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A State-Space Analysis and an Optimal Control Synthesis for a Class of Linear Time-Optimal Control Problems

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Abstract. The paper deals with two new properties of the linear time-optimal control problems with real non-positive eigenvalues of the system. They are the foundation of a new method of synthesizing the time-optimal control without the need to describe the switching hyper-surfaces. The so called "axes initialization" and the synthesis technique are illustrated on the already classical example of the time-optimal control of a double integrator.

Keywords: time-optimal control; minimum time control; Pontryagin's maximum principle; synthesis of optimal systems; linear systems; switching surface

MSC: 49N35; 93B50; 49N05; 93B52; 93C05

1. Introduction

In [1] a new property of the linear time-optimal control problem for the case of real non-positive eigenvalues of the system is derived. In Section 4 – Discussion [1] (p. 13) it is said that „Besides the property proved here, there is still a rigorous need to prove other properties of the problem in the case of its expansion." The necessary theoretical elements of the new method include the questions about the relationship between the coordinate axes in the state-space of the system of the considered Problem $\mathrm{P}(n)$ and its switching hyper-surface, as well as the synthesis of the optimal control based on the solution of the easier Problem $\mathrm{P}(n-1)$ generated by the original Problem $\mathrm{P}(n)$. The present work is a direct continuation of the previous work [1] and deals precisely with these necessary theoretical questions. It should be noted that in the period since the publication of the previous work [1], there are no other published studies on the linear time-optimal control problem developing the idea of the proposed method.

Works [2] – [5] consider time-optimal trajectory planning of systems in the presence of limitations or time-varying parameters. They apply Pontryagin's maximum principle with still other optimization techniques. In [2] (p. 506) it is stated that „In control engineering, effective implementations require to take into account the constraints for both the inputs and the outputs of the controlled system." Some of the studies [2] – [4] on time-optimal control of linear systems with constraints offer solutions where "the proposed idea is to discretize the continuous-time system and to solve the resulting discrete-time problem by means of linear programming." ([4], p. 2234) so that "By time discretization and linear programming, an approximation of the generalized bang-bang control can be computed." [2] (p. 511). In [6] (p. 2) the authors note: „We observe that for many systems, time-optimal trajectories can be represented as a concatenation of multiple trajectory segments. … Take a double integrator as an example. It is well-known that its optimal control policy is bang-bang with at most one switch". The direction of their research leads to „a polynomial-based quadrotor motion planner that can generate time-optimal trajectories for a wide range of scenarios" [6] (p. 17). In [4] (p. 2234) it is noted that „in the control engineering literature, it has been emphasized that to achieve high performances in real applications, due attention has to be paid to the constraints which all the plant variables must comply with." The development and the maturity reached in the theory of time-optimal control [7] – [15] as well as the contribution of the contemporary researchers [16] – [18], [2] – [6] are a direct confirmation of the long-standing efforts in this field. The research in this field is accompanied and enriched in the years of its development and with theoretical attempts to justify the existence of closed-loop time-optimal control [19], [20]. The difficulties in this topic are reason to note that "… there is still no complete time-optimal analytical solution for systems higher than second order" [18] (p. 1). The approach of the authors of [18] (pp. 1–2) „can be considered in two steps: (1) the time optimal control of multiple integrator with only input saturation; (2) the time optimal control of multiple integrator with input saturation and full state constraints". For their research „based on Bellman's principle of optimality, a series of switching surfaces and curves in phase space is generated using dynamic programming method" (p. 8).

The approach chosen in [1] and developed in the present work with regard to the classical linear time-optimal control problem of a controllable linear system with one input and real non-positive eigenvalues is based on the following main ideas:

Pontryagin's original solution [9] (Chapter 3, § 20, § 21, Example 3) with regard to the synthesizing function

$$u(x) = \begin{cases} +1 & in\ all\ areas \quad M_i^+, \\ -1 & in\ all\ areas \quad M_i^-, \end{cases}$$

where M_i^+ and M_i^- are the respective manifolds. The manifold M_i is of dimension $(n-i+1)$; M_{i+1} is entirely in M_i and divides it into two areas M_i^+ and M_i^-; M_i^+ consists of all the trajectories under the control $+1$ ending at a point of M_{i+1}^-, while M_i^- consists of all the trajectories under the control -1 ending at a point of M_{i+1}^+. The last manifold M_1 coincides with the whole state-space of the system;

New geometric properties of the original time-optimal control problem with respect to the system state-space;

Solution of the synthesis problem based on the new properties of the time-optimal control problem and carried out without the need to describe or generate the respective manifolds – the switching hyper-surfaces, but by solving the simpler lower-order problem generated by the original time-optimal control problem.

The current paper is organized in the following way. In Section 2 two new properties of the linear time-optimal control problem are theoretically represented. In Section 3 the author carries out the so called "axes initialization" of the time-optimal control problem of a double integrator as well as the synthesis of the optimal control and obtaining the optimal trajectory. Section 4 presents some concluding remarks on the obtained results.

2. Formulation of the Problem and Solution

Let us consider the linear time-optimal control problem of order n, $n \geq 2$, called Problem P(n) as well as its sub-problem Problem P$(n-1)$ presented in [1] (pp. 2–4).

