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Article

On Duality Principles and Concerned Convex Dual Formulations Applied to a Non-Linear Plate Theory and Related Models

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Abstract: This article develops duality principles applicable to originally non-convex primal variational formulations. More specifically, as a first application, we establish a convex dual approximate variational formulation for a non-linear Kirchhoff-Love plate model. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. We emphasize such a convex dual approximate formulation obtained may be applied to a large class of similar models in the calculus of variations. Finally, in the last section, we present a duality principle and respective convex dual formulation for a Ginzburg-Landau type equation.

Keywords: duality principle; non-linear plate model; convex dual approximate formulation

MSC: 49N15

1. Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. Specifically in this text, we present applications to the non-linear Kirchhoff-Love plate model.

We emphasize the results on duality theory here addressed and developed are inspired mainly in the approaches of J.J. Telega, W.R. Bielski and co-workers presented in the articles [1–4]. Other main reference is the article by Toland, [5].

Moreover, details on the Sobolev spaces involved may be found in [6].

Similar results and models are addressed in [7–11].

Basic results on convex analysis are addressed in [12]. Other similar results and approaches may be found in [13–15].

Now we start to describe the primal variational formulation for the plate model in question.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

We assume such a Ω set represents the middle surface of a thin plate with a constant thickness $h > 0$.

Moreover, we suppose such a plate is subject to a external load $(P_\alpha, P) \in L^2(\Omega; \mathbb{R}^3)$ resulting a field of displacements denoted by

$$(u_\alpha, w) = (u_1, u_2, w) \in W_0^{1,2}(\Omega; \mathbb{R}^2) \times W_0^{2,2}(\Omega) = V.$$

Both the load and displacements fields refers to a cartesian system $(0, x_1, x_2, x_3)$ and related canonical basis in \mathbb{R}^3 .

Finally, we denote $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^4)$ and $Y_2 = Y_2^* = L^2(\Omega; \mathbb{R}^2)$.

We also emphasize the boundary conditions in question refer to a clamped plate.

The strain tensors are defined by

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2}w_{,\alpha}w_{,\beta},$$

and

$$\kappa_{\alpha\beta}(w) = -w_{,\alpha\beta}.$$

The plate total energy functional is defined by

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\Omega} H_{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(u) \gamma_{\lambda\mu}(u) \, dx \\ &+ \frac{1}{2} \int_{\Omega} h_{\alpha\beta\lambda\mu} \kappa_{\alpha\beta}(u) \kappa_{\lambda\mu}(u) \, dx \\ &- \langle w, P \rangle_{L^2} - \langle u_{\alpha}, P_{\alpha} \rangle_{L^2}. \end{aligned} \quad (1)$$

Here $\{H_{\alpha\beta\lambda\mu}\}$ is a fourth order positive definite symmetric constant tensor.

Moreover

$$\{h_{\alpha\beta\lambda\mu}\} = \frac{h^2}{12} \{H_{\alpha\beta\lambda\mu}\}$$

and we denote

$$\{\bar{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$$

and

$$\{\bar{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$$

in an appropriate tensor sense.

2. The Main Duality Principle and Related Convex Dual Approximate Formulation

We start by defining the approximate functional $J_1 : V \rightarrow \mathbb{R}$ by

$$J_1(u) = J(u) + \sum_{\alpha=1}^2 \frac{\varepsilon_1}{2} \int_{\Omega} (u_{\alpha})^2 \, dx,$$

and considering an appropriate real constant $K > 0$, the functionals $F_1 : V \rightarrow \mathbb{R}$, $F_2 : V \times Y_1^* \rightarrow \mathbb{R}$, $F_3 : V \rightarrow \mathbb{R}$ and $F_4 : V \rightarrow \mathbb{R}$, by

