Article

Not peer-reviewed version

Analysis of Error-Based Switched Fractional Order Adaptive Systems: An Error Model Approach

Norelys Aguila-Camacho * , Javier A. Gallegos , YangQuan Chen , Juan C. Travieso-Torres

Posted Date: 13 September 2024

doi: 10.20944/preprints202409.0949.v1

Keywords: fractional calculus; switched fractional order adaptive systems; switched fractional order error models

Article

Analysis of Error-Based Switched Fractional Order Adaptive Systems: An Error Model Approach

Norelys Aguila-Camacho [1,*], Javier A. Gallegos [2], YangQuan Chen [3] and Juan C. Travieso-Torres [4]

[1] Department of Electricity, Universidad Tecnológica Metropolitana, Av. José Pedro Alessandri 1242, Santiago 7800002, Chile
[2] Department of Electrical Engineering, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago 7820436, Chile
[3] School of Engineering, University of California Merced, 5200 Lake Rd, Merced, CA 95343, USA
[4] Department of Industrial Technologies, Universidad de Santiago de Chile, El Belloto 3735, Santiago 9170125, Chile
* Correspondence: norelys.aguila@utem.cl

Abstract: This paper presents the analysis of four error models that can appear in the field of adaptive systems, where adaptive laws for parameter estimation are represented as fractional order differential equations whose order can switch between an order within the range $(0, 1)$ and 1. These switched adaptive laws have been proposed in the past few years to improve the balance between control energy and system performance, which is often a challenge when using traditional integer-order or fractional-order adaptive laws in adaptive identification and control. Boundedness of the solutions is proved for all cases, together with the convergence to zero of the estimation/tracking error. Additionally, sufficient conditions for parameter convergence are presented, showing that the excitation condition required for parameter convergence in the vector case is also sufficient for parameter estimation in the matrix case.

Keywords: fractional calculus; switched fractional order adaptive system; switched fractional order error models

---

1. Introduction

The main goal in developing adaptive methods is to address problems such as estimation or tracking in the presence of parametric uncertainties [1]. Many different adaptive schemes has been proposed and improved over the years, such as Model Reference Adaptive Control (MRAC) [2], Self Tuning Regulators [3], Adaptive Pole Placement Control [4], Adaptive Passivity Based Control [5], Adaptive Observers [2], among others. Applications span process control, automotive systems, positioning systems, propulsion systems, and a huge effort in flight control (see [6] and the references therein).

Technically speaking, the term *adaptive system* has been adopted to refer to the wide set of feedback systems involving estimation and control, regardless of the specific adaptive scheme used. For analysis purposes, adaptive systems are commonly represented in the form of differential and algebraic equations, referred as *error models*, which contain two different type of errors. The first error (e) represents a tracking or estimation error and the second error (ϕ) represents a parameter estimation error. The tracking or estimation error e is measurable, and it is used to tune the parameter error ϕ, which is unknown and adjustable [7], often stated as the difference between a designed parameter estimate θ and a true value θ^* in the form $\phi = \theta - \theta^*$.

For continuous-time and standard-derivative dynamics, a general enough description of an error model can be given as follows [2]:

$$\dot{e} = f(e, \phi, \omega, t) \tag{1}$$

$$\dot{\phi} = g(e, \omega, t) \tag{2}$$

where ω contains measurable input-output data.

In early 2000's, fractional order operators (FO) [8] started to be introduced to model real systems, due to the link between these operators and certain physical phenomena such as damping coefficients in a variety of viscoelastic materials, difussion, and electrical impedance of biological tissues [9].

 doi:10.20944/preprints202409.0949.v1

Since then, application of FO to modelling real systems has been widely reported in literature. Just to mention some of them, we can find fractional order models for food science [10], signal/speech processing [11,12], biology [13], disease/infection rates [14] and economics [15]. FO have been also introduced in the control field, leading to fractional order controllers with better performance than classic integer order controllers. The reader can check many survey papers detailing these controllers and the reported results, see for instance [16–18].Adaptive control has not been the exception, with a wide amount of works reporting the use of FO in the adaptive controller/observer design, and/or to model the system to be controlled or identified (see for instance [19–22]). Within that, the analytical analysis of the resulting fractional order adaptive schemes became relevant, leading to the adoption of the error model concept in a generalized way, e.g., Fractional Order Error Models (FOEM). The general representation of a FOEM is the same as in (1),(2), but using fractional orders derivatives for e, ϕ, instead of the classic integer order derivatives.

Dealing with the analysis of FOEM was and is still a challenge, mainly due to inherent characteristics of fractional operators and the lack of proper tools to analyzing them. Still, significant advances have been made over the years, such as the analysis of the four more common FOEM (e.g., FOEM1 to FOEM4) in [23], for those cases where the fractional order is the same for differential equations of e and ϕ. Modifications of these FOEM were introduced in [24], addressing the analysis of FOEM1 to FOEM4 in the presence of bounded disturbances and parameter variations. Later in [25], the mixed order cases were analyzed for FOEM2 and FOEM3, where fractional order for differential equations of e and ϕ do not need to be the same, as long as the fractional order in ϕ equation is lower or equal than the fractional order in e equation. Although these advances have been made, still there are many open questions to the analysis of FOEM, such as how to choose the fractional orders for the adaptive laws (which leads to the fractional order in the ϕ equation), how to prove the convergence to zero of the estimation or control error e for some cases, how to deal with FOEM where all the components of differential equations for e,ϕ can have different fractional orders, among others.

A particular open problem has arisen recently due to the introduction of switched adaptive laws in adaptive schemes, aimed at improving system performance and controlling energy use and leading to a differential equation for ϕ (e.g., (2)) whose order can switch among a fractional order α and the integer order 1, according to certain switching rule. Indeed, the problem of analyzing the boundedness and error convergence in these switched equations has been only addressed for particular adaptive schemes. For instance, in [26], an identification problem has been addressed as well as a Switched Fractional Order Model Reference Adaptive Control (SFOMRAC) scheme to control Linear Time Invariant (LTI) Single Input Single Output (SISO) plant. In this work, the switching rule in the adaptive law is time-based, that is, at some time instant defined by the designer, the order changes from fractional to integer. Recently, a couple of works have been published where the order of the adaptive law uses an error-based switching rule, that is, the order switches between a fractional order and the integer order according to the magnitude of the control error e. These two works correspond to a High-gain adaptive control scheme [27] and a SFOMRAC for Multiple Input Multiple Output (MIMO) systems [28]. Although these works have provided analytical guarantees on the performance of the resulting switched schemes, the potential use of switched adaptive laws in different problems and applications leads to the need of a general framework to establish boundedness of the signals and convergence of the errors.

This is precisely the main goal of this paper, where the error model approach is adopted to analyze adaptive schemes using switched fractional order adaptive laws. The main contributions are stated in the following.

- In the context of the error model approach, and including the four error equations for e that appear more often in the study of control and estimation adaptive systems [2], this papers proposes differential equations for the parameter error ϕ that can switch between a fractional order and the integer order, using an error-based switching mechanism. This leads to the analysis of four Switched Fractional Order Error Models (SFOEM), specifically SFOEM1 to SFOEM4.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 September 2024 doi:10.20944/preprints202409.0949.v1

The only previous work addressing something within this scope is [28], where a SFOMRAC is proposed and analyzed (without proving parametric convergence), whose structure coincides with the SFOEM2 in this paper, constituting a particular case of this paper.
- A complete analytical proof of stability and convergence is provided for each of the SFOEM presented, allowing its future application to any switched adaptive scheme that can be put in their form, which is the fundamental advantage of the error model approach.
- In contrast to all revised literature, such as [2,3,28,29], where the four classic error models are analyzed for the case when ϕ is a vector, in this paper the analysis is made considering ϕ as a matrix (multi variable case) for three out of four error models. We found that the same excitation condition for the vector case is sufficient to estimate the matrix parameters. Roughly speaking, this is due to a sharing of the excitation for each column vector of the parameter matrix.