We have obtained that in the n-dimensional state-space of Problem P(n) the difference between the vector-function $\boldsymbol{x}(t)$ (28) of [1] with an initial point representing the initial state $\boldsymbol{x}_0$ (2) or (12) of [1] of Problem P(n) under the optimal control $u_{n-1}^o(t), t \in \left[0, t_{(n-1)f}^o\right]$, of Problem P$(n-1)$ and the vector-function $\boldsymbol{x}^1(t)$ with an initial point $\boldsymbol{x}_0^1$ (20) – (21) of [1] under the optimal control $u_{n-1}^o(t), t \in \left[0, t_{(n-1)f}^o\right]$, of Problem P$(n-1)$ represents (31) of [1]:

$$\boldsymbol{x}(t) - \boldsymbol{x}^1(t) = \begin{pmatrix} \left(\underbrace{0 \quad \cdots \quad 0}_{n-1}\right)^T \\ e^{\lambda_n t}(x_{n0} - x_{n0}^1) \end{pmatrix} \tag{1}$$

$$\text{for } t \in \left[0, t_{(n-1)f}^o\right].$$

Let us continue the consideration of the second case of (31) of [1] or the above expression with regard to its last n-th coordinate $e^{\lambda_n t}(x_{n0} - x_{n0}^1)$ for $t \in [0, t_{(n-1)f}^o]$, i.e. when $x_{n0} \neq x_{n0}^1$. In case

$$x_{n0} - x_{n0}^1 > 0 \tag{2}$$

and x_{n0} describes sequentially all the values of the interval (x_{n0}^1, ∞), then by (31) of [1] all the points of the line passing through the point $\boldsymbol{x}_0^1$ with coordinates (20) – (21) of [1] parallel to the axis $0x_n$ and located above the point $\boldsymbol{x}_0^1$ by (31) are transformed at the moment $t_{(n-1)f}^o$ successively into the points of the positive part of the axis $0x_n$ of the n-dimensional state-space of Problem $\mathrm{P}(n)$, whereby all these trajectories are located outside the switching hyper-surface of Problem $\mathrm{P}(n)$ having one and the same relation to it, simultaneously above or simultaneously below the switching hyper-surface of Problem $\mathrm{P}(n)$. Thus the positive semi-axis $0x_n$ is outside the switching hyper-surface of Problem $\mathrm{P}(n)$, being wholly below or wholly above the switching hyper-surface of Problem $\mathrm{P}(n)$. When

$$x_{n0} - x_{n0}^1 < 0 \tag{3}$$

based on the same reasoning we get that the negative semi-axis $0x_n$ is outside the switching hyper-surface of Problem $\mathrm{P}(n)$, being wholly below or wholly above the switching hyper-surface of Problem $\mathrm{P}(n)$ but in the opposite relation to the switching hyper-surface of Problem $\mathrm{P}(n)$ relative to the positive semi-axis $0x_n$. Based on this conclusion for the considered class of problems we could define the term for the variable x_{n+} spreading the definition of the term about the variable x_{k+} mentioned in [1] (p. 11) as a term introduced in [14] of [1] (p. 38) and [15] of [1] (pp. 319–320) "which defines the relation of the points of the axis x_k of the state-space of the system of Problem $\mathrm{P}(k)$ from the considered class of problems to the switching hyper-surface of the same Problem $\mathrm{P}(k)$."

Definition. We define x_{n+} as a variable accepting values $+1$ or -1, i.e.

$$x_{n+} \in \{-1, \quad +1\}, \tag{4}$$

so that for the optimal control – the solution of Problem $\mathrm{P}(n)$ it must be satisfied:

$$u^o(0) = u_0 \quad \text{when} \quad \begin{aligned} \boldsymbol{x}_0 &= (x_{10} \;\cdots\; x_{(n-1)0} \;\; x_{n0})^T \\ &= (\underbrace{0 \;\cdots\; 0}_{n-1} \;\; x_{n0})^T, \; x_{n0}: sign(x_{n0}) = x_{n+}. \end{aligned} \tag{5}$$

As it is mentioned in [1] (p. 11) the value of the variable x_{k+} is determined by a procedure called "axes initialization" [14] of [1] (Chapter 3, Section 3.3, pp. 60–88), [16] of [1] (pp. 41–45).

In order to determine x_{n+}, let us now perform the following few constructions.

Construction 1

First, let us first choose $(n-1)$ random positive finite numbers $t_1, t_2, \ldots, t_{(n-1)}$, i.e.

$$t_i \in (0, \infty) \;\; \text{for} \;\; i = 1, 2, \ldots, (n-1). \tag{6}$$

Let us form the following piecewise constant function $u_1(t)$ as an admissible function for the considered class of problems in [1] – "the admissible control $u(t)$ is a piecewise continuous function that takes its values from the range (3), which is continuous on the boundaries of the set of allowed values (3) and in the points of discontinuity τ we have (4)", where its value in the i-th interval, $i = 1, 2, \ldots, (n-1)$, is

$$u_1(t) = (-1)^i u_0 \;\; \text{when} \;\; t \in \left[\textstyle\sum_{j=1}^{i-1} t_j, \sum_{j=1}^{i} t_j\right) \tag{7}$$

$$\text{for} \;\; i = 1, 2, \ldots, (n-1).$$

Note that the value of $u_1(t)$ in the first interval of constancy is $-u_0$.

Let us denote by $t_{(n-1)f}^1$ the length of the function $u_1(t)$ or the sum of the lengths of all intervals of constancy of $u_1(t)$

$$t^1_{(n-1)f} = \sum_{i=1}^{(n-1)} t_i. \tag{8}$$

Let the initial state of Problem P$(n-1)$ be the following point $\boldsymbol{x}^1_{(n-1)0}$ in the $(n-1)$-dimensional state-space of Problem P$(n-1)$

$$\boldsymbol{x}^1_{(n-1)0} = -e^{-A_{n-1}t^1_{(n-1)f}} \int_0^{t^1_{(n-1)f}} e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau. \tag{9}$$