$$\begin{aligned} F_1(u) &= \frac{1}{2} \int_{\Omega} h_{\alpha\beta\lambda\mu} \kappa_{\alpha\beta}(u) \kappa_{\lambda\mu}(u) \, dx + \sum_{\alpha=1}^2 \frac{K}{2} \int_{\Omega} w_{,\alpha}^2 \, dx \\ &+ \sum_{\alpha=1}^2 \varepsilon \int_{\Omega} w_{,\alpha}^2 \, dx - \langle w, P \rangle_{L^2} \end{aligned} \quad (2)$$

$$\begin{aligned} F_2(u, N) &= \frac{1}{2} \int_{\Omega} N_{\alpha\beta} w_{,\alpha} w_{,\beta} \, dx \\ &- \sum_{\alpha=1}^2 \frac{K}{2} \int_{\Omega} w_{,\alpha}^2 \, dx \end{aligned} \quad (3)$$

$$F_3(u) = \sum_{\alpha=1}^2 \frac{\varepsilon_1}{2} \int_{\Omega} u_{\alpha}^2 \, dx \quad (4)$$

$$F_4(u) = \sum_{\alpha=1}^2 \varepsilon \int_{\Omega} w_{,\alpha}^2 \, dx. \quad (5)$$

Moreover, we define the polar functionals $F_1^* : [Y_2^*]^2 \rightarrow \mathbb{R}$, $F_2^* : [Y_2^*]^2 \times Y_1^* \rightarrow \mathbb{R}$, $F_3^* : Y_1^* \rightarrow \mathbb{R}$ and $F_4^* : [Y_2^*]^2 \rightarrow \mathbb{R}$, by

$$F_1^*(R, L) = \sup_{u \in V} \{ \langle w_{,\alpha}, R_\alpha - L_\alpha \rangle_{L^2} - F_1(u) \}, \quad (6)$$

$$\begin{aligned} F_2^*(Q, L) &= \inf_{v \in Y_2} \left\{ \langle v_\alpha, Q_\alpha + L_\alpha \rangle_{L^2} - \frac{1}{2} \int_{\Omega} N_{\alpha\beta} v_\alpha v_\beta \, dx \right. \\ &\quad \left. + \sum_{\alpha=1}^2 \frac{K}{2} \int_{\Omega} v_\alpha^2 \, dx + \frac{1}{2} \int_{\Omega} \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx \right\} \\ &= \frac{1}{2} \int_{\Omega} \overline{N_{\alpha\beta}^{(-K)}} (Q_\alpha + L_\alpha) (Q_\beta + L_\beta) \, dx \\ &\quad + \frac{1}{2} \int_{\Omega} \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx, \end{aligned} \quad (7)$$

if $N = \{N_{\alpha\beta}\} \in B^*$, where

$$B^* = \{N \in Y_1^* : \|N_{\alpha\beta}\|_\infty \leq K/8, \forall \alpha, \beta \in \{1, 2\}\},$$

$$\{N_{\alpha\beta}^{(-K)}\} = \{N_{\alpha\beta} - K\delta_{\alpha\beta}\},$$

and

$$\{\overline{N_{\alpha\beta}^{(-K)}}\} = \{N_{\alpha\beta}^{(-K)}\}^{-1}.$$

Furthermore,

$$\begin{aligned} F_3^*(N) &= \sup_{u \in V} \{ \langle u_\alpha, N_{\alpha\beta,\beta} + P_\alpha \rangle_{L^2} - F_3(u) \} \\ &= \sum_{\alpha=1}^2 \frac{1}{2\varepsilon_1} \int_{\Omega} (N_{\alpha\beta,\beta} + P_\alpha)^2 \, dx, \end{aligned} \quad (8)$$

$$\begin{aligned} F_4^*(R, Q) &= \sup_{(v_1, v_2) \in Y_2^*} \left\{ \langle (v_1)_\alpha, R_\alpha \rangle_{L^2} - \frac{\varepsilon}{2} \int_{\Omega} (v_1)_\alpha^2 \, dx \right. \\ &\quad \left. + \langle (v_2)_\alpha, Q_\alpha \rangle_{L^2} - \frac{\varepsilon}{2} \int_{\Omega} (v_2)_\alpha^2 \, dx \right\} \\ &= \sum_{\alpha=1}^2 \left(\frac{1}{2\varepsilon} \int_{\Omega} R_\alpha^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} Q_\alpha^2 \, dx \right). \end{aligned} \quad (9)$$

