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A Duality Principle and Related Convex Dual Approximate Formulation Applied to a Non-Linear Plate Model

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A Duality Principle and Related Convex Dual Approximate Formulation Applied to a Non-Linear Plate Model

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Abstract: This article develops a duality principle applicable to originally non-convex primal variational formulations. More specifically, as a first application, we establish a convex dual approximate variational formulation for a non-linear Kirchhoff-Love plate model. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. We emphasize such a final convex dual approximate formulation obtained may be applied to a large class of similar models in the calculus of variations.

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MSC: 49N15

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1. Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. Specifically in this text, we present applications to the non-linear Kirchhoff-Love plate model.

We emphasize the results on duality theory here addressed and developed are inspired mainly in the approaches of J.J.Telega, W.R. Bielski and co-workers presented in the articles [1–4]. Other main reference is the article by Toland, [5].

Moreover, details on the Sobolev spaces involved may be found in [6].

Similar results and models are addressed in [7–11].

Basic results on convex analysis are addressed in [12]. Other similar results and approaches may be found in [13–15].

Now we start to describe the primal variational formulation for the plate model in question.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

We assume such a Ω set represents the middle surface of a thin plate with a constant thickness $h > 0$.

Moreover, we suppose such a plate is subject to a external load $(P_\alpha, P) \in L^2(\Omega; \mathbb{R}^3)$ resulting a field of displacements denoted by

$$(u_\alpha, w) = (u_1, u_2, w) \in W_0^{1,2}(\Omega; \mathbb{R}^2) \times W_0^{2,2}(\Omega) = V.$$

Both the load and displacements fields refers to a cartesian system $(0, x_1, x_2, x_3)$ and related canonical basis in $\mathbb{R}^3$.

Finally, we denote $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^4)$ and $Y_2 = Y_2^* = L^2(\Omega; \mathbb{R}^2)$.

We also emphasize the boundary conditions in question refer to a clamped plate.

The strain tensors are defined by

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta},$$

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and

$$\kappa_{\alpha\beta}(w) = -w_{,\alpha\beta}.$$

The plate total energy functional is defined by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)\,dx \\ &+\frac{1}{2}\int_\Omega h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)\,dx \\ &-\langle w,P\rangle_{L^2} - \langle u_\alpha, P_\alpha\rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\{H_{\alpha\beta\lambda\mu}\}$ is a fourth order positive definite symmetric constant tensor. Moreover

$$\{h_{\alpha\beta\lambda\mu}\} = \frac{h^2}{12}\{H_{\alpha\beta\lambda\mu}\}$$

and we denote

$$\{\overline{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$$

and

$$\left\{\overline{h}_{\alpha\beta\lambda\mu}\right\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$$

in an appropriate tensor sense.

2. The Main Duality Principle and Related Convex Dual Approximate Formulation

We start by defining the approximate functional $J_1 : V \to \mathbb{R}$ by

$$J_1(u) = J(u) + \sum_{\alpha=1}^{2}\frac{\varepsilon_1}{2}\int_\Omega (u_\alpha)^2\,dx,$$

and considering an appropriate real constant $K > 0$, the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y_1^* \to \mathbb{R}$, $F_3 : V \to \mathbb{R}$ and $F_4 : V \to \mathbb{R}$, by

$$\begin{aligned} F_1(u) &= \frac{1}{2}\int_\Omega h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)\,dx + \sum_{\alpha=1}^{2}\frac{K}{2}\int_\Omega w_{,\alpha}^2\,dx \\ &+\sum_{\alpha=1}^{2}\varepsilon\int_\Omega w_{,\alpha}^2\,dx - \langle w,P\rangle_{L^2} \end{aligned} \tag{2}$$

$$\begin{aligned} F_2(u,N) &= \frac{1}{2}\int_\Omega N_{\alpha\beta}w_{,\alpha}w_{,\beta}\,dx \\ &-\sum_{\alpha=1}^{2}\frac{K}{2}\int_\Omega w_{,\alpha}^2\,dx \end{aligned} \tag{3}$$

$$F_3(u) = \sum_{\alpha=1}^{2}\frac{\varepsilon_1}{2}\int_\Omega u_\alpha^2\,dx \tag{4}$$

$$F_4(u) = \sum_{\alpha=1}^{2}\varepsilon\int_\Omega w_{,\alpha}^2\,dx. \tag{5}$$