Let us now consider the trajectory in the $(n-1)$-dimensional state-space of Problem P$(n-1)$ with an initial state $\boldsymbol{x}^1_{(n-1)0}$ (9) under the control $u_1(t)$ (7)

$$\boldsymbol{x}^1_{n-1}(t) = e^{A_{n-1}t}\boldsymbol{x}^1_{(n-1)0} + \int_0^t e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau \tag{10}$$

$$\text{for } t \in \left[0, t^1_{(n-1)f}\right].$$

It is easy seen that the control $u_1(t)$ (7) transfers the initial state $\boldsymbol{x}^1_{(n-1)0}$ (9) of the system of Problem P$(n-1)$, (7) of [1], at the moment $t^1_{(n-1)f}$ at the origin of the $(n-1)$-dimensional state-space of Problem P$(n-1)$:

$$\boldsymbol{x}^1_{n-1}\left(t^1_{(n-1)f}\right) = e^{A_{n-1}t^1_{(n-1)f}}\boldsymbol{x}^1_{(n-1)0} + \int_0^{t^1_{(n-1)f}} e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau\,; \tag{11}$$

$$\boldsymbol{x}^1_{n-1}\left(t^1_{(n-1)f}\right) = e^{A_{n-1}t^1_{(n-1)f}}\left(-e^{-A_{n-1}t^1_{(n-1)f}} \int_0^{t^1_{(n-1)f}} e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau\right) + \int_0^{t^1_{(n-1)f}} e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau\,; \tag{12}$$

$$\boldsymbol{x}^1_{n-1}\left(t^1_{(n-1)f}\right) = -I_{n-1} \int_0^{t^1_{(n-1)f}} e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau + \int_0^{t^1_{(n-1)f}} e^{A_{n-1}(t-\tau)} B_{n-1} u_1(t) d\tau\,; \tag{13}$$

$$\boldsymbol{x}^1_{n-1}\left(t^1_{(n-1)f}\right) = (\underbrace{0 \quad \cdots \quad 0}_{n-1})^T. \tag{14}$$

Thus we can draw the following conclusion.

Corollary 1. The piecewise constant function $u_1(t)$ (7) with an amplitude u_0, $(n-1)$ non-zero intervals of constancy $t_1, t_2, \ldots, t_{(n-1)}$, and a value in the first interval of constancy $-u_0$ transfers the system of Problem P$(n-1)$ from its initial state $\boldsymbol{x}^1_{(n-1)0}$ (9) at the final moment $t^1_{(n-1)f}$ (8) at the origin of the $(n-1)$-dimensional state-space of Problem P$(n-1)$. The function $u_1(t)$ represents the optimal control of Problem P$(n-1)$ with an initial state of the system $\boldsymbol{x}^1_{(n-1)0}$. The point $\boldsymbol{x}^1_{(n-1)0}$ is located outside the switching hyper-surface of Problem P$(n-1)$.

Let us denote by $y^1_{n-1}(t)$ the output of the system of Problem P$(n-1)$, (7) of [1], with an initial state $\boldsymbol{x}^1_{(n-1)0}$ (9) under the control $u_1(t)$ (7) for $t \in \left[0,\ t^1_{(n-1)f}\right]$:

$$y^1_{n-1}(t) = C_{n-1}\boldsymbol{x}^1_{n-1}(t) \ \text{ for } t \in \left[0, t^1_{(n-1)f}\right]. \tag{15}$$

$$y^1_{n-1}\left(t^1_{(n-1)f}\right) = C_{n-1}\boldsymbol{x}^1_{n-1}\left(t^1_{(n-1)f}\right) = C_{n-1}\left(\underbrace{0 \quad \cdots \quad 0}_{n-1}\right)^T = 0. \tag{16}$$

Construction 2

Secondly, let us now consider Problem $\mathrm{P}(n)$ with an initial state of the system (1) of [1] in form (10) of [1] at the point $\boldsymbol{x}^1_0$ with coordinates

$$\boldsymbol{x}^1_0 = \begin{pmatrix} \boldsymbol{x}^1_{(n-1)0} \\ x^1_{n0} \end{pmatrix}, \tag{17}$$

$$x^1_{n0} = -\frac{\int_0^{t^1_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^1_{n-1}(\tau)+b_n u_1(\tau)\right)d\tau}{e^{\lambda_n t^1_{(n-1)f}}}, \tag{18}$$

where $\boldsymbol{x}^1_{(n-1)0}$ is (9), $y^1_{n-1}(t)$ is (15), $u_1(t)$ is (7), but x^1_{n0} (18) is the initial state of the n-th coordinate of the state-space vector $\boldsymbol{x}$ of system (1) or (10) of [1]. Let us consider the trajectory $\boldsymbol{x}^1(t)$ in the n-dimensional state-space of Problem $\mathrm{P}(n)$ with an initial state at the point $\boldsymbol{x}^1_0$ with coordinates (17) and (18) under the control $u_1(t), t \in \left[0, t^1_{(n-1)f}\right]$, (7), which according to the above Corollary 1 represents the optimal control of Problem $\mathrm{P}(n-1)$ with an initial state $\boldsymbol{x}^1_{(n-1)0}$ (9). The vector-function $\boldsymbol{x}^1(t)$ based on the representation of the system (1) in form (10) is described as

$$\boldsymbol{x}^1(t) = \begin{pmatrix} e^{A_{n-1}t}\boldsymbol{x}^1_{(n-1)0} + \displaystyle\int_0^t e^{A_{n-1}\tau}B_{n-1}u_1\,(t-\tau)d\tau \\ e^{\lambda_n t}x^1_{n0} + \displaystyle\int_0^t e^{\lambda_n(t-\tau)}\left(y^1_{n-1}(\tau) + b_n u_1(\tau)\right)d\tau \end{pmatrix} \tag{19}$$

$$\text{for } t \in \left[0, t^1_{(n-1)f}\right].$$

For the state $\boldsymbol{x}^1(t)$ at the moment $t^1_{(n-1)f}$:

$$\boldsymbol{x}^1\left(t^1_{(n-1)f}\right) = \begin{pmatrix} e^{A_{n-1}t^1_{(n-1)f}}\boldsymbol{x}^1_{(n-1)0} + \int_0^{t^1_{(n-1)f}} e^{A_{n-1}\tau}B_{n-1}u_1\,(t-\tau)d\tau \\ e^{\lambda_n t^1_{(n-1)f}}x^1_{n0} + \int_0^{t^1_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^1_{n-1}(\tau) + b_n u_1(\tau)\right)d\tau \end{pmatrix}, \tag{20}$$

bearing in mind (11) – (14) and replacing x^1_{n0} with the expression for it (18), we obtain consecutively:

$$\boldsymbol{x}^1\left(t^1_{(n-1)f}\right) = \begin{pmatrix} \boldsymbol{x}^1_{n-1}\left(t^1_{(n-1)f}\right) \\ e^{\lambda_n t^1_{(n-1)f}}\left(-\frac{\int_0^{t^1_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^1_{n-1}(\tau)+b_n u_1(\tau)\right)d\tau}{e^{\lambda_n t^1_{(n-1)f}}}\right) \\ +\int_0^{t^1_{(n-1)f}} e^{\lambda_n(t-\tau)}\left(y^1_{n-1}(\tau) + b_n u_1(\tau)\right)d\tau \end{pmatrix}. \tag{21}$$

$$\boldsymbol{x}^1(t^1_{(n-1)f}) = \begin{pmatrix} (\underbrace{0 \quad \cdots \quad 0}_{n-1})^T \\ -\int_0^{t^1_{(n-1)f}} e^{\lambda_n(t-\tau)}(y^1_{n-1}(\tau) + b_n u_1(\tau))d\tau \\ + \int_0^{t^1_{(n-1)f}} e^{\lambda_n(t-\tau)}(y^1_{n-1}(\tau) + b_n u_1(\tau))d\tau \end{pmatrix}. \tag{22}$$

$$\boldsymbol{x}^1(t^1_{(n-1)f}) = \begin{pmatrix} (\underbrace{0 \quad \cdots \quad 0}_{n-1})^T \\ 0 \end{pmatrix}. \tag{23}$$

$$\boldsymbol{x}^1(t^1_{(n-1)f}) = (\underbrace{0 \quad \cdots \quad 0}_{n})^T. \tag{24}$$

Thus, we can draw the following conclusion.

Corollary 2. The trajectory $\boldsymbol{x}^1(t)$ in the n-dimensional state-space of the system (1) or (10) of [1] with an initial point $\boldsymbol{x}^1_0$ (17) and (18) under the control $u_1(t), t \in [0, t^1_{(n-1)f}]$, (7) ends at the moment $t = t^1_{(n-1)f}$ at the origin of the n-dimensional state-space of Problem $\mathrm{P}(n)$. Taking into account Corollary 1 that the function $u_1(t), t \in [0, t^1_{(n-1)f}]$, is a piecewise constant function with amplitude u_0 and exactly $(n-1)$ intervals of constancy, and represents the optimal control of Problem $\mathrm{P}(n-1)$, whose purpose built initial state $\boldsymbol{x}^1_{(n-1)0}$ is generated also as initial state of the sub-problem by the initial point $\boldsymbol{x}^1_0$ (17) of Problem $\mathrm{P}(n)$, then it follows that in accordance with [13] (Chapter 2, §6, Theorem 2.11, p. 116) the trajectory $\boldsymbol{x}^1(t)$ lies wholly on the switching hyper-surface of Problem $\mathrm{P}(n)$. The value of the optimal control for this point $\boldsymbol{x}^1_0$ is $-u_0 = u_1(0)$, as is the control in the first interval of constancy of $u_1(t)$. The point $\boldsymbol{x}^1_0$ is a point from the manifold M_2^- according to the exposition in Section 1 "Introduction" of [1] based on Pontryagin's original sources [9] (Chapter 3, § 20, § 21, Example 3), where the solution of the problem of a linear time-optimal control system fulfilling the condition of normality with real non-positive eigenvalues and one control input is described.

Construction 3

Let us employ the matrix representation of the equations (1) of [1] of the original system of Problem $\mathrm{P}(n)$ and the relation by the form (1) of [1] with the system (7) – (9) of [1] of order $(n-1)$ of its sub-problem Problem $\mathrm{P}(n-1)$, which matrix description is also depicted in Figure 1 of [1].

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{25}$$

$$A = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix} & 0_{n-1,1} \\ \underbrace{a_{n,1} \quad a_{n,2} \quad \cdots \quad a_{n,n-1}}_{n-1} & \lambda_n \end{pmatrix}, \quad B = \begin{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix} \\ b_n \end{pmatrix}. \tag{26}$$

$$A = \begin{pmatrix} A_{n-1} & 0_{n-1,1} \\ C_{n-1} & \lambda_n \end{pmatrix}, \quad B = \begin{pmatrix} B_{n-1} \\ b_n \end{pmatrix}. \tag{27}$$

$$\boldsymbol{x} = \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} \tag{28}$$

$$\begin{pmatrix} \dot{x}_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} A_{n-1} & 0_{n-1,1} \\ C_{n-1} & \lambda_n \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} B_{n-1} \\ b_n \end{pmatrix} u. \tag{29}$$

Let now t_n be an arbitrary positive number, i.e.

$$t_n \in (0, \infty) \tag{30}$$

and consider the state $\boldsymbol{x}_0$ in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ with coordinates

$$\boldsymbol{x}_0 = e^{-At_n}\left(\boldsymbol{x}_0^1 - \int_0^{t_n} e^{A\tau} B u_0 \, d\tau\right), \tag{31}$$

as well as the trajectory starting at this point under the constant control $+u_0$ when $t \in [0, t_n]$

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} B u_0 \, d\tau \tag{32}$$

for $t \in [0, t_n]$.

The final point of this trajectory at the moment t_n is the state $\boldsymbol{x}_0^1$ (17) – (18) in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$.

$$\boldsymbol{x}(t_n) = e^{At_n}\boldsymbol{x}_0 + \int_0^{t_n} e^{A\tau} B u_0 \, d\tau. \tag{33}$$

$$\boldsymbol{x}(t_n) = e^{At_n}\left(e^{-At_n}\left(\boldsymbol{x}_0^1 - \int_0^{t_n} e^{A\tau} B u_0 \, d\tau\right)\right) + \int_0^{t_n} e^{A\tau} B u_0 \, d\tau. \tag{34}$$

$$\boldsymbol{x}(t_n) = I_n\left(\boldsymbol{x}_0^1 - \int_0^{t_n} e^{A\tau} B u_0 \, d\tau\right) + \int_0^{t_n} e^{A\tau} B u_0 \, d\tau. \tag{35}$$

$$\boldsymbol{x}(t_n) = \boldsymbol{x}_0^1. \tag{36}$$

We get that the points of the trajectory starting at $\boldsymbol{x}_0$ under the control $+u_0$ fall into the point $\boldsymbol{x}_0^1$, which is a point from the switching hyper-surface of Problem $\mathrm{P}(n)$. Having in mind Corollary 2 and following Pontryagin [3] (Chapter 3, § 20, § 21, Example 3), we can draw the following conclusion.