At this point, denoting

$$D^* = \{Q = \{Q_\alpha\} \in Y_2^* : \|Q_\alpha\| \leq 5, \forall \alpha \in \{1, 2\}\},$$

we define $J_1^* : (D^*)^2 \times B^* \times Y_2^* \rightarrow \mathbb{R}$ by

$$J_1^*(R, Q, N, L) = -F_1^*(R, L) - F_2^*(Q, L, N) - F_3^*(N) + F_4^*(R, Q),$$

and $J_2^* : (D^*)^2 \times B^* \rightarrow \mathbb{R}$ by

$$J_2^*(R, Q, N) = \text{sta}_{L \in Y_2^*} J_1^*(R, Q, N, L) = J_1^*(R, Q, N, \hat{L}(R, Q, N)),$$

where $\hat{L} = \hat{L}(R, Q, N) \in Y_2^*$ is the only solution of the linear equation in L

$$\frac{\partial J_1^*(R, Q, N, \hat{L})}{\partial L} = \mathbf{0}.$$

Moreover, we define $J_3^* : (D^*)^2 \times B^* \rightarrow \mathbb{R}$, by

$$\begin{aligned} & J_3^*(R, Q, N) \\ = & J_2^*(R, Q, N) \\ & - \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \frac{K_1}{2} \left\| \bar{H}_{\alpha\beta\lambda\mu} N_{\lambda\mu} - \frac{(N_{\alpha\rho,\rho} + P_\alpha)_{,\beta} + (N_{\beta\rho,\rho} + P_\alpha)_{,\alpha}}{2\varepsilon_1} - \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\beta \right\|_{0,2}^2, \end{aligned} \quad (10)$$

where

$$\tilde{v}_\alpha = \overline{N_{\alpha\beta}^{(-K)}}(Q_\beta + L_\beta(R, Q, N)), \quad \forall \alpha \in \{1, 2\}.$$

Here, we assume

$$K_1 \gg \max\{1, K, \max\{\bar{h}_{\alpha\beta\lambda\mu}, \alpha, \beta, \lambda, \mu \in \{1, 2\}\}\},$$

$$0 < \varepsilon, \varepsilon_1 \ll 1$$

and

$$\frac{1}{\varepsilon} \gg \max\left\{K_1, \frac{1}{\varepsilon_1}\right\}.$$

Observe that

$$\frac{\partial^2 J_3^*(R, Q, N)}{\partial Q_\alpha^2} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) > \mathbf{0},$$

$$\frac{\partial^2 J_3^*(R, Q, N)}{\partial R_\alpha^2} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) > \mathbf{0},$$

$\forall \alpha \in \{1, 2\}$.

Thus, considering also the remaining mixed variations in Q_α and R_α , we may infer that

$$\det\left\{\frac{\partial^2 J_3^*(R, Q, N)}{\partial Q_\alpha \partial R_\beta}\right\} > \mathbf{0},$$

in $(D^*)^2 \times B^*$.

Moreover, by direct computation, clearly

$$\frac{\partial^2 J_3^*(R, Q, N)}{\partial (N_{\alpha\beta})^2} < \mathbf{0}, \quad \forall \alpha, \beta \in \{1, 2\}$$

and considering the remaining mixed variations of J_3^* in N_{11} , N_{22} , N_{12} and N_{21} and the concerned remaining minor determinants, we may infer that

$$\det\left\{\frac{\partial^2 J_3^*(R, Q, N)}{\partial N_{\alpha\beta} \partial N_{\lambda\mu}}\right\} > \mathbf{0},$$

in $(D^*)^2 \times B^*$.