The paper is organized as follows. Section 2 contains some concepts, definitions and analytical tools to be used in the analysis throughout the paper. Sections 3 to 6 presents the Switched Fractional Order Error Models 1 to 4 with their corresponding analytical analysis. Finally, Section 7 presents the main conclusions of the work.

2. Basic Concepts

This section presents some concepts of fractional calculus and analytical tools that are used along the paper. In what follows, $\mathbb{R}_{>(\geq)0}$ denotes the set of positive (non-negative) real numbers, and $\mathbb{R}^n$ the Euclidean space of dimension n. For $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, a positive definite matrix, $|x|_A := x^\top Ax$ defines a norm. In the case of matrices, $|B|_A := B^\top AB$ for $B \in \mathbb{R}^{n\times n}$. PE$(n)$ denotes the set of persistently exciting signals of order n [2].

2.1. Definitions from Fractional Calculus

Fractional calculus studies integrals and derivatives of orders that can be any real or complex numbers [8]. The Riemann-Liouville fractional integral is one of the main concepts of fractional calculus, and is presented in Definition 1.

Definition 1 (Riemann-Liouville fractional integral [30])**.**

$$I_a^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{3}$$

where $\alpha > 0$ *and* $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ *is the Gamma function [8].*

There are some alternative definitions regarding fractional derivatives. Definition 2 corresponds to the fractional derivative according to Caputo, which is the one most frequently used in engineering problems and the one used in this paper.

Definition 2 (Caputo fractional derivative [30])**.**

$$D_a^\alpha f(t) := I_a^{m-\alpha} D^m f(t), \tag{4}$$

where $\alpha > 0$ *and* $m = \lceil \alpha \rceil$.

2.2. Analytical Tools

Property 1. *For any continuous function* $f : [0,\infty) \to \mathbb{R}$, *any real numbers* $a, \alpha, \beta > 0$, *and for any real number* $t > 0$, *the following holds [30]*

$$I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t), \tag{5}$$

and

$$I_a^\alpha D_a^\alpha f(t) = f(t) - f(a). \tag{6}$$

The following result establishes existence and uniqueness in the set of continuous functions for the solutions of a kind of fractional equations.

Theorem 1 ([31]). *Consider the system of fractional order integral equations*

$$y(t) = p(t) + I^\alpha[f(.,y(.))](t), \tag{7}$$

where α is to be seen as a vector with components α_i, and y, f, p are vectors of components y_i, f_i, p_i respectively, for $i = 1, \ldots, n$. If $p : [0,T] \to \mathbb{R}^n$ is a continuous function and $f_i(.,.)$ are continuous functions in their first variable and Lipschitz continuous functions in their second variables for $i = 1, \ldots, n$, then

i. There exists a unique continuous solution $y \in \mathcal{C}[0,T]$ to system (7).
ii. $y \in \mathcal{C}[0,T]$ is a solution to system (7) for

$$p_i(t) := \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_k}{k!} y_{i_0}^{(k)} \tag{8}$$

if and only if each of its components y_i is a solution to $D^{\alpha_i} y_i = f_i(t,y)$ with initial condition $y_i^{(k)}(0) = y_{i_0}^{(k)}$ for $k = 1, \ldots, \lceil\alpha\rceil - 1$ and $i = 1, \ldots, n$.

Finally, a useful property for the derivative of composite functions is presented.

Theorem 2 ([32]). *For a given $x \in \mathbb{R}^d$, let $u \in \{x\} + I_{0+}^\alpha \mathcal{C}([0,T]), \mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions*

- *The function V is convex on $\mathbb{R}^d$ and $V(0) = 0$.*
- *The function V is differentiable on $\mathbb{R}^d$.*

Then the following inequality holds for all $t \in [0,T]$

$$D_{0+}^\alpha V(u(t)) \leq \langle \nabla V(u(t)), D_{0+}^\alpha u(t) \rangle. \tag{9}$$

3. Analysis of Switched Fractional Order Error Model 1

Many problems in adaptive estimation and control may be expressed as

$$y(t) = \theta^{*T} \omega(t), \tag{10}$$

where $\theta^* \in \mathbb{R}^{m \times n}$ represents an unknown parameter matrix, $\omega \in \mathbb{R}^m$ is a measurable $\mathcal{C}^1-$function called regressor, and $y \in \mathbb{R}^n$ is a measurable signal, called *plant output* hereafter, that can be determined at each time instant [6]. For instance, when designing adaptive observers for linear plants, the plant output can be represented as an algebraic combination of filtered inputs and outputs [2]. Also, the plant model used in some combined/composite approaches to adaptive control can be written in this form in (10), as reported for instance in [29,33].

In these problems, usually an estimator for plant output is formulated in the form $\hat{y}(t) = \theta^T(t)\omega(t)$, with $\theta \in \mathbb{R}^{m \times n}$ the estimated parameter matrix. If the estimation error is defined as $e = \hat{y} - y$ and the parameter error as $\phi = \theta - \theta^*$, it can be easily derived that the relationship between e and ϕ can be stated as

$$e(t) = \phi^T(t)\omega(t) \tag{11}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 September 2024 doi:10.20944/preprints202409.0949.v1

The most common approach for adjusting the estimate θ at each time t is to use an adaptive law that is obtained using the gradient-descent approach to minimize $e^T e$ [2,34], leading to an equation for the parameter error ϕ as

$$\dot{\phi}(t) = -\gamma\omega(t)e^T(t), \tag{12}$$

with $\gamma \in \mathbb{R}^+$ an adaptive gain. Equations (11),(12) constitute the classic Error Model 1, completely studied in [2, Chapter 7] for the case when $e \in \mathbb{R}$. The case when $e \in \mathbb{R}^n$ is briefly analyzed in [2, Chapter 10]. Fractional Order Error Model 1 (FOEM1) uses a fractional order derivative instead of the integer one in Equation (12), whose analysis for the scalar case $e \in \mathbb{R}$ can be found in [23].

In this paper, a switched derivation order approach is used to adjust the estimated parameter θ, resulting in a different error dynamics, called hereafter Switched Fractional Order Error Model 1 (SFOEM1), that has not been previously analyzed. Definition of SFOEM1 corresponds to

$$e(t) = \phi^T(t)\omega(t), \quad e(0) = e_0 \tag{13}$$

$$D_{a(t)}^{\alpha(t)}\phi(t) = -\gamma(t)\omega(t)e^T(t), \quad \phi(0) = \phi_0 \tag{14}$$

$$\gamma(t) = \frac{\gamma_i}{1 + tr[\theta(t_i)\theta^T(t_i)]}, \quad \gamma(0) = \gamma_0 \tag{15}$$

This yields

$$D_{a(t)}^{\alpha(t)}\phi(t) = -\gamma(t)\omega(t)\omega^T(t)\phi(t). \tag{16}$$

The adaptive gain γ, aimed to manipulate the speed of (14) and to normalize, takes a constant value $\gamma_i/(1 + tr[\theta(t_i)\theta^T(t_i)])$ in each switching interval $t \in [t_i, t_{i+1})$, with $\{\gamma_i\}_{i\in\mathbb{N}}$ a designer chosen sequence of real numbers such that $0 < \gamma_l < \gamma_i$ for any i.