Corollary 3.1 The entire trajectory $\boldsymbol{x}(t)$ (32) with an initial point $\boldsymbol{x}_0$ (31) under the constant control $+u_0$ with a duration t_n, ending at $\boldsymbol{x}_0^1$ (17) – (18), but without the endpoint itself $\boldsymbol{x}_0^1 \in M_2^-$, is a part of the manifold M_1^+, which is the part of the state-space of Problem $\mathrm{P}(n)$ outside the switching hyper-surface $M_2 = M_2^- \cup M_2^+$ of Problem $\mathrm{P}(n)$ with a value of the optimal control for the points of this manifold $+u_0$.

Let us say for convenience that the points of the manifold M_1^+ are above the switching hyper-surface of Problem $\mathrm{P}(n)$ while the points of M_1^- are below the switching hyper-surface of Problem $\mathrm{P}(n)$.

Let us form the following piecewise constant function $u_2(t)$ with n intervals of constancy, where the duration of the first interval is t_n and the value of the function there is u_0, while from the second till the n-th interval the function $u_2(t)$ represents the shifted to the right by a distance t_n function $u_1(t)$ (7), i.e. the duration of the second interval is $t_1$while the value there is $(-1)^1 u_0$ and so on for the intervals after the second one till the n-th one, which is with a length t_{n-1} and a value of the function within it $(-1)^{n-1} u_0$. We write the function $u_2(t)$ in the following two ways.

$$u_2(t) = \begin{cases} u_0 & \text{при} \quad t \in [0, t_n) \\ u_1(t - t_n) & \text{при} \quad t \in \left[t_n,\ t_n + \sum_{j=1}^{i} t_j\right] \end{cases}. \tag{37}$$

$$u_2(t) = \begin{cases} u_0 & \text{при} \quad t \in [0, t_n) \\ (-1)^i u_0 & \text{при} \quad t \in \left[t_n + \sum_{j=1}^{i-1} t_j \,,\; t_n + \sum_{j=1}^{i} t_j \right). \\ & \quad \text{за } i = 1, 2, \dots, (n-1). \end{cases} \tag{38}$$

Then the point $\boldsymbol{x}_0$ (31) in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ expressed in (31) by $\boldsymbol{x}_0^1$ (17) and (18) is also expressed as:

$$\boldsymbol{x}_0 = e^{-A(t_1+t_2+\cdots+t_n)} \left(-\int_0^{(t_1+t_2+\cdots+t_n)} e^{A\tau} B u_2(t-\tau)\, d\tau \right). \tag{39}$$

Corollary 3.1 can be amended with the following corollary.

Corollary 3.2 The piecewise constant function $u_2(t)$, $t \in [0, \sum_{i=1}^{n} t_i]$, (37) or (38) with n non-zero intervals of constancy represents the optimal control of Problem $\mathrm{P}(n)$ with an initial state $\boldsymbol{x}_0$ (39) or (31).

Construction 4

Let us now consider Problem $\mathrm{P}(n)$ with an initial state $\boldsymbol{x}_0$ (39) or (31) and its respective sub-problem Problem $\mathrm{P}(n-1)$. Let us form the trajectory in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ with an initial state $\boldsymbol{x}_0$ under the optimal control $u_{n-1}^o(t), t \in [0, t_{(n-1)f}^o]$, of Problem $\mathrm{P}(n-1)$:

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} B u_{n-1}^o(t-\tau)\, d\tau \tag{40}$$

$$\text{for } t \in [0, t_{(n-1)f}^o].$$

According to Corollary 3 $\boldsymbol{x}_0 \in M_1^+$. Therefore, by Theorem 1 of [1] this trajectory also lies in M_1^+ – lies above the switching hyper-surface of Problem $\mathrm{P}(n)$ nowhere intersecting it and ends at the moment $t = t_{(n-1)f}^o$ at a point of the coordinate axis x_n different from zero.

Considering the relation (12) of [1] between the initial state $\boldsymbol{x}_0$ of Problem $\mathrm{P}(n)$, (2) of [1], and the initial state $\boldsymbol{x}_{(n-1)0}$ of Problem $\mathrm{P}(n-1)$, (11) of [1],

$$\boldsymbol{x}_0 = \begin{pmatrix} \boldsymbol{x}_{(n-1)0} \\ x_{n0} \end{pmatrix}, \tag{41}$$

the trajectory $\boldsymbol{x}(t)$ (40), according to (29) of [1], represents:

$$\boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{x}_{n-1}^o(t) \\ e^{\lambda_n t} x_{n0} + \int_0^t e^{\lambda_n (t-\tau)} \left(y_{n-1}^o(\tau) + b_n u_{n-1}^o(\tau) \right) d\tau \end{pmatrix} \in M_1^+ \tag{42}$$

$$\text{for } t \in [0, t_{(n-1)f}^o].$$

The final point $\boldsymbol{x}(t_{(n-1)f}^o)$ of this trajectory lies on the axis Ox_n:

$$\boldsymbol{x}\left(t^o_{(n-1)f}\right) = \begin{pmatrix} \boldsymbol{x}^o_{n-1}\left(t^o_{(n-1)f}\right) \\ e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int\limits_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right) d\tau \end{pmatrix} \in M_1^+. \tag{43}$$

$$\boldsymbol{x}\left(t^o_{(n-1)f}\right) = \begin{pmatrix} \left(\underbrace{0 \quad \cdots \quad 0}_{n-1}\right)^T \\ e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int\limits_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right) d\tau \end{pmatrix} \in M_1^+. \tag{44}$$

Thus, the value, more precisely, the sign of the last n-th coordinate

$$e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int\limits_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right) d\tau \neq 0 \tag{45}$$

indicates whether the positive or negative part of the Ox_n axis is a part of the manifold M_1^+, i.e. is above the switching hyper-surface. We obtain

$$x_{n+} = sign\left(e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int\limits_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \left(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\right) d\tau \right). \tag{46}$$

Thus, if $x_{n+} = 1$, then the positive part of the Ox_n axis is a part of the manifold M_1^+, i.e. is located above the switching hyper-surface, and accordingly the negative part of the Ox_n axis is below the switching hyper-surface or is a part of the manifold M_1^-.