From such results, we may also infer that J_3^* is convex in (R, Q) and concave in N in $(D^*)^2 \times B^*$. Let $(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \in (D^*)^2 \times B^* \times Y_2^*$ be such that

$$\delta J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) = \mathbf{0}.$$

Let $u_0 = ((u_0)_\alpha, w_0) \in V$ be such that

$$(u_0)_\alpha = \frac{\hat{N}_{\alpha\beta,\beta} + P_\alpha}{\varepsilon_1},$$

and

$$(w_0)_{,\alpha} = \frac{\hat{Q}_\alpha}{\varepsilon},$$

$\forall \alpha \in \{1, 2\}$.

From standard results in Duality Theory and the Legendre Transform properties, we may obtain

$$\delta J_1(u_0) = \mathbf{0},$$

$$\delta J_2^*(\hat{R}, \hat{Q}, \hat{N}) = \mathbf{0},$$

$$\delta J_3^*(\hat{R}, \hat{Q}, \hat{N}) = \mathbf{0}$$

and

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}). \end{aligned} \tag{11}$$

From such results and the Min-Max Theorem, we have

$$J_3^*(\hat{R}, \hat{Q}, \hat{N}) = \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\}.$$

Joining the pieces, we have got

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\} \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}). \end{aligned} \tag{12}$$

Remark 2.1. Defining $J_5^* : (D^*)^2 \rightarrow \mathbb{R}$ by

$$J_5^*(R, Q) = \sup_{N \in B^*} J_3^*(R, Q, N),$$

we have that such a functional J_5^* is convex in $(D^*)^2$ as the supremum in $N \in B^*$ of a family of convex functionals in (R, Q) .

In such a case, we have also obtained

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\} \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} J_5^*(R, Q) \\ &= J_5^*(\hat{R}, \hat{Q}). \end{aligned} \tag{13}$$

3. One More Duality Principle and Related Convex Dual Formulation

In this section we develop another new duality principle with a related convex dual functional applied to a Ginzburg-Landau type equation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \rightarrow \mathbb{R}$ where

$$J(u) = \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 \, dx - \langle u, f \rangle_{L^2}, \quad (14)$$

where $\gamma > 0, \alpha > 0, \beta > 0$ and $f \in L^2(\Omega)$.

Here $u \in V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y^* \rightarrow \mathbb{R}, F_2 : V \times Y^* \rightarrow \mathbb{R}$ and $F_3 : V \rightarrow \mathbb{R}$ by

$$F_1(u, v_0^*) = \frac{\gamma}{4} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{1}{2} \langle u^2, v_0^* \rangle_{L^2} + \frac{K}{2} \int_{\Omega} u^2 \, dx, \quad (15)$$

$$F_2(u, v_0^*) = \frac{\gamma}{4} \int_{\Omega} \nabla u \cdot \nabla u \, dx + \frac{1}{2} \langle u^2, v_0^* \rangle_{L^2} + \frac{K}{2} \int_{\Omega} u^2 \, dx - \langle u, f \rangle_{L^2} - \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx - \beta \int_{\Omega} v_0^* \, dx, \quad (16)$$

$$F_3(u) = K \int_{\Omega} u^2 \, dx, \quad (17)$$

where $K > 0$ is an appropriate real constant.

Define also the polar functionals $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}, F_2^* : [Y^*]^3 \rightarrow \mathbb{R}$ and $F_3^* : Y^* \rightarrow \mathbb{R}$ by

$$F_1^*(v_1^*, v_0^*, z^*) = \sup_{u \in V} \{ \langle u, v_1^* + z^*/2 \rangle_{L^2} - F_1(u, v_0^*) \} \\ = \frac{1}{2} \int_{\Omega} \frac{(v_1^* + z^*/2)^2}{((-\gamma \nabla^2 + 2v_0^*)/2 + K)} \, dx \quad (18)$$

