamma distribution in (18).
- For EB, we use MLE $(\hat{\delta}_1, \hat{\delta}_2)$ to compute $\hat{\theta}_{GE}$, where the results MLE $(\hat{\delta}_1, \hat{\delta}_2)$ based on exponential and Rayliegh distributions are shown in Table 1.
- For the Monte Carlo simulations we use $M = 10{,}000$ replicates, therefore the estimator $\hat{\theta} = \frac{\sum_{i=1}^{M}\hat{\theta}_i}{M}$, and the estimated risk, $ER = \sqrt{\frac{\sum_{i=1}^{M}(\theta_i-\hat{\theta}_i)^2}{M}}$.
- Using (20), (24), (27) and (31), the estimation results are obtained and expressed by the estimator $\hat{\theta}$ and ER for different values of LRP, where $\eta = 0.1, 0.5, 1$.
- The results of GBE and GEBE for exponential and Rayliegh distributions are shown in Table 2 and Table 4.
- Prediction results are based on one sample from each distribution with size $n = 20$, the number of observations is $r = 15$, we then compute the GBP, GEBP bounds and its lengths at $\alpha = 0.05$, for the future values with $s = 16, 18, 20$ using (28) and (32).
- The results of GBP and GEBP for exponential and Rayliegh distributions are shown in Table 3 and Table 5.

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Table 1. The MLEs of the hyperparameters $\hat{\delta}_1, \hat{\delta}_2$ under different data from the two distributions.

(n, r)	$Exp(\theta)$ $(\hat{\delta}_1, \hat{\delta}_2)$	$Ray(\theta)$ $(\hat{\delta}_1, \hat{\delta}_2)$
(20, 15)	(10.497, 4.1)	(10.95,4.625)
(50, 30)	(11.31, 4.926)	(11, 4.4252)
(50, 40)	(10.9, 5.218)	(10.995, 5.1456)
(50, 50)	(10.3, 5.289)	(10.5, 5.2727)

Table 2. GBE and GEBE for the parameter of exponential distribution.

r	η	$\hat{\theta}_{GB}$	ER_{GB}	$\hat{\theta}_{GE}$	ER_{GE}
30	0.1	2.0491	0.0033	2.2308	0.0070
40		2.0436	0.0027	2.0687	0.0064
50		2.0295	0.0023	1.9677	0.0067
30	0.5	2.0611	0.0078	2.1430	0.0058
40		2.0461	0.0052	2.0526	0.0053
50		2.0401	0.0046	2.0052	0.0047
30	1	2.0655	0.0127	2.1119	0.0051
40		2.0495	0.0111	2.0495	0.0046
50		2.0414	0.0078	2.0187	0.0035

From Table 2. According to $\hat{\theta}$ and *ER*, GBE becomes better for small value of LRP but for the large value of r, that means getting the best result at $\eta = 0.1$ and $r = 50$ (complete sample). GEBE becomes better for large value of LRP and for the large value of r, that means getting the best result at $\eta = 1$ and $r = 50$. In general the result of GBE is better than that of GEBE.

Table 3. GBP and GEBP bound for exponential future values.

s	η	$(L, U)_{GB}$	*length*	$(L, U)_{GE}$	*length*
16	0.1	(0.6596, 1.1826)	0.5230	(0.6591, 1.0062)	0.3471
18		(0.7282, 2.1919)	1.4637	(0.7178, 1.6180)	0.9002
20		(0.9338, 4.9415)	4.0077	(0.8992, 3.3181)	2.4189
16	0.5	(0.6597, 1.0920)	0.4323	(0.6592, 1.0142)	0.3550
18		(0.7321,1.8567)	1.1246	(0.7237, 1.6225)	0.8988

20		(0.9557, 3.9800)	3.0243	(0.9251, 3.3245)	2.3994
16	1	(0.6596, 1.0642)	0.4046	(0.6593, 1.0181)	0.3588
18		(0.7337, 1.7554)	1.0217	(0.7273, 1.6225)	0.8952
20		(0.9653, 3.6914)	2.7261	(0.9411, 3.3220)	2.3809

From Table 3. According to the length of the interval, GBP and GEBP becomes better for large value of $\boldsymbol{r}$ and LRP, that means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$ (complete sample). In general, the result of GEBP is better than that of GBP.