When $x_{n+} = -1$, then the negative part of the Ox_n axis is a part of the manifold M_1^+, i.e. is located above the switching hyper-surface, while the positive part of the Ox_n axis is below the switching hyper-surface or is a part of the manifold M_1^-.

Thus, the following theorem with regard to the "axes initialization" has been proven.

Theorem 1. If the initial state of Problem $\mathrm{P}(n)$ represents the point $\boldsymbol{x}_0$ (39), where t_1, t_2, …, $t_{(n-1)}, t_n$ are n random finite positive numbers, the function $u_2(t)$, $t \in [0, (t_1 + t_2 + \cdots + t_n)]$, is a piecewise constant function with n non-zero intervals of constancy (38) with an amplitude u_0, and starts with a value u_0 on its first interval, then:

• The point $\boldsymbol{x}_0 \in M_1^+$ and the function $u_2(t)$, $t \in [0, (t_1 + t_2 + \cdots + t_n)]$, represents the optimal control of Problem $\mathrm{P}(n)$;

• The trajectory (40) in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ with an initial point $\boldsymbol{x}_0$ under the optimal control $u^o_{n-1}(t), t \in \left[0, t^o_{(n-1)f}\right]$, of Problem $\mathrm{P}(n-1)$ is also located in the manifold M_1^+ – lies above the switching hyper-surface of Problem $\mathrm{P}(n)$ nowhere intersecting it and ends at the moment $t = t^o_{(n-1)f}$ at the point (44) of the coordinate axis Ox_n different from zero;

• The sign of this n-th coordinate x_{n+} (46) indicates whether the positive or the negative part of the Ox_n axis is a part of the manifold M_1^+, i.e. is above the switching hyper-surface. Thus, if $x_{n+} = 1$, then the positive part of the Ox_n axis is a part of the manifold M_1^+, i.e. is located above the

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switching hyper-surface, and accordingly the negative part of the Ox_n axis is below the switching hyper-surface or is a part of the manifold M_1^-. When $x_{n+} = -1$, then the negative part of the Ox_n axis is a part of the manifold M_1^+, i.e. is located above the switching hyper-surface, while the positive part of the Ox_n axis is below the switching hyper-surface or is a part of the manifold M_1^-.

Construction 5

Let us now consider Problem $\mathrm{P}(n)$ with an initial state at the point $\boldsymbol{x}_0$ which can now be any point in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$. Suppose that we have solved the simpler sub-problem Problem $\mathrm{P}(n-1)$ and obtained $u_{n-1}^o(t), t \in [0, t_{(n-1)f}^o]$, of Problem $\mathrm{P}(n-1)$. Let us consider the trajectory in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ with an initial point $\boldsymbol{x}_0$ under the optimal control $u_{n-1}^o(t), t \in [0, t_{(n-1)f}^o]$, of Problem $\mathrm{P}(n-1)$:

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)} B u_{n-1}^o(\tau)\, d\tau \tag{47}$$

$$\text{for } t \in [0, t_{(n-1)f}^o].$$

Based on the relation (12) of [1] between the initial state $\boldsymbol{x}_0$ of Problem $\mathrm{P}(n)$, (2) of [1], and the initial state $\boldsymbol{x}_{(n-1)0}$ of Problem $\mathrm{P}(n-1)$, (11) of [1],

$$\boldsymbol{x}_0 = \begin{pmatrix} \boldsymbol{x}_{(n-1)0} \\ x_{n0} \end{pmatrix}, \tag{48}$$

the trajectory $\boldsymbol{x}(t)$ (47) according to (29) of [1] is described as:

$$\boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{x}_{n-1}^o(t) \\ e^{\lambda_n t} x_{n0} + \int_0^t e^{\lambda_n (t-\tau)} \left(y_{n-1}^o(\tau) + b_n u_{n-1}^o(\tau)\right) d\tau \end{pmatrix} \tag{49}$$

$$\text{for } t \in [0, t_{(n-1)f}^o].$$

The final point of this trajectory $\boldsymbol{x}(t_{(n-1)f}^o)$ lies on the x_n axis:

$$\boldsymbol{x}(t_{(n-1)f}^o) = \begin{pmatrix} \boldsymbol{x}_{n-1}^o(t_{(n-1)f}^o) \\ e^{\lambda_n t_{(n-1)f}^o} x_{n0} + \int_0^{t_{(n-1)f}^o} e^{\lambda_n (t-\tau)} \left(y_{n-1}^o(\tau) + b_n u_{n-1}^o(\tau)\right) d\tau \end{pmatrix}. \tag{50}$$

$$\boldsymbol{x}(t_{(n-1)f}^o) = \begin{pmatrix} (\underbrace{0 \ \cdots \ 0}_{n-1})^T \\ e^{\lambda_n t_{(n-1)f}^o} x_{n0} + \int_0^{t_{(n-1)f}^o} e^{\lambda_n (t-\tau)} \left(y_{n-1}^o(\tau) + b_n u_{n-1}^o(\tau)\right) d\tau \end{pmatrix}. \tag{51}$$

According to Theorem 1 of [1] this entire trajectory $\boldsymbol{x}(t)$ (47) or (49), i.e. all its points, have the same relation to the switching hyper-surface of Problem $\mathrm{P}(n)$. Let us now consider the possible cases.