$$F_2^*(v_1^*, v_0^*, z^*) = \sup_{u \in V} \{ \langle u, -v_1^* + z^*/2 \rangle_{L^2} - F_2(u, v_0^*) \} \\ = \frac{1}{2} \int_{\Omega} \frac{(-v_1^* + z^*/2 + f)^2}{((-\gamma \nabla^2 + 2v_0^*)/2 + K)} \, dx \\ + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 \, dx + \beta \int_{\Omega} v_0^* \, dx, \quad (19)$$

if $v_0^* \in B^*$, where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\| \leq K/4\},$$

$$F_3^*(z^*) = \sup_{w \in L^2} \{ \langle w, z^* \rangle_{L^2} - F_3(w) \} \\ = \frac{1}{4K} \int_{\Omega} (z^*)^2 \, dx. \quad (20)$$

Moreover, define

$$D^* = \{v_1^* \in Y^* : \|2v_1^* - f\|_\infty \leq 5\},$$

$$D_1^+ = \{z^* \in Y^* : z^* f \geq 0, \text{ in } \Omega\},$$

and

$$D_1^* = \{z^* \in D^+ : \|z^*\|_\infty \leq (5/2)K\}.$$

Assuming $K_1 \gg \max\{\gamma, \alpha, \beta, 1/\alpha, 1, K\}$ and

$$K \gg \alpha,$$

define $J^* : D^* \times B^* \times D_1^* \rightarrow \mathbb{R}$ by

$$J^*(v_1^*, v_0^*, z^*) = -F_1^*(v_1^*, v_0^*, z^*) - F_2^*(v_1^*, v_0^*, z^*) + F_3^*(z^*). \quad (21)$$

From the variation of J^* in v_1^* , we have

$$-\frac{v_1^* + z^*/2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} + \frac{-v_1^* + z^*/2 + f}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} = 0, \text{ in } \Omega,$$

so that

$$-2v_1^* + f = 0, \text{ in } \Omega.$$

Moreover, denoting

$$u_0 = \frac{v_1^* + z^*/2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)},$$

from the variation of J^* in z^* , we obtain

$$u_0 = \frac{z^*}{2K}.$$

From such results, we may also obtain

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

so that

$$-\gamma\nabla^2 \left(\frac{z^*}{2K}\right) + 2\alpha \left(\left(\frac{v_1^* + z^*/2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} \right)^2 - \beta \right) \left(\frac{z^*}{2K}\right) - f = \mathbf{0}$$

in Ω .

With such results in mind, we define also the exactly penalized functional $J_1^* : D^* \times B^* \times D_1^* \rightarrow \mathbb{R}$, where

$$\begin{aligned} & J_1^*(v_1^*, v_0^*, z^*) \\ &= J^*(v_1^*, v_0^*, z^*) - \frac{K_1}{2} \|2v_1^* - f\|_{0,2}^2 \\ & \quad + 30\frac{K}{2} \left\| -\gamma\nabla^2 \left(\frac{z^*}{2K}\right) + 2\alpha \left(\left(\frac{v_1^* + z^*/2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} \right)^2 - \beta \right) \left(\frac{z^*}{2K}\right) - f \right\|_{0,2}^2 \end{aligned} \quad (22)$$

Clearly, we have

$$\frac{\partial^2 J_1^*(v_1^*, v_2^*, v_0^*)}{\partial (v_1^*)^2} = -\mathcal{O}(K_1) < \mathbf{0},$$

and

$$\frac{\partial^2 J_1^*(v_1^*, v_2^*, v_0^*)}{\partial (v_0^*)^2} = -\mathcal{O}(1/\alpha) < \mathbf{0},$$

so that considering also the mixed variations of J_1^* in v_1^* and v_0^* , we may infer that

$$\det \left\{ \frac{\partial^2 J_1^*(v_1^*, v_0^*, z^*)}{\partial v_1^* \partial v_0^*} \right\} > \mathbf{0}, \text{ in } D^* \times B^* \times D_1^*.$$

Hence, J_1^* is concave in (v_1^*, v_0^*) in $D^* \times B^* \times D_1$.