The fractional order $\alpha(t)$ is varied using the following switched strategy for any $t > 0$

$$\alpha(t) = \begin{cases} \alpha_0 & if \quad \|e(t)\| > \epsilon \quad \text{and} \quad h(t) \\ 1 & if \qquad \text{otherwise} \end{cases} \tag{17}$$

where $\epsilon > 0$ and $\alpha_0 \in (0,1)$ are designer choices. The function $h(\cdot)$ is a logic function (i.e., taking true/false values) defined for any $t \in [t_i, t_{i+1})$ as

$$h(t) = \begin{cases} \text{False} & if \quad \left(i > C, \quad \|e(t)\| < \epsilon + \delta \quad \text{and} \quad \mathcal{T}(t) \le i \cdot \delta \right) \\ \text{True} & if \qquad \text{otherwise} \end{cases} \tag{18}$$

where $C \ge 1$ is a designer constant, and $\mathcal{T}$ is the largest interval of time in which the fractional mode is active at time t, i.e.,

$$\mathcal{T}(t) := \max_j\{|t_{j+1} - t_j| : t_{j+1} \le t \quad \& \quad \alpha(t_j) = \alpha_0\}. \tag{19}$$

Definition of function h encodes, on the one hand, a hysteresis mechanism to avoid Zeno solutions. This is accomplished by fixing a small enough $\delta > 0$, so the switching from the integer mode to the fractional mode occurs only when $\|e(t_i)\| = \epsilon + \delta$, while the switching from the fractional mode to the integer mode will occur only when $\|e(t_j)\| = \epsilon$. On the other hand, function h encodes also a mechanism to promote that in disturbed or transient stages ($\|e\| > \epsilon$) the adaptation with α_0 should occur, and conversely, that when staying close to the aim ($\|e\| < \epsilon$) the lesser should be the need of switching to α_0. To this aim, the condition $\mathcal{T}(t) \le i \cdot \delta$ means that the transitions to the fractional mode remain active as long as the fractional mode is needed, as measured in terms of time quantify $\mathcal{T}$ and relative to a measure of the overall time given by $i \cdot \delta$, where we use the same δ for simplicity but it can be scaled differently. The use of a large enough constant C in (18) ensures that both mechanisms previously explained are triggered after a finite number C of switches, to let the transient as unaffected as possible.

The differential Equation (14) is understood in the resetting mode; that is, every time a switch occurs, the initial time a of the fractional operator is set equal to the switching time t_i. This defines the initial time function $a = a(t)$ in (14). Moreover, every time a switch occurs, $\theta(t_i) = \theta(t_i^-)$, which implies $\phi(t_i) = \phi(t_i^-)$ and thus, discontinuities are excluded.

3.1. Boundedness of the Signals and Convergence of the Estimation Error in SFOEM1

The following result states boundedness of the signals and convergence of the estimation error e in SFOEM1.

Theorem 3. *Consider SFOEM1 (13), (14) specified by (15), (17), (18) and (19). Then, θ remains bounded and $e \in \mathcal{L}^2$. If in addition ω is assumed a bounded function, then the error e is bounded. If $\dot{\omega}$ is also assumed bounded, then the error e converges asymptotically to zero, i.e., $\lim_{t\to\infty} e(t) = 0$. Moreover, if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty} \phi(t) = 0$, i.e., $\lim_{t\to\infty} \theta(t) = \theta^*$.*

Proof of Theorem 3. The proof relies on proving several claims as in the follows.

i. *There exists $T_i \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that the existence and uniqueness of continuous solutions holds on $[t_i, t_i + T_i)$, where t_i is any switching time.*

Proof of claim i. Since $\omega \in \mathcal{C}^1$ and γ is constant between switches times, the right hand side of (16) is locally Lipschitz continuous as a function of ϕ. Then, we can find small enough T_i such that the right hand side is Lipschitz continuous with respect to ϕ and continuous with respect to t when restricted to $\mathcal{C}[t_i, t_i + T]$. This enable us to apply Theorem 1 on the space $\mathcal{C}[t_i, t_i + T]$, by writing (16) in the form (7), and to conclude that there is a unique continuous solution ϕ on $[t_i, t_i + T_i)$. Since ω is also continuous, by (13) the estimation error e is also a unique continuous function on $[t_i, t_i + T_i)$. □

ii. *The fractional mode, i.e., $\alpha = \alpha_0 < 1$, can remain activated only on time intervals of finite lengths.*

Proof of claim ii. By part (i), we can consider that a solution to (16) exist, is unique and continuous at least at a small neighborhood $[t_i, t_i + T_i)$ after the switching time. Let us consider that the fractional mode is active in the interval $[t_i, t_i + T_i)$ with $T_i \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$. Note that, since (16) operates in the resetting mode, the fractional derivative must be started in each t_i. For notation convenience, we set $I_i^\alpha = I_{t_i}^\alpha$ and $D_i^\alpha = D_{t_i}^\alpha$.

Since the solution of (16) is continuous on $[t_i, t_i + T_i)$ and γ remains constant in each switching interval, we can applying Theorem 2 to the function $tr\left[\phi^T\phi\right]$ and Property 6. Then, by using (16), it follows that

$$\frac{1}{\gamma} D_i^{\alpha_0} tr\left[\phi^T\phi\right] \leq \frac{1}{\gamma} 2tr\left[\phi^T D_i^{\alpha_0}\phi\right] = -2tr\left[ee^T\right] \tag{20}$$

Applying Property 6 on (20), we obtain for any $t \in [t_i, t_i + T_i)$

$$\frac{1}{\gamma} tr\left[\phi^T(t)\phi(t)\right] - \frac{1}{\gamma} tr\left[\phi^T(t_i)\phi(t_i)\right] \leq -2I_i^{\alpha_0} tr[ee^T](t) \tag{21}$$

From (21), since right hand side is always lesser or equal than zero, then it can be concluded that $tr\left[\phi^T(t)\phi(t)\right] \leq tr\left[\phi^T(t_i)\phi(t_i)\right]$, $\forall t > t_i$. Note that this will hold also for those intervals in the integer order mode, because an expression similar to (21) can be established, just using the integer order integral in the right-hand side. By the resetting mode and the no jump condition in switching times on θ, for any time interval it holds that $tr\left[\phi^T(t)\phi(t)\right] \leq tr\left[\phi^T(0)\phi(0)\right]$. Since $\phi(0)$ is bounded, it

 doi:10.20944/preprints202409.0949.v1

follows that $\phi(t)$ is bounded and $I_i^{\alpha_0} tr[ee^T](t)$ is bounded by a number C depending only on the initial condition $\phi(0)$.

Now, we need to prove that there exists a time instant $\bar{t} < \infty, \bar{t} \geq t_i$ such that $\|e(\bar{t})\| = \epsilon$ so that the fractional mode is deactivated in finite time. If that would not be the case and $\|e(t)\| > \epsilon$ for all $t \geq t_i$, then $I_i^{\alpha_0} tr[ee^T](t) \to \infty$, contradicting the fact stated in the previous paragraph, namely, that $I_i^{\alpha_0} tr[ee^T](t) < C < \infty$ for any t. This proves claim (ii).
□

iii. *There is no finite escape time in each mode of operation.*

Proof of claim iii. To prove this claim, we will differentiate between the intervals in fractional mode and the intervals in the integer mode.

When fractional mode is active, then the following candidate Lyapunov function can be proposed

$$V = \frac{1}{\gamma} tr\left[\phi^T \phi\right] \tag{22}$$

By applying the α_0 fractional order derivative to (22), and using Theorem 2 and (16), it follows that

$$D_i^{\alpha_0} V = \frac{1}{\gamma} D_i^{\alpha_0} tr\left[\phi^T \phi\right] \leq \frac{1}{\gamma} 2tr\left[\phi^T D_i^{\alpha_0} \phi\right] = -2tr\left[ee^T\right] \tag{23}$$

From (23), it can be seen that $D_i^{\alpha_0} V \leq 0$ for all $t \in [t_i, t_i + 1]$, then using [32, Theorem 3], it can be concluded that ϕ remains bounded and thus there is no finite time escape in fractional mode.