Table 4. GBE and GEE for the parameter of Rayleigh distribution.

r	$\boldsymbol{\eta}$	$\hat{\boldsymbol{\theta}}_{GB}$	$\boldsymbol{ER_{GB}}$	$\hat{\boldsymbol{\theta}}_{GE}$	$\boldsymbol{ER_{GE}}$
30	0.1	2.0131	0.0028	2.3676	0.0129
40		2.0124	0.0023	2.1026	0.0082
50		2.0122	0.0013	1.9992	0.0013
30	0.5	2.0415	0.0060	2.2088	0.0095
40		2.0321	0.0026	2.0677	0.0048
50		2.0311	0.0024	2.0166	0.0015
30	1	2.0514	0.0075	2.1532	0.0074
40		2.0436	0.0059	2.0603	0.0020
50		2.0349	0.0041	2.0260	0.0018

From Table 4. GBE becomes better for small value of LRP but for the large value of $\boldsymbol{r}$, that means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$ (complete sample). GEBE becomes better for large value of LRP and for the large value of $\boldsymbol{r}$, except for the complete sample, the result becomes better for small value of LRP that means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. The result of GBE is better than that of GEBE at $\boldsymbol{r} = 30, 40$, but GEBE is better than GBE for the complete sample.

Table 5. GBP and GEBP bound for Rayleigh future values.

s	$\boldsymbol{\eta}$	$(\boldsymbol{L}, \boldsymbol{U})_{GB}$	*length*	$(\boldsymbol{L}, \boldsymbol{U})_{GE}$	*length*
16	0.1	(1.1379, 1.5297)	0.3918	(1.1376, 1.4252)	0.2876
18		(1.1967, 2.0880)	0.8913	(1.1914, 1.8238)	0.6324
20		(1.3577, 3.1400)	1.7823	(1.3448, 2.6322)	1.2874
16	0.5	(1.1379, 1.4695)	0.3316	(1.1377, 1.4253)	0.2876
18		(1.2000, 1.9209)	0.7209	(1.1954, 1.8145)	0.6191

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20		(1.3738, 2.8175)	1.4437	(1.3597, 2.6121)	1.2524
16	1	(1.1379, 1.4507)	0.3128	(1.1378, 1.4251)	0.2873
18		(1.2013, 1.8674)	0.6661	(1.976, 1.8084)	0.6108
20		(1.3807, 2.7132)	1.3325	(1.3689, 2.5993)	1.2304

From Table 5. GBP and GEBP becomes better for large value of $\boldsymbol{r}$ and LRP, that means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$ (complete sample). In general, the result of GEBP is better than that of GBP.

4. Discussion and Conclusion

In this study, a one-parameter model belonging to the class of exponential models is considered. Two well-known models $Exp(\theta)$ and $Ray(\theta)$ are examined based on a censored type-II sample. GB, GEB, GBP and GEBP are discussed for the two distributions with different values of LRP η. From the results in Table 2 to Table 5, we can summarize the results of the two distributions as follows:

4.1. The result of exponential model

- GBE becomes better for small value of LRP but for the large value of $\boldsymbol{r}$, that means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. GEBE becomes better for large value of LRP and for the large value of $\boldsymbol{r}$, that means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$.
- GBP and GEBP becomes better for large value of $\boldsymbol{r}$ and LRP, that means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$.
- The result of GBE is better than that of GEBE but the result of GEBP is better than that of GBP.
- Small values of LRP give the best result for GBE but vice versa for GEBP.

4.2. The result of Rayleigh model

- GBE becomes better for small value of LRP but for the large value of $\boldsymbol{r}$, that means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. GEBE becomes better for large value of LRP and for the large value of $\boldsymbol{r}$, except for the complete sample the result becomes better for small value of LRP, that means getting the best result at $\eta = 0.1$ and $\boldsymbol{r} = 50$. The result of GBE is better than that of GEBE at $\boldsymbol{r} = 30, 40$, but GEBE is better than GBE for the complete sample.
- GBP and GEBP becomes better for large value of $\boldsymbol{r}$ and LRP, that means getting the best result at $\eta = 1$ and $\boldsymbol{r} = 50$.
- The result of GBE is better than that of GEBE but the result of GEBP is better than that of GBP.
- Small values of LRP for the complete sample give the best result for GBE but vice versa for GEBP.