Moreover, we define the polar functionals $F_1^* : [Y_2^*]^2 \to \mathbb{R}$, $F_2^* : [Y_2^*]^2 \times Y_1^* \to \mathbb{R}$, $F_3^* : Y_1^* \to \mathbb{R}$ and $F_4^* : [Y_2^*]^2 \to \mathbb{R}$, by

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$$F_1^*(R,L) = \sup_{u \in V}\{\langle w_{,\alpha}, R_\alpha - L_\alpha\rangle_{L^2} - F_1(u)\}, \tag{6}$$

$$\begin{aligned} F_2^*(Q,L) &= \inf_{v \in Y_2}\left\{\langle v_\alpha, Q_\alpha + L_\alpha\rangle_{L^2} - \frac{1}{2}\int_\Omega N_{\alpha\beta} v_\alpha v_\beta \, dx \right. \\ &\left. + \sum_{\alpha=1}^{2} \frac{K}{2}\int_\Omega v_\alpha^2 \, dx + \frac{1}{2}\int_\Omega \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx \right\} \\ &= \frac{1}{2}\int_\Omega \overline{N_{\alpha\beta}^{(-K)}} (Q_\alpha + L_\alpha)(Q_\beta + L_\beta) \, dx \\ &+ \frac{1}{2}\int_\Omega \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx, \end{aligned} \tag{7}$$

if $N = \{N_{\alpha\beta}\} \in B^*$, where

$$B^* = \{N \in Y_1^* \; : \; \|N_{\alpha\beta}\|_\infty \leq K/8, \; \forall \alpha, \beta \in \{1,2\}\},$$

$$\left\{N_{\alpha\beta}^{(-K)}\right\} = \{N_{\alpha\beta} - K\delta_{\alpha\beta}\},$$

and

$$\left\{\overline{N_{\alpha\beta}^{(-K)}}\right\} = \left\{N_{\alpha\beta}^{(-K)}\right\}^{-1}.$$

Furthermore,

$$\begin{aligned} F_3^*(N) &= \sup_{u \in V}\{\langle u_\alpha, N_{\alpha\beta,\beta} + P_\alpha\rangle_{L_2} - F_3(u)\} \\ &= \sum_{\alpha=1}^{2} \frac{1}{2\varepsilon_1}\int_\Omega (N_{\alpha\beta,\beta} + P_\alpha)^2 \, dx, \end{aligned} \tag{8}$$

$$\begin{aligned} F_4^*(R,Q) &= \sup_{(v_1,v_2) \in Y_2^*}\left\{\langle (v_1)_\alpha, R_\alpha\rangle_{L^2} - \frac{\varepsilon}{2}\int_\Omega (v_1)_\alpha^2 \, dx \right. \\ &\left. + \langle (v_2)_\alpha, Q_\alpha\rangle_{L^2} - \frac{\varepsilon}{2}\int_\Omega (v_2)_\alpha^2 \, dx \right\} \\ &= \sum_{\alpha=1}^{2}\left(\frac{1}{2\varepsilon}\int_\Omega R_\alpha^2 \, dx + \frac{1}{2\varepsilon}\int_\Omega Q_\alpha^2 \, dx\right). \end{aligned} \tag{9}$$

At this point, denoting

$$D^* = \{Q = \{Q_\alpha\} \in Y_2^* \; : \; \|Q_\alpha\| \leq 5, \; \forall \alpha \in \{1,2\}\},$$

we define $J_1^* : (D^*)^2 \times B^* \times Y_2^* \to \mathbb{R}$ by

$$J_1^*(R,Q,N,L) = -F_1^*(R,L) - F_2^*(Q,L,N) - F_3^*(N) + F_4^*(R,Q),$$

and $J_2^* : (D^*)^2 \times B^* \to \mathbb{R}$ by

$$J_2^*(R,Q,N) = \text{sta}_{L \in Y_2^*} J_1^*(R,Q,N,L) = J_1^*(R,Q,N,\hat{L}(R,Q,N)),$$