The proof in the integer order mode is similar, using the same candidate Lyapunov function (22) and taking its first order derivative, leading to $\dot{V} = -2tr\left[ee^T\right] \leq 0$. This proves the claim. □

iv. *There exists a finite number of switches, after which the mode becomes integer. In particular, there is no Zeno solution.*

Proof of claim iv. It follows from (ii) that if the number of switches is finite, then the final mode is necessarily integer. Also, if number of switches is finite, then there is no Zeno solution. Thus, it remains to prove that the number of switches is finite.

To this aim, it is enough proving that $\mathcal{T}$ (19) is bounded by a constant that does not depend on i, because due to the choice of the hysteresis function h (18), this would imply that there exists large enough i such that $\mathcal{T}(t) \leq i\delta$ for any $t > 0$, making h false after finite switches and consequently the fractional mode would be no longer activated.

To prove that $\mathcal{T}$ is bounded, let us use (21), which holds for every time interval when the fractional order mode is active. Since in fractional mode $\|e\| > \epsilon$, then from (21) we can state

$$tr\left[\phi^T(t)\phi(t)\right] - tr\left[\phi^T(t_i)\phi(t_i)\right] \leq -2\gamma I_i^{\alpha_0} \epsilon^2 = \frac{-2\gamma\epsilon^2}{\alpha_0 \Gamma(\alpha_0)} (t - t_i)^{\alpha_0}. \tag{24}$$

Since $\mathcal{T}$ is the time the fractional mode remains active and it was proved in (ii) that for every time interval, it holds that $tr\left[\phi^T(t)\phi(t)\right] \leq tr\left[\phi^T(0)\phi(0)\right]$, then it can be stated from (24) that

$$\mathcal{T}^{\alpha_0} \leq \frac{\alpha_0 \Gamma(\alpha_0)}{2\gamma_I \epsilon^2} tr\left[\phi^T(0)\phi(0)\right]. \tag{25}$$

Inequality (25) establishes an upper bound for $\mathcal{T}$ that does not depend on the number of switches i, and the the claim follows. □

v. *Statement of Theorem 3 holds.*

Proof of claim v. From claims (iii) and (iv), and since ϕ remains bounded by a bound independent of i, it is enough to prove the statement for the integer mode. As in the proof of item (ii), with V as in

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 September 2024 doi:10.20944/preprints202409.0949.v1

(22), one obtains $\dot{V} = -2tr\left[ee^T\right] \leq 0$ for the integer mode. This implies $\phi \in \mathcal{L}_\infty$ and, by integrating this inequality, $e \in \mathcal{L}_2$ since $tr\left[ee^T\right] = |e|^2$. If $\omega \in \mathcal{L}_\infty$, then $e \in \mathcal{L}_\infty$ and, hence, $\dot{\phi} \in \mathcal{L}_\infty$. Therefore, if in addition $\dot{\omega} \in \mathcal{L}_\infty$, then $\dot{e} \in \mathcal{L}_\infty$. Applying Barbalat Lemma, we get $\lim_{t\to\infty} e(t) = 0$. On the other hand, according to (16), the vector obtained from the $i-$th column of ϕ, denoted as $\phi^{(i)}$, satisfies for $i = 1, \ldots, n$

$$\dot{\phi}^{(i)} = \gamma\omega\omega^T\phi^{(i)}.$$

So, by [2, Theorem 2.16] and [2, Comment 2.10], if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty} \phi(t) = 0$. □

This completes the proof of Theorem 3. □

4. Analysis of Switched Fractional Order Error Model 2

Switched Fractional Order Error Model 2 (SFOEM2) arises when the whole state vector of the system to be controlled/identified is accessible and the adaptive laws use a switched approach. In contrast to SFOEM1, where the relationship between $\phi^T\omega$ and e is algebraic, in SFOEM2 it is dynamic.

Structure of the SFOEM2 corresponds to

$$\dot{e}(t) = A_m e(t) + B_m\phi^T(t)\omega(t), \quad e(0) = e_0. \tag{26}$$

$$D_{a(t)}^{\alpha(t)}\phi(t) = -\gamma(t)B_m Pe(t)\omega^T, \quad \phi(0) = \phi_0 \tag{27}$$

$$\gamma(t) = \frac{\gamma_i}{1 + tr[\theta(t_i)\theta^T(t_i)]}, \quad \gamma(0) = \gamma_0 \tag{28}$$

$$\alpha(t) = \begin{cases} \alpha_0 & if \quad \|e(t)\| > \epsilon \quad \text{and} \quad h(t) \\ 1 & if \quad\quad \text{othewise} \end{cases} \tag{29}$$

$$h(t) = \begin{cases} \text{False} & if \quad \left(i > C, \quad \|e(t)\| < \epsilon + \delta \quad \text{and} \quad \mathcal{T}(t) \leq i \cdot \delta \right) \\ \text{True} & if \quad\quad \text{otherwise} \end{cases} \tag{30}$$

$$\mathcal{T}(t) := \max_j\{|t_{j+1} - t_j| : t_{j+1} \leq t \quad \& \quad \alpha(t_j) = \alpha_0\} \tag{31}$$

In Equation (26) and (27), $e \in \mathbb{R}^n$ is the tracking/estimation error, assumed to be accessible. $A_m \in \mathbb{R}^{n\times n}$ is a known Hurwitz matrix, e.g., a square matrix whose eigenvalues have negative real parts. $B_m \in \mathbb{R}^{n\times q}$ is a constant matrix, either positive or negative definite, the sign of which is assumed known and, without loss of generality, taken positive. $\phi \in \mathbb{R}^{m\times q}$ is the parameter error, given by $\phi(t) = \theta(t) - \theta^*$, with $\theta^* \in \mathbb{R}^{m\times q}$ the ideal (true) parameter, which is assumed to be unknown and $\theta \in \mathbb{R}^{m\times q}$ the adjustable parameter that estimates θ^*. $\omega \in \mathbb{R}^m$ is a measurable $\mathcal{C}^1$ function. $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix such that $A_m^T P + PA_m = -Q$, with $Q \in \mathbb{R}^{n\times n}$ an arbitrary positive definite matrix. The existence of such P is ensured by the Hurwitz property of A_m. The structure and parameters of (28), (29), (30) and (31) are the same as explained in SFOEM1 and thus not repeated here.

As in the case of SFOEM1, the differential Equation (27) is understood in the resetting mode; that is, every time a switch occurs, the initial time a of the fractional operator is set equal to the switching time t_i. Also, every time a switch occurs, $\phi(t_i) = \phi(t_i^-)$ to avoid discontinuities.

4.1. Boundedness of the Signals and Convergence of the Estimation Error in SFOEM2

In [28], a particular adaptive control problem was addressed, namely a Switched Fractional Order Model Reference Adaptive Controller for Multiple Input Multiple Output systems, where the plant and the reference model were described by integer order differential equations and the order of the adaptive laws switched between a fractional order and the integer order. The closed loop description of the adaptive system in [28] is identical to the structure of SFOEM2 (26)-(31), but the claim below is more general than [28, Theorem 3], particularly, in providing conditions for parametric convergence.

Theorem 4. *Consider the Switched Fractional Order Error Model 2 defined by (26), (27), with (28), (29), (30) and (31). Then, θ and e remain bounded and $e \in \mathcal{L}^2$. If in addition ω is assumed bounded function, then the error e asymptotically converges to zero, i.e., $\lim_{t\to\infty} e(t) = 0$, and if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty} \phi(t) = 0$, i.e., $\lim_{t\to\infty} \theta(t) = \theta^*$.*

Proof of Theorem 4. The reader is referred to the proof of Theorem 5, as SFOEM2 is a particular case of SFOEM3 by taking $C_m = I$ in the latter. □

5. Analysis of Switched Fractional Order Error Model 3

Switched Fractional Order Error Model 3 (SFOEM3) has a very similar structure than SFOEM2. However, there is a fundamental difference between them, because while the whole estimation/tracking error e is accessible in SFOEM2, only an algebraic combination of the components of e, namely e_o, is accessible in SFOEM3. This error model usually arises when only the output of the plant to be controlled or identified is accessible, rather than its whole state vector. This makes SFOEM3 applicable to a much wider class of problems than SFOEM2.