Furthermore,

$$\frac{\partial^2 J_2^*(v_1^*, v_0^*, z^*)}{\partial (z^*)^2} = \mathcal{O}(30/(4K)) > \mathbf{0},$$

in $D^* \times B^* \times D_1^*$, so that J_1^* is convex in z^* in $D^* \times B^* \times D_1^*$.

Let $(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \in D^* \times B^* \times D_1^*$ be such that

$$\delta J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \mathbf{0}.$$

From this, the last previous results and the Min-Max theorem we may infer that

$$J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \inf_{z^* \in D_1^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*) \right\}.$$

Let $u_0 \in V$ be such that

$$u_0 = \frac{\hat{z}^*}{2K}.$$

From fundamentals of duality theory and the Legendre Transform properties, we may obtain

$$\delta J(u_0) = \mathbf{0},$$

and

$$J(u_0) = J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*).$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ &= \inf_{z^* \in D_1^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*) \right\}. \end{aligned} \quad (23)$$

Remark 3.1. Defining $J_3^* : D_1^* \rightarrow \mathbb{R}$ by

$$J_3^*(z^*) = \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*),$$

we have that J_3^* is convex in D_1^* as a supremum of a family of convex functionals in z^* .

In such case, we have

$$\begin{aligned} J(u_0) &= J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ &= \inf_{z^* \in D_1^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*) \right\} \\ &= \inf_{z^* \in D_1^*} J_3^*(z^*) \\ &= J_3^*(\hat{z}^*). \end{aligned} \quad (24)$$

The objective of this section is complete.

4. An Approximate Procedure for Improving the Convexity Conditions for an Originally Non-Convex Primal Formulation

In this section we obtain an approximate procedure for improving the convexity conditions for an originally non-convex variational formulation.

In this new version we present some corrections and improvements concerning the previous one.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Let $V = W_0^{1,2}(\Omega)$ and consider a continuously twice Fréchet differentiable functional $J : V \rightarrow \mathbb{R}$ such that

$$-50 I_d \leq \frac{\partial^2 J(u)}{\partial u^2} \leq 50 I_d,$$

in V_1 , where the set V_1 will be specified in the next lines.

Let $K = 500000 \cdot 2\pi$ and $K_3 = 1.0$.

Define

$$V_1 = \{u \in V : 0 \leq u \leq K_3\}.$$

and the functional $J_2 : V_1 \rightarrow \mathbb{R}$ by

$$J_2(u) = -\frac{10^{16}}{K^3} \int_{\Omega} \cos\left(\frac{K^4}{u/10^{10} + K}\right) dx.$$

Define also the functional $J_1 : V_1 \rightarrow \mathbb{R}$ by

$$J_1(u) = J(u) + J_2(u).$$

Observe that, with a help of the software MAT-LAB, we may obtain

$$\begin{aligned} \frac{\partial J_1(u)}{\partial u} &= \frac{\partial J(u)}{\partial u} + \frac{\partial J_2(u)}{\partial u} \\ &= \frac{\partial J(u)}{\partial u} + \mathcal{O}(0.3). \end{aligned} \quad (25)$$

Moreover,

$$\begin{aligned} \frac{\partial^2 J_1(u)}{\partial u^2} &= \frac{\partial^2 J(u)}{\partial u^2} + \frac{\partial^2 J_2(u)}{\partial u^2} \\ &= \frac{\partial^2 J(u)}{\partial u^2} + \mathcal{O}(116) \\ &> \mathbf{0}, \text{ in } V_1. \end{aligned} \quad (26)$$

Thus, such an approximate functional J_1 is convex in V_1 and its first variation results, for a large class of appropriate numerical parameters, in a very close approximation for the first variation of the original functional J .