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where $\hat{L} = \hat{L}(R, Q, N) \in Y_2^*$ is the only solution of the linear equation in L

$$\frac{\partial J_1^*(R, Q, N, \hat{L})}{\partial L} = \mathbf{0}.$$

Moreover, we define $J_3^* : (D^*)^2 \times B^* \to \mathbb{R}$, by

$$\begin{aligned} & J_3^*(R, Q, N) \\ = \; & J_2^*(R, Q, N) \\ & - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} \frac{K_1}{2} \left\| \overline{H}_{\alpha\beta\lambda\mu} N_{\lambda\mu} - \frac{(N_{\alpha\rho,\rho} + P_\alpha)_{,\beta} + (N_{\beta\rho,\rho} + P_\alpha)_{,\alpha}}{2\varepsilon_1} - \frac{1}{2}\tilde{v}_\alpha \tilde{v}_\beta \right\|_{0,2}^2, \end{aligned} \tag{10}$$

where

$$\tilde{v}_\alpha = \overline{N_{\alpha\beta}^{(-K)}}(Q_\beta + L_\beta(R, Q, N)), \; \forall \alpha \in \{1, 2\}.$$

Here, we assume

$$K_1 \gg \max\{1, K, \max\{\overline{h}_{\alpha\beta\lambda\mu}, \; \alpha, \beta, \lambda, \mu \in \{1, 2\}\}\},$$

$$0 < \varepsilon, \varepsilon_1 \ll 1$$

and

$$\frac{1}{\varepsilon} \gg \max\left\{K_1, \frac{1}{\varepsilon_1}\right\}.$$

Observe that

$$\frac{\partial^2 J_3^*(R, Q, N)}{\partial Q_\alpha^2} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) > \mathbf{0},$$

$$\frac{\partial^2 J_3^*(R, Q, N)}{\partial R_\alpha^2} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) > \mathbf{0},$$

$\forall \alpha \in \{1, 2\}$.

Thus, considering also the remaining mixed variations in Q_α and R_α, we may infer that

$$\det\left\{\frac{\partial^2 J_3^*(R, Q, N)}{\partial Q_\alpha \, \partial R_\beta}\right\} > \mathbf{0},$$

in $(D^*)^2 \times B^*$.

Moreover, by direct computation, clearly

$$\det\left\{\frac{\partial^2 J_3^*(R, Q, N)}{\partial N_{\alpha\beta} \, \partial N_{\lambda\mu}}\right\} < \mathbf{0},$$

in $(D^*)^2 \times B^*$.

From such results, we may infer that J_3^* is convex in (R, Q) and concave in N in $(D^*)^2 \times B^*$.

Let $(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \in (D^*)^2 \times B^* \times Y_2^*$ be such that

$$\delta J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) = \mathbf{0}.$$

Let $u_0 = ((u_0)_\alpha, w_0) \in V$ be such that

$$(u_0)_\alpha = \frac{\hat{N}_{\alpha\beta,\beta} + P_\alpha}{\varepsilon_1},$$

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and

$$(w_0)_{,\alpha} = \frac{\hat{Q}_\alpha}{\varepsilon},$$

$\forall \alpha \in \{1, 2\}$.

From standard results in Duality Theory and the Legendre Transform properties, we may obtain

$$\delta J_1(u_0) = \mathbf{0},$$

$$\delta J_2^*(\hat{R}, \hat{Q}, \hat{N}) = \mathbf{0}$$

,

$$\delta J_3^*(\hat{R}, \hat{Q}, \hat{N}) = \mathbf{0}$$

and

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}). \end{aligned} \tag{11}$$

From such results and the Min-Max Theorem, we have

$$J_3^*(\hat{R}, \hat{Q}, \hat{N}) = \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\}.$$

Joining the pieces, we have got

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\} \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}). \end{aligned} \tag{12}$$

Remark 2.1. *Defining* $J_5^* : (D^*)^2 \to \mathbb{R}$ *by*

$$J_5^*(R, Q) = \sup_{N \in B^*} J_3^*(R, Q, N),$$

we have that such a functional J_5^* *is convex in* $(D^*)^2$ *as the supremum in* $N \in B^*$ *of a family of convex functionals in* (R, Q).

In such a case, we have also obtained

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\} \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} J_5^*(R, Q) \\ &= J_5^*(\hat{R}, \hat{Q}). \end{aligned} \tag{13}$$

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3. Conclusion

In this article, we have developed a duality principle and related convex dual approximate variational formulation for an originally non-convex primal one.

We highlight the results here obtained are applicable to a large class of models in the calculus of variations, including other plate and shell non-linear theories, models in superconductivity, phase transition and micro-magnetism, among many others.

In a near future research we intend to apply such results to some of these mentioned related models.

Conflicts of Interest: The author declares no conflict of interest concerning this article.

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