Structure of SFOEM3 has the following form

$$\begin{aligned} \dot{e}(t) &= A_m e(t) + B_m \phi^T(t)\omega(t), \quad e(0) = e_0. \\ e_o(t) &= C_m\, e(t) \end{aligned} \tag{32}$$

$$D^{\alpha(t)}_{a(t)} \phi(t) = -\gamma(t)\omega(t)e_o^T(t), \quad \phi(0) = \phi_0 \tag{33}$$

$$\gamma(t) = \frac{\gamma_i}{1 + tr[\theta(t_i)\theta^T(t_i)]}, \quad \gamma(0) = \gamma_0 \tag{34}$$

$$\alpha(t) = \begin{cases} \alpha_0 \quad if \quad \|e_o(t)\| > \epsilon \quad \text{and} \quad h(t) \\ 1 \quad if \quad \text{othewise} \end{cases} \tag{35}$$

$$h(t) = \begin{cases} \text{False} \quad if \quad \left(i > C, \quad \|e_o(t)\| < \epsilon + \delta \quad \text{and} \quad \mathcal{T}(t) \le i \cdot \delta \right) \\ \text{True} \quad if \quad \text{otherwise} \end{cases} \tag{36}$$

$$\mathcal{T}(t) := \max_j\{|t_{j+1} - t_j| : t_{j+1} \le t \quad \& \quad \alpha(t_j) = \alpha_0\} \tag{37}$$

In Equations (32) and (33), $e \in \mathbb{R}^n$ is the tracking/estimation error, which is not accessible in this case, while $e_o \in \mathbb{R}^q$ is the output error, which is accessible. $A_m \in \mathbb{R}^{n\times n}$ is a known Hurwitz matrix, $C_m \in \mathbb{R}^{q\times n}$ is a constant matrix, $B_m \in \mathbb{R}^{n\times q}$ is a constant matrix, either positive or negative definite, the sign of which is assumed known and, without loss of generality, taken positive. $\phi \in \mathbb{R}^{m\times q}$ is the parameter error, e.g., $\phi(t) = \theta(t) - \theta^*$, with $\theta^* \in \mathbb{R}^{m\times q}$ the unknown parameter and $\theta \in \mathbb{R}^{m\times q}$ the adjustable parameter. $\omega \in \mathbb{R}^m$ is the measurable $\mathcal{C}^1$ regressor function. On the other hand, $\gamma \in \mathbb{R}^+$ is an adaptive gain. The structure and parameters of (34), (35), (36) and (37) are the same as explained in SFOEM1 and thus not repeated here.

As stated in [2], it can be expected that having no access to e will imply that more stringent conditions must be imposed on the transfer function between $\phi^T(t)\omega(t)$ and $e_o(t)$. In that sense, it is assumed that positive definite matrices $P = P^T \in \mathbb{R}^{n\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$ [2] exist such that

$$\begin{aligned} A_m^T P + P A_m &= -Q \\ P B_m &= C_m^T \end{aligned} \tag{38}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 13 September 2024 doi:10.20944/preprints202409.0949.v1

which is equivalent to ask that the transfer function $C_m(sI - A_m)^{-1}B_m$ is Strictly Positive Real.

5.1. Boundedness of the Signals and Convergence of the Estimation Error in SFOEM3

The following result states boundedness of the signals and convergence of the tracking/estimation error e and output error e_o in SFOEM3.

Theorem 5. *Consider the Switched Fractional Order Error Model 3 defined by (32), (33), with (34), (35), (36) and (37). Then, θ, e_o and e remain bounded and $e_o, e \in \mathcal{L}^2$. If in addition ω is assumed bounded function, then the error e asymptotically converges to zero, i.e., $\lim_{t\to\infty} e(t) = 0$, and if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty} \phi(t) = 0$, i.e., $\lim_{t\to\infty} \theta(t) = \theta^*$.*

Proof of Theorem 5. As in the case of SFOEM1, the proof consists of several claims, as it is detailed in the following.

i. *There exists $T_i \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that the existence and uniqueness of continuous solutions holds on $[t_i, t_i + T_i)$, where t_i is any switching time.*

Proof of claim i. Since ω is $\mathcal{C}^1$, the right hand side of (32), (33) is locally Lipschitz continuous as a function of e, ϕ and we can find small enough T_i such that the right hand side is Lipschitz continuous with respect to e, ϕ and continuous with respect to t. Then, Theorem 1 can be applied to (32) and (33), allowing to write them in the form (7) and concluding that e, ϕ are unique and continuous in $[t_i, t_i + T]$. □

ii. *The fractional mode, i.e., $\alpha = \alpha_0 < 1$, can only be activated on time intervals of finite lengths.*

Proof of claim ii. As in the proof of claim ii for SFOEM1, let us analyze what happens in those intervals when the fractional mode is active, e.g., $[t_i, t_i + T_i)$ with $T_i \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$. Since the solution of (32) and (33) was proven to be continuous, the right-hand of (32) and (33) are continuous too, and we can take the first order derivative of $e^T P e$ and use (32) together with trace property $tr\left[yx^T\right] = x^T y$ to establish that

$$\frac{d}{dt}e^T P e = 2e^T P\dot{e} = e^T(A_m^T P + PA_m)e + 2e_o^T\phi^T\omega = -|e|_Q + 2tr\left[\phi^T\omega e_o^T\right] \tag{39}$$

In the same way, we can take the α_0 derivative of $tr\left[\phi^T\phi\right]$ and use Theorem 2 together with (33) to establish that

$$\frac{1}{\gamma}D_i^{\alpha_0} tr\left[\phi^T\phi\right] \leq \frac{1}{\gamma}2tr\left[\phi^T D_i^{\alpha_0}\phi\right] = -2tr\left[\phi^T\omega e_o^T\right] \tag{40}$$

Applying the first order integral to (39) and the α_0-integral to (40) and using Properties 5 and 6, we obtain for any $t \in [t_i, t_i + T_i)$

$$I_i^{1-\alpha_0} I_i^{\alpha_0}|e|_Q = |e(t_i)|_P - |e(t)|_P + I_i^{1-\alpha_0} I_i^{\alpha_0} 2tr\left[\phi^T\omega e_o^T\right] \tag{41}$$

$$I_i^{\alpha_0} 2tr\left[\phi^T\omega e_0^T\right] \leq \frac{1}{\gamma}tr[|\phi(t_i)|] - \frac{1}{\gamma}tr[|\phi(t)|] \tag{42}$$

Combining (41) and (42), the following inequality can be established

$$I_i^{1-\alpha_0}\left(I_i^{\alpha_0}|e|_Q - \frac{1}{\gamma}tr[|\phi(t_i)|]\right) \leq |e(t_i)|_P \tag{43}$$