The functions

$$\frac{\partial J_2(x)}{\partial x}$$

and

$$\frac{\partial^2 J_2(x)}{\partial x^2}$$

on the interval $[0, K_3 = 1]$ stands for

$$\begin{aligned}\frac{\partial J_2(x)}{\partial x} &= \frac{10^{16}}{K^3} \sin\left(\frac{K^4}{x/10^{10} + K}\right) \left(\frac{-K^4}{(x/10^{10} + K)^2}\right) \left(\frac{1}{10^{10}}\right), \\ \frac{\partial^2 J_2(x)}{\partial x^2} &= \frac{10^{16}}{K^3} \cos\left(\frac{K^4}{x/10^{10} + K}\right) \left(\frac{-K^4}{(x/10^{10} + K)^2}\right)^2 \left(\frac{1}{10^{10}}\right)^2 \\ &\quad + 2 \frac{10^{16}}{K^3} \sin\left(\frac{K^4}{x/10^{10} + K}\right) \left(\frac{K^4}{(x/10^{10} + K)^3}\right) \left(\frac{1}{10^{10}}\right),\end{aligned}\quad (27)$$

respectively.

For their graphs, please see Figures 1 and 2, respectively.

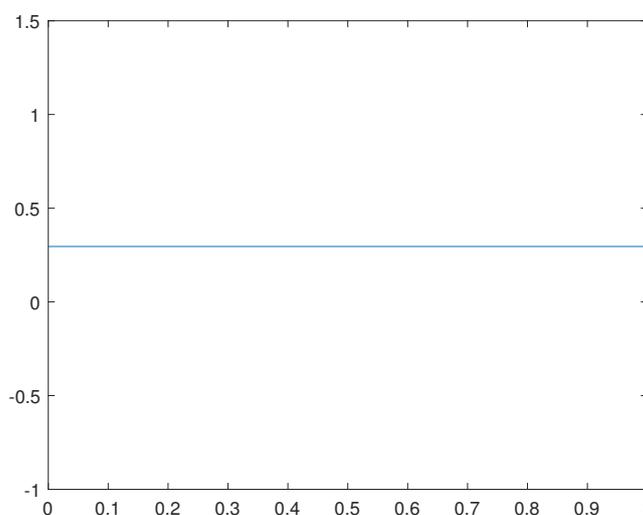


Figure 1. Graph of function $J'_2(x)$ on the interval $[0, 1]$.

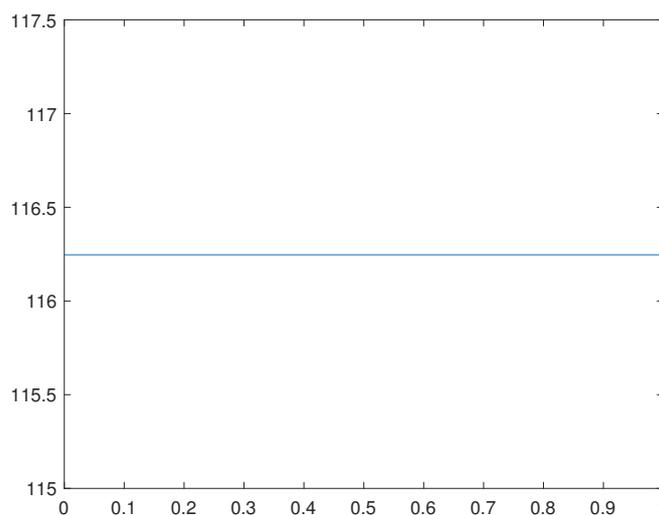


Figure 2. Graph of function $J''_2(x)$ on the interval $[0, 1]$.

Remark 4.1. The reason the graphs of $J'_2(x)$ and $J''_2(x)$ are straight lines is because $\frac{x}{10^{10}}$ varies very little on the interval $[0, 1]$.

The objective of this section is complete.

5. Conclusion

In this article, we have developed duality principles and related convex dual variational formulations for originally non-convex primal ones.

We highlight the results here obtained are applicable to a large class of models in the calculus of variations, including other plate and shell non-linear theories, models in superconductivity, phase transition and micro-magnetism, among many others.

In a near future research we intend to apply such results to some of these mentioned related models.

Conflicts of Interest: The author declares no conflict of interest concerning this article.

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