Case 1. The trajectory lies entirely on the switching hyper-surface of Problem P(n) and the final point $\boldsymbol{x}(t^o_{(n-1)f})$ at the moment $t^o_{(n-1)f}$ is the origin of the n-dimensional state-space of the system of Problem P(n). Therefore, the n-th coordinate of $\boldsymbol{x}(t^o_{(n-1)f})$ (51) is also 0:

$$0 = e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \big(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\big) d\tau. \tag{52}$$

In this case the optimal control $u^o_{n-1}(t), t \in [0, t^o_{(n-1)f}]$ – the solution of the sub-problem Problem P $(n-1)$ which is a piecewise constant function with at most $(n-1)$ intervals of constancy, i.e. with at most $(n-2)$ switchings, is also the solution of Problem P(n):

$$u^o_n(t) \equiv u^o_{n-1}(t), \quad t^o_{nf} \equiv t^o_{(n-1)f}. \tag{53}$$

Case 2. The trajectory lies wholly above or below the switching hyper-surface of Problem P(n) nowhere intersecting it and the final point $\boldsymbol{x}(t^o_{(n-1)f})$ at the moment $t^o_{(n-1)f}$ is a point of the coordinate axis Ox_n different from zero. Therefore, the n-th coordinate of $\boldsymbol{x}(t^o_{(n-1)f})$ (51) satisfies:

$$e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \big(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\big) d\tau \neq 0. \tag{54}$$

Case 2.1 If the sign of the specified above n-th coordinate and the sign of the variable x_{n+} (46) are the same, then according to the proved above Theorem 1 the entire trajectory $\boldsymbol{x}(t)$ (47) or (49) belongs to the manifold M_1^+, i.e. all the point of $\boldsymbol{x}(t)$ are points of the manifold M_1^+ or are located above the switching hyper-surface of Problem P(n):

$$\text{If } \left(e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \big(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\big) d\tau\right) x_{n+} > 0 \quad \text{then } \boldsymbol{x}(t) \in M_1^+ \text{ for } t \in [0, t^o_{(n-1)f}]. \tag{55}$$

Case 2.2 If the sign of the specified above n-th coordinate and the sign of the variable x_{n+} (46) are opposite to each other, then according to the proved above Theorem 1 the entire trajectory $\boldsymbol{x}(t)$ (47) or (49) belongs to the manifold M_1^-, i.e. all the point of $\boldsymbol{x}(t)$ are points of the manifold M_1^- or are located below the switching hyper-surface of Problem P(n):

$$\text{If } \left(e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \big(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\big) d\tau\right) x_{n+} < 0 \quad \text{then } \boldsymbol{x}(t) \in M_1^- \text{ for } t \in [0, t^o_{(n-1)f}]. \tag{56}$$

Let us define the variable $x_{n,v}$, which is the value of the n-th coordinate of the final state $\boldsymbol{x}(t^o_{(n-1)f})$ (51) of the trajectory $\boldsymbol{x}(t)$ (47) or (49), obtained under the optimal control $u^o_{n-1}(t), t \in [0, t^o_{(n-1)f}]$, of Problem P$(n-1)$, at the moment $t^o_{(n-1)f}$ in the n-dimensional state-space of the system of Problem P(n) with an initial state $\boldsymbol{x}_0$:

$$x_{n,v} = e^{\lambda_n t^o_{(n-1)f}} x_{n0} + \int_0^{t^o_{(n-1)f}} e^{\lambda_n (t-\tau)} \big(y^o_{n-1}(\tau) + b_n u^o_{n-1}(\tau)\big) d\tau. \tag{57}$$

Thus, the following theorem with regard to the synthesis of the time-optimal control at the initial state $\boldsymbol{x}_0$ of Problem P(n) has been proven.

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Theorem 2. If the optimal control of the sub-problem Problem $\mathrm{P}(n-1)$ is found – the solution $u^o_{n-1}(t), t \in [0, t^o_{(n-1)f}]$, then the optimal control $u^o_n(0)$ at the initial state $\boldsymbol{x}_0$ of Problem $\mathrm{P}(n)$ can be determined as:

$$u^o_n(0) = \begin{cases} +u_0 & when \quad x_{n,v}x_{n+} > 0 \\ u^o_{n-1}(0) & when \quad x_{n,v} = 0 \\ -u_0 & when \quad x_{n,v}x_{n+} < 0 \end{cases} \tag{58}$$

The following consequence of the above Theorem 2 holds.

Consequence 1. In the case that $x_{n,v} = 0$, then the solution of Problem $\mathrm{P}(n)$ represents the already found solution of the sub-problem Problem $\mathrm{P}(n-1)$:

$$\begin{aligned} u^o_n(t) &\equiv u^o_{n-1}(t), \\ t^o_{nf} &\equiv t^o_{(n-1)f} \end{aligned} \quad when \quad x_{n,v} = 0. \tag{59}$$

Let us consider the case when $x_{n,v} \neq 0$. This means that the initial state $\boldsymbol{x}_0$ of the system of Problem $\mathrm{P}(n)$ is outside the switching hyper-surface of Problem $\mathrm{P}(n)$, i.e. it is a point of the manifold M_1, M_1^+ or M_1^-. In this case, assume that the optimal control at the initial state $\boldsymbol{x}_0$ of Problem $\mathrm{P}(n)$ and the optimal control at the initial state of Problem $\mathrm{P}(n-1)$ are the same, i.e. both have the value $+u_0$ or $-u_0$ simultaneously:

$$u^o_n(0) = u^o_{n-1}(0). \tag{60}$$

It follows from the above that the optimal control of Problem $\mathrm{P}(n-1)$ as a piecewise constant function with at most $(n-1)$ intervals of constancy with an amplitude u_0 has at least one interval of constancy. Let us denote its length as $t^o_{(n-1),1}$. So it is valid:

$$0 < t^o_{(n-1),1} \le t^o_{(n-1)f}. \tag{61}$$

Let us consider the trajectory $\boldsymbol{x}(t)$ (47) or (49) in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ with an initial state $\boldsymbol{x}_0$ obtained under the optimal control $u^o_{n-1}(t), t \in [0, t^o_{(n-1)f}]$, of the sub-problem Problem $\mathrm{P}(n-1)$. According to Theorem 1 of [1] this trajectory $\boldsymbol{x}(t)$ (47) or (49) lies entirely in one of the two manifolds M_1^+ or M_1^- of the state-space of Problem $\mathrm{P}(n)$. Let us consider the first section ot this trajectory formed under the constant control with a value $u^o_{n-1}(0)$ and a duration $t^o_{(n-1),1}$:

$$u^o_{n-1}(t) = u^o_{n-1}(0) \quad \text{for} \quad t \in [0, t^o_{(n-1),1}]. \tag{62}$$