Since $e_o = C_m e$, we claim that inequality (43) implies the existence of $\bar{t} < \infty$, $\bar{t} \geq t_i$ such that $\|e_o(\bar{t})\| = \epsilon$. By contradiction and recalling that we are in the fractional mode, if $\|e_o(t)\| > \epsilon$ for all $t \geq t_i$, then there exists $\bar{\epsilon} > 0$ such that $\|e(t)\| > \bar{\epsilon}$ for all $t \geq t_i$, since $\|e_o(t)\| \leq \|C_m\|\|e(t)\|$ by using the induced matrix norm for C_m. This implies that the left-hand side of (43) goes to $+\infty$ because the integrand of $I_i^{1-\alpha_0}$ goes to $+\infty$. This contradicts inequality (43). Therefore, the existence of $\bar{t} < \infty$ such that $\|e_o(\bar{t})\| = \epsilon$ is guaranteed, meaning that the integer mode is triggered some finite time after the fractional mode started. This proves claim (ii). □

iii. *There is no finite escape time in each mode of operation.*

Proof of claim iii. In the integer order mode, the claim can be easily proved by constructing a Lyapunov function $V = e^T Pe + tr\left[\frac{1}{\gamma_1}\phi^T\phi\right]$ and using (39),(40) with $\alpha = 1$, leading to $\dot{V} \leq 0$. When the fractional order mode is active, by claim (ii), and after finite time, the switching condition $\|e_o(t_{i+1})\| = \epsilon$ must hold, implying e_o cannot escape at $[t_i, t_{i+1}]$. Then $e_o \in \mathcal{C}[t_i, t_{i+1}]$, and since $\omega \in \mathcal{C}^1$, the right-hand side of (33) is bounded at $[t_i, t_{i+1}]$. This implies, by $\alpha-$integration, that ϕ is bounded $[t_i, t_{i+1}]$, i.e., ϕ does not escape. Then, the forcing function in (39) is bounded, and since A_m is stable, e remains bounded at $[t_i, t_{i+1}]$ and the claim is proved. □

iv. *There exists a finite number of switches, after which the mode becomes integer. In particular, there is no Zeno solution.*

Proof of claim iv. According to claim (ii), if the number of switches is finite, then the final mode is necessarily integer and there is no Zeno solution. Then, to prove claim (iv) it is enough to prove that the number of switches is finite. For this, as in the case of SFOEM1, it is enough to show that $\mathcal{T}$ given by (37) is bounded by a constant that does not depend on i.

As in the proof of claim (ii), in fractional mode, the condition $\|e_o\| \geq \epsilon$, the equivalence of norms in $\mathbb{R}^n$, and $e_o = C_m e$ imply the existence of a constant C_1, independent of t_i, such that $|e|_Q \geq C_1$ for all t when fractional order mode is active. By using the normalization factor in (34) (recall that $\phi(t) = \theta(t) - \theta^*$) and the equivalence of norms in finite dimension spaces, there exists a constant C_2, independent of t_i, such that $\frac{1}{\gamma}tr[|\phi(t_i)|] \leq C_2$. Also, due to the choice of the hysteresis function h and the equivalence of norms, there exists constant $C_3 > 0$, independent of t_i, such that $|e(t_i)| \leq C_3(\epsilon + \delta)$. Using these constants C_1, C_2, C_3 to bound (43) and solving the resulting integrals, it can be written that, for all t when the fractional order mode is active, it holds that

$$C_1(t - t_i) - \frac{C_2}{(1-\alpha_0)\Gamma(1-\alpha_0)}(t - t_i)^{1-\alpha_0} \leq C_3(\epsilon + \delta). \tag{44}$$

Inequality (44) establishes an upper bound for $\mathcal{T}$ (lenght of the time interval $t - t_i$ when the fractional order mode is active) that does not depend on i, and the the claim follows. □

v. *Statement of Theorem 5 holds.*

Proof of claim v. From claims (iii) and (iv), it is enough to prove the statement for the integer mode. By considering positive definite function V as defined in the proof of claim (iii), we get

$$\dot{V} = -|e|_Q. \tag{45}$$

This implies $\phi, e, e_o, \theta \in \mathcal{L}_\infty$ and $e, e_o \in \mathcal{L}_1$. By (32), if $\omega \in \mathcal{L}_\infty$, then we also have $\dot{e} \in \mathcal{L}_\infty$. By Barbalat Lemma, we conclude $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} e_o(t) = 0$. By (33), we also have $\lim_{t\to\infty} \dot{\phi}(t) = 0$. Hence, for any $T > 0$, $\phi(t+T) - \phi(t) = \int_t^{t+T} \dot{\phi} d\tau$ goes to zero as $t \to \infty$. This implies that in

$$\int_t^{t+T} |(\phi(\tau) - \phi(t))^T \omega(\tau)|\, d\tau \geq \int_t^{t+T} |\phi(t)^T \omega(\tau)| d\tau - \int_t^{t+T} |\phi(\tau)^T \omega(\tau)| d\tau, \tag{46}$$

the left-hand side goes to zero when $\omega \in \mathcal{L}_\infty$ and $t \to \infty$. By assuming in addition that $\omega \in \mathrm{PE}(m)$, which means that there exists $T_0, \epsilon_1 > 0$ such that $|\int_t^{t+T_0} u^T\omega(\tau)d\tau| > \epsilon_1|u|$ for any $t \geq 0$ and $u \in \mathbb{R}^m - \{0\}$, we can find $T_0, \epsilon_2 > 0$ such that $|\int_t^{t+T_0} M^T\omega(\tau)d\tau| > \epsilon_2|M|$ for any $t \geq 0$ and

$M \in \mathbb{R}^{m\times m} - \{0\}$, simply by considering each component of vector $M^T\omega$ as the product of a column of M with ω. Therefore, if $\omega \in \mathrm{PE}(m)$, we obtain from (46), for t large enough and $\epsilon_0 > 0$

$$\int_t^{t+T_0} |\phi(\tau)^T\omega(\tau)|d\tau \geq \epsilon_0|\phi(t)|. \tag{47}$$

On the other hand, integrating (32) on $[t, t+T]$ for arbitrary $T > 0$, we have

$$e(t+T) = e(t) + \int_t^{t+T} A_m e\, d\tau + \int_t^{t+T} B_m\phi^T\omega d\,\tau. \tag{48}$$

By sending $t \to \infty$, and since e goes to zero, it follows from (48) that

$$\int_t^{t+T} \phi^T\omega d\,\tau \to 0. \tag{49}$$

However, if $\omega \in \mathrm{PE}(m)$, a contradiction is obtained with (47) whenever ϕ does not converges to zero. Therefore, if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty}\phi(t) = 0$ and the claim is proved. □
□

Remark 1. *In practical applications of Theorems 4 and 5, the bounded assumption on ω can be relaxed to ω being a bounded function of e and/or θ, in the sense that boundedness of e and/or θ imply bounded ω. For instance, in the adaptive control setup of [28], it holds $\omega = e + \eta$, with η a bounded signal. This is because the boundedness of e and θ was established regardless of the boundedness of ω in Theorem 5, and hence, if e and/or ϕ bounded imply ω bounded, then Theorem 5 guarantees convergence of e to zero.*

6. Analysis of Switched Fractional Order Error Model 4

SFOEM4 arises when neither vector $e(t)$ is accessible nor condition (38) holds. In such situation, for the particular case when the parameter error ϕ is a vector ($q = 1$) rather than a matrix, the solution consists on introducing an additional signal e_a in the adaptive scheme, as detailed below.

Structure of SFOEM4 is the following

$$e_o(t) = W(s)\phi^T(t)\omega(t) \tag{50}$$

$$e_a(t) = \theta^T(t)W(s)I_m\omega(t) - W(s)\theta^T(t)\omega(t) \tag{51}$$

$$\varepsilon(t) = e_o(t) + e_a(t) \tag{52}$$

$$D_{a(t)}^{\alpha(t)}\phi(t) = -\gamma(t)\,(W(s)I_m\omega(t))\varepsilon(t) \tag{53}$$

$$\gamma(t) = \frac{\gamma_i}{1+\theta^T(t_i)\theta(t_i)}, \quad \gamma(0) = \gamma_0 \tag{54}$$

$$\alpha(t) = \begin{cases} \alpha_0 \quad if \quad |\varepsilon(t)| > \epsilon \quad \text{and} \quad h(t) \\ 1 \quad if \qquad \text{othewise} \end{cases} \tag{55}$$

$$h(t) = \begin{cases} \text{False} \quad if \quad \left(i > C, \quad \|\varepsilon(t)| < \epsilon + \delta \quad \text{and} \quad \mathcal{T}(t) \leq i\cdot\delta \right) \\ \text{True} \quad if \qquad \text{otherwise} \end{cases} \tag{56}$$

$$\mathcal{T}(t) := \max_j\{|t_{j+1} - t_j| : t_{j+1} \leq t \quad \& \quad \alpha(t_j) = \alpha_0\} \tag{57}$$

In Equation (50) to (53), variable $e_o \in \mathbb{R}$ is the output error, $e_a(t) \in \mathbb{R}$ is an auxiliary error, $\varepsilon(t) \in \mathbb{R}$ is the augmented error, $W(s)$ is the transfer function between $\phi^T\omega$ and e_o, an exponentially stable time-invariant linear filter which operates component-wise when applied to a vector or a matrix.