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References

1. Miller, J. W. and Dunson, D. B. Robust Bayesian inference via coarsening. Journal of the American Statistical Association, 2019, 114(527): 1113-1125.
2. Grünwald, P. The safe Bayesian: learning the learning rate via the mixability gap. In Algorithmic Learning Theory, 2012, volume 7568 of Lecture Notes in Computer Science, 169-183. Springer, Heidelberg. MR3042889.
3. Grünwald, P. and van Ommen, T. Inconsistency of Bayesian inference for misspecified linear models, and a proposal for repairing it. Bayesian Analysis, 2017, 12(4): 1069-1103.
4. Grünwald, P. Safe probability. Journal of Statistical Planning and Inference, 2018, 47-63. MR3760837.
5. De Heide, R., Kirichenko, A., Grünwald, P., and Mehta, N. Safe-Bayesian generalized linear regression. In International Conference on Artificial Intelligence and Statistics, 2020, 2623-2633. PMLR. 106, 113.
6. Holmes, C. C. and Walker, S. G. Assigning a value to a power likelihood in a general Bayesian model. Biometrika, 2017, 497-503.
7. Lyddon, S. P., Holmes, C. C., and Walker, S. G. General Bayesian updating and the loss-likelihood bootstrap. Biometrika, 2019, 465-478.
8. Martin, R. Invited comment on the article by van der Pas, Szabó, and van der Vaart. Bayesian Analysis, 2017, 1254-1258.
9. Martin, R. and Ning, B. Empirical priors and coverage of posterior credible sets in a sparse normal mean model. Sankhyā Series A, 2020, 477-498. Special issue in memory of Jayanta K. Ghosh.
10. Wu, P. S., Martin, R. "A Comparison of Learning Rate Selection Methods in Generalized Bayesian Inference." Bayesian Anal., 2023,18 (1) 105 - 132. https://doi.org/10.1214/21-BA1302
11. Abdel-Aty, Y., Kayid, M., and Alomani, G. Generalized Bayes estimation based on a joint type-II censored sample from k-exponential populations. Mathematics, 2023, 11, 2190. https://doi.org/10.3390/math11092190.
12. Abdel-Aty, Y.; Kayid, M.; Alomani, G. Generalized Bayes Prediction Study Based on Joint Type-II Censoring. Axioms 2023, 12, 716. https://doi.org/10.3390/axioms12070716.
13. *Abdel-Aty, Y.; Kayid, M.; Alomani, G.* Selection effect of learning rate parameter on estimators of k exponential populations under the joint hybrid censoring. Heliyon, 2024(10), e34087. https://doi.org/10.1016/j.heliyon.2024.e34087.
14. Shafay, A.R., Balakrishnan, N. Y. Abdel-Aty, Y. Bayesian inference based on a jointly type-II censored sample from two exponential populations, Journal of Statistical Computation and Simulation, 2014, 2427-2440.
15. Abdel-Aty, Y., Franz, J., Mahmoud, M.A.W. Bayesian prediction based on generalized order statistics using multiply type-II censoring. *Statistics*, 2007, 495-504.
16. Shafay, A. R. , Mohie El-Din, M. M. and Abdel-Aty, Y. Bayesian inference based on multiply type-II censored sample from a general class of distributions. Journal of Statistical Theory and Applications, 2018, 17(1), 146–157.
17. Mohie El-Din, M. M, Okasha, H. and B. Al-Zahrani, B. Empirical Bayes estimators of reliability performances using progressive type-II censoring from Lomax model, Journal of Advanced Research in Applied Mathematics, 2013, 1(5), 74-83.
18. Kumar, M., Singh, S., Singh, U. and Pathak, A. Empirical Bayes estimator of parameter, reliability and hazard rate for Kumaraswamy distribution. Life Cycle Reliability and Safety Engineering, 2019, 1(14).
19. Al-Ameen, M. and Abdel-Aty, Y. Empirical Bayes Inference for Rayleigh Distribution, Journal of Statistics Applications & Probability, 2022, 11(2), 695-708.