$\phi \in \mathbb{R}^m$ is the parameter error, e.g., $\phi(t) = \theta(t) - \theta^*$, with $\theta^* \in \mathbb{R}^m$ the unknown parameter and $\theta \in \mathbb{R}^m$ the adjustable parameter. On the other hand $\omega \in \mathbb{R}^m$ is a measurable $\mathcal{C}^1$ function, I_m is the identity matrix of order m and $\gamma \in \mathbb{R}^+$ is an adaptive gain. The structure and parameters of (54), (55), (56) and (57) are the same as explained in previous sections.

6.1. Boundedness of the Signals and Convergence of the Estimation Error in SFOEM4

The following result states boundedness of the signals and convergence of the output error e_o in SFOEM4.

Theorem 6. *Consider the Switched Fractional Order Error Model 4 defined by (50), (51), (52), (53), with (54), (55), (56) and (57). Then, θ remains bounded. If in addition ω is a bounded function, then the error e is bounded and if $\dot{\omega}$ is also assumed bounded then e converges asymptotically to zero, i.e., $\lim_{t\to\infty} e(t) = 0$. Moreover, if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty} \phi(t) = 0$, i.e., $\lim_{t\to\infty} \theta(t) = \theta^*$.*

Proof of Theorem 6. If we recall that $\phi(t) = \theta(t) - \theta^*$, Equation (51) can be written as

$$\begin{aligned} e_a(t) &= \theta^T(t)W(s)I_m\omega(t) + {\theta^*}^T W(s)I_m\omega(t) - {\theta^*}^T W(s)I_m\omega(t) - W(s)\theta^T(t)\omega(t) \\ &\quad + W(s){\theta^*}^T\omega(t) - W(s){\theta^*}^T\omega(t) \\ &= \phi^T(t)W(s)I_m\omega(t) - W(s)\phi^T(t)\omega(t) + {\theta^*}^T W(s)I_m\omega(t) - W(s){\theta^*}^T\omega(t) \end{aligned} \tag{58}$$

Since θ^* is a constant vector and W is an exponentially stable time-invariant linear filter, the term $\eta(t) = {\theta^*}^T W(s)I_m\omega(t) - W(s){\theta^*}^T\omega(t)$ is exponentially decaying. Substituting (58),(50) in (52) and using the notation $\xi = W(s)I_m\omega(t)$ in both (52) and(53), it follows that

$$\varepsilon(t) = \phi^T(t)\xi(t) + \eta(t) \tag{59}$$

$$D_{a(t)}^{\alpha(t)}\phi(t) = -\gamma(t)\,\varepsilon(t)\xi(t) \tag{60}$$

As can be observed, Equations (59) and (60) has the same structure than SFOEM1 studied in Section 3.1, for the case when ϕ is a vector rather than a matrix, but for the term η. We show that the items of the proof of Theorem 3 still hold in this case with the following key additional elements. Since W is a time-invariant linear filter, it maps $\mathcal{C}^1$ to $\mathcal{C}^1$ so that claim (i) holds. Items (ii), (iii) and (iv) follow by using [35, Theorem 1], recalling that the $\alpha-$fractional integral of an exponentially decaying function like η is bounded and converges to zero when $\alpha < 1$. Finally, item (v) follows by applying the result in [2, p. 282]. In this way, we obtain that $\phi \in \mathcal{L}_\infty$. Moreover, if $\omega \in \mathcal{L}_\infty$ then $e_o \in \mathcal{L}_\infty$, if $\dot{\omega} \in \mathcal{L}_\infty$ then $\lim_{t\to\infty} e_o(t) = 0$, and if $\omega \in \mathrm{PE}(m)$, then $\lim_{t\to\infty} \phi(t) = 0$. To get these conclusions, note that boundedness of $\omega, \dot{\omega}$ implies boundedness of $\xi, \dot{\xi}$, and that boundedness and convergence to zero of ε implies boundedness and convergence to zero of e_o, and consequently of e. □

Remark 2. *To address the case when ϕ is a matrix ($q > 1$) rather than a vector, some difficulties arise and the complexity of the resulting adaptive scheme increases dramatically. This case is currently under research and will be addressed in a future work. Still, the particular case when $\phi \in \mathbb{R}^{m\times q}$ and the transfer matrix $W(s) \in \mathbb{R}^{q\times q}$ between $\phi^T\omega$ and $e_0 \in \mathbb{R}^q$ is diagonal, can be solved in a similar way to that stated above. Specifically, the problem can be decomposed as follows*

$$e_{o,i}(t) = W_i(s)\phi_i^T(t)\omega(t) \tag{61}$$

$$e_{a,i}(t) = \theta_i^T(t)W_i(s)I_m\omega(t) - W_i(s)\theta_i^T(t)\omega(t) \tag{62}$$

$$\varepsilon_i(t) = e_{o,i}(t) + e_{a,i}(t) \tag{63}$$

where $W_i \in \mathbb{R}$ are the diagonal elements of $W(s)$ and ϕ_i is a column vector of matrix ϕ. The proof follows as in the case of Theorem 6.

7. Conclusions

In this paper, the analysis of four switched fractional order error models has been presented, allowing to prove boundedness of the solutions and convergence to zero of the estimation/tracking error in every adaptive scheme that can be put in one of these forms. The results also states sufficient conditions to achieve parameter convergence in these schemes.

In contrast to previous works, the four switched error models are analyzed considering the parameter error ϕ as a matrix (multi variable case) for three out of four error models. It was found that the same excitation condition for the vector case is sufficient to estimate the matrix parameters.

Author Contributions: "Conceptualization, N.A-C. and J.G.; methodology, N.A-C. and J.G.; formal analysis, N.A-C. an J.G.; investigation, N.A-C., J.G., J.C.T-T and Y.Q.Ch; resources, N.A-C. and Y.Q.Ch.; writing—original draft preparation, N.A-C. and J.G.; writing—review and editing, N.A-C., J.G., J.C.T-T and Y.Q.Ch; visualization, N.A-C.; supervision, N.A-C.; project administration, N.A-C.; funding acquisition, N.A-C, J.G., J.C.T-T and Y.Q.Ch. All authors have read and agreed to the published version of the manuscript."

Funding: This research was funded by ANID CHILE under grant numbers Fondecyt 1220168, Fondecyt 3230674, FONDEF IT23I0117 and Basal Project AFB230001.

Conflicts of Interest: "The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results".

References

1. Gaudio, J.E.; Gibson, T.E.; Annaswamy, A.M.; Bolender, M.A.; Lavretsky, E. Connections Between Adaptive Control and Optimization in Machine Learning, 2019, [1904.05856].
2. Narendra, K.; Annaswamy, A. *Stable Adaptive Systems*; Dover Publications: Mineola, N.Y., USA, 2005.
3. Åström, K.J.; Wittenmark, B. *Adaptive Control*; Dover Publications Inc.: NY, USA, 2008.
4. Sastry, K.; Bodson, M. *Adaptive Control: Stability, Convergence and Robustness*; Prentice Hall: NY, USA, 1989.
5. Duarte-Mermoud, M.A.; Castro-Linares, R.; Facuse-Castillo, A. Adaptive Passivity of Nonlinear Systems Using Time-Varying Gains. *Dynamics and Control* **2001**, *11*, 333–351. doi:10.1023/A:1020862904420.
6. Annaswamy, A.M.; Fradkov, A.L. A historical perspective of adaptive control and learning. *Annual Reviews in Control* **2021**, *52*, 18–41. doi:10.1016/j.arcontrol.2021.10.014.
7. Gaudio, J.; Annaswamy, A.; Bolender, M.; Lavretsky, E.; Gibson, T. A Class of High Order Tuners for Adaptive Systems. *IEEE Control Systems Letters* **2021**, *5*, 391–3961. doi:10.1109/LCSYS.2020.3002513.
8. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations, Volume 204 (North-Holland Mathematics Studies)*; Elsevier Science Inc.: USA, 2006.
9. Shastri, S.V.; Narendra, K.S. Fractional Order Derivatives in Systems Theory. American Control Conference, Denver, CO, USA, July 1-3, 2020, pp. 5076–5081.
10. Kaur, A.; Takhar, P.; Smith, D.; Mann, J.; Brashears, M. Fractional differential equations based modeling of microbial survival and growth curves: model development and experimental validation. *Journal of Food Science* **2008**, *73*, 403–414. doi:10.1111/j.1750-3841.
11. Ozaktas, H.M.; Zalevsky, Z.; Kutay, M.A. *The Fractional Fourier Transform: with Applications in Optics and Signal Processing*; John Wiley & Sons: Chichester, UK, 2001.
12. Assaleh, K.; Ahmad, W.M. Modeling of speech signals using fractional calculus. 9th International Symposium on Signal Processing and Its Applications, Sharjah, United Arab Emirates, 2007, pp. 1–4.
13. Dai, Z.; Peng, Y.; Mansy, H.; Sandler, R.; Royston, T. A model of lung parenchyma stress relaxation using fractional viscoelasticity. *Medical Engineering & Physics* **2015**, *37*, 752–758. doi:10.1016/j.medengphy.2015.05.003.
14. Arafa, A.A.M.; Rida, S.Z.; Khalil, M. A fractional-order model of HIV infection: Numerical solution and comparisons with data of patients. *International Journal of Biomathematics* **2014**, *07*. doi:10.1142/S1793524514500363.
15. Tarasov, V.E.; Tarasova, V.V. Long and Short Memory in Economics: Fractional-Order Difference and Differentiation, 2017, [1612.07903].
16. Dastjerdi, A.A.; Vinagre, B.M.; Chen, Y.; HosseinNia, S.H. Linear fractional order controllers; A survey in the frequency domain. *Annual Reviews in Control* **2019**, *47*, 51–70. doi:10.1016/j.arcontrol.2019.03.008.
17. Cajo, R.; Mac, T.T.; Plaza, D.; Copot, C.; De Keyser, R.; Ionescu, C. A Survey on Fractional Order Control Techniques for Unmanned Aerial and Ground Vehicles. *IEEE Access* **2019**, *7*, 66864–66878. doi:10.1109/ACCESS.2019.2918578.

18. de Almeida, A.M.; Lenzi, M.K.; Lenzi, E.K. A Survey of Fractional Order Calculus Applications of Multiple-Input, Multiple-Output (MIMO) Process Control. *Fractal and Fractional* **2020**, *4*. doi:10.3390/fractalfract4020022.
19. Vinagre, B.; Petrás, I.; Podlubny, I.; Chen, Y. Using fractional order adjustment rules and fractional order reference models in model-reference adaptive control. *Nonlinear Dynamics* **2002**, *29*, 269–279. doi:10.1023/A:1016504620249.
20. Tejado, I.; HosseinNia, S.; Vinagre, B. Adaptive gain-order fractional control for network-based applications. *Journal of Fractional Calculus and Applied Analysis* **2014**, *17*, 462–482. doi:10.2478/s13540-014-0180-6.
21. Aguila-Camacho, N.; Duarte-mermoud, M.A. Fractional Adaptive Control for an Automatic Voltage Regulator. *ISA Transactions* **2013**, *52*, 807–815. doi:10.1016/j.isatra.2013.06.005.
22. Aguila-Camacho, N.; Duarte-mermoud, M.A. Improving the control energy in model reference adaptive controllers using fractional adaptive laws. *IEEE/CAA Journal of Automatica Sinica* **2016**, *3*, 332–337. doi:10.1109/JAS.2016.7508809.
23. Aguila-Camacho, N.; Duarte-mermoud, M.A. Boundedness of the solutions for certain classes of fractional differential equations with application to adaptive systems. *ISA Transactions* **2016**, *60*, 82–88. doi:10.1016/j.isatra.2015.11.013.
24. Duarte-mermoud, M.A.; Aguila-Camacho, N. Fractional adaptive systems in the presence of bounded disturbances and parameter variations. *International Journal of Adaptive Control and Signal Processing* **2017**, *31*, 1273–1284. doi:10.1002/acs.2763.
25. Aguila-Camacho, N.; Gallegos, J.; Duarte-mermoud, M.A. Analysis of fractional order error models in adaptive systems: Mixed order cases. *Journal of Fractional Calculus and Applied Analysis* **2019**, *22*, 1113–1132. doi:10.1515/fca-2019-0058.
26. Gallegos, J.A.; Aguila-Camacho, N.; Duarte-Mermoud, M.A.; Travieso-Torres, J.C.; Ceballos-Benavides, G.E. Switched systems with changing derivation order: Stability and applications. *Journal of the Franklin Institute* **2021**, *358*, 3943–3963. doi:10.1016/j.jfranklin.2021.02.033.
27. Gallegos, J.; Aguila-Camacho, N.; Núñez, F. High-Gain Adaptive Control With Switching Derivation Order and Its Application to a Class of Multiagent Systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems* **2024**, *Early Access*, 1–12. doi:10.1109/TSMC.2024.3376698.
28. Aguila-Camacho, N.; Gallegos, J.A. Error-Based Switched Fractional Order Model Reference Adaptive Control for MIMO Linear Time Invariant Systems. *Fractal and Fractional* **2024**, *8*. doi:10.3390/fractalfract8020109.
29. Ortega, R.; Aranovskiy, S.; Pyrkin, A.A.; Astolfi, A.; Bobtsov, A.A. New Results on Parameter Estimation via Dynamic Regressor Extension and Mixing: Continuous and Discrete-Time Cases. *IEEE Transactions on Automatic Control* **2021**, *66*, 2265–2272. doi:10.1109/TAC.2020.3003651.
30. Diethelm, K. *The analysis of fractional differential equations: An application-oriented exposition using operators of Caputo type*; Springer: Berlin, Germany, 2004.
31. Gallegos, J.; Aguila-Camacho, N.; Duarte-Mermoud, M. Smooth Solutions to mixed-order fractional differential systems with applications to stability analysis. *Journal of Integral Equations and Applications* **2019**, *31*, 59–84. doi:10.1216/JIE-2019-31-1-59.
32. Tuan, H.; Trinh, H. Stability of fractional-order nonlinear systems by Lyapunov direct method. *IET Control Theory and Applications* **2018**, *12*, 2417–2422. doi:10.1049/iet-cta.2018.5233.
33. Lavretsky, E. Combined/Composite Model Reference Adaptive Control. *IEEE Transactions on Automatic Control* **2009**, *54*, 2692–2697. doi:10.1109/TAC.2009.2031580.
34. Annaswamy, A.M. Adaptive Control and Intersections with Reinforcement Learning. *Annual Review of Control, Robotics, and Autonomous Systems* **2023**, *6*, 65–93. doi:10.1146/annurev-control-062922-090153.
35. J. A. Gallegos, M. A. Duarte-Mermoud, R.C.L. Mixed order robust adaptive control for general linear time invariant systems. *Journal of the Franklin Institute* **2018**, *355*, 3399–3422. doi:https://doi: 10.1016/j.jfranklin.2018.02.022.