Then the first section of the trajectory $\boldsymbol{x}(t)$ (47) or (49) for $t \in [0, t^o_{(n-1),1}]$ in the n-dimensional state-space of the system of Problem $\mathrm{P}(n)$ is described as:

$$\begin{aligned} \boldsymbol{x}(t) &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)} B u^o_{n-1}(\tau)\, d\tau \\ &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)} B u^o_{n-1}(0)\, d\tau \\ &\text{for} \quad t \in [0, t^o_{(n-1),1}]. \end{aligned} \tag{63}$$

Since (60) is fulfilled in this case, then for (63) or the first section of the trajectory $\boldsymbol{x}(t)$ (47) or (49) it is valid that:

$$\begin{aligned} \boldsymbol{x}(t) &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)} B u^o_n(0)\, d\tau \\ &\text{for} \quad t \in [0, t^o_{(n-1),1}]. \end{aligned} \tag{64}$$

The possible options for the optimal control at the point $\boldsymbol{x}_0$ can only be $u_n^o(0) = +u_0$ or $u_n^o(0) = -u_0$. If $\boldsymbol{x}_0 \in M_1^+$, then $u_n^o(0) = +u_0$, and if $\boldsymbol{x}_0 \in M_1^-$, then $u_n^o(0) = -u_0$.

We can represent this section (64) of the trajectory $\boldsymbol{x}(t)$ (47) or (49) in terms of $u_n^o(0)$ as:

$$\begin{aligned} \boldsymbol{x}(t) &= \left(e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}Bu_n^o(0)\, d\tau\right) \\ &= \left(e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}Bu_0\, d\tau\right) \in M_1^+ \quad if \quad u_n^o(0) = +u_0 \end{aligned} \tag{65}$$

$$\text{for } t \in \left[0, t_{(n-1),1}^o\right],$$

$$\begin{aligned} \boldsymbol{x}(t) &= \left(e^{At}\boldsymbol{x}_0 + \int\limits_0^t e^{A(t-\tau)}Bu_n^o(0)\, d\tau\right) \\ &= \left(e^{At}\boldsymbol{x}_0 + \int\limits_0^t e^{A(t-\tau)}B(-u_0)\, d\tau\right) \in M_1^- \quad if \quad u_n^o(0) = -u_0 \end{aligned} \tag{66}$$

$$\text{for } t \in \left[0, t_{(n-1),1}^o\right].$$

But (65) describes the part of the optimal trajectory of the system in the manifold M_1^+ of the n-dimensional state-space of the system of Problem P(n) with an initial point $\boldsymbol{x}_0$ obtained under the optimal control for the points of this manifold $+u_0$ for $t \in \left[0, t_{(n-1),1}^o\right]$ in the case when $u_n^o(0) = u_{n-1}^o(0) = +u_0$, while (66) describes the part of the optimal trajectory of the system in the manifold M_1^- of the n-dimensional state-space of the system of Problem P(n) with an initial point $\boldsymbol{x}_0$ obtained under the optimal control for the points of this manifold $-u_0$ for $t \in \left[0, t_{(n-1),1}^o\right]$ in the case when $u_n^o(0) = u_{n-1}^o(0) = -u_0$. Thus, we proved the following corollary of Theorem 2.

Consequence 2. In case when $x_{n,v} \neq 0$ and $u_n^o(0) = u_{n-1}^o(0)$, then the first section of the trajectory in the n-dimensional state-space of the system of Problem P(n) with an initial point $\boldsymbol{x}_0$ obtained under the optimal control $u_{n-1}^o(t), t \in \left[0, t_{(n-1)f}^o\right]$, of the sub-problem Problem P$(n-1)$ and formed under the constant control with a value $u_{n-1}^o(0)$ and a duration $t_{(n-1),1}^o$, in addition to being outside – above or below and nowhere intersecting the switching hyper-surface of Problem P(n), is also a part of the optimal trajectory of Problem P(n) located in the manifold M_1^+ when $u_n^o(0) = u_{n-1}^o(0) = +u_0$ or located in the manifold M_1^- when $u_n^o(0) = u_{n-1}^o(0) = -u_0$.

Example

Let us carry out an "axes initialization" for the double integrator from the example of [1] (Section 3. Example, pp. 8–13, 3.2. Synthesis Based on the New Property and the Method [14], pp. 10–13). The equations of the system of the considered Problem P(2) there in [1] are (45) and (46). The equations of the system of the sub-problem Problem P(1) are there (47) и (48).

Equation (57) of [1] (p. 11) simply points out the fact that $x_{2+} = -1$. We will derive this result based on the theoretical conclusions obtained here.

For $n = 2$, according to Construction 1, we choose an arbitrary finite positive t_1 (6), for example

$$t_1 = 2. \tag{67}$$

The piecewise constant function $u_1(t)$ (7), in this case with only one interval of constancy with a length $t_1 = 2$ (67), represents

$$u_1(t) = (-1)^1 u_0 = -u_0 \ \text{ for } t \in [0, t_1). \tag{68}$$

The duration $t_{(n-1)f}^1$ of $u_1(t)$ is

$$t_{(n-1)f}^1 = t_{1f}^1 = t_1 = 2. \tag{69}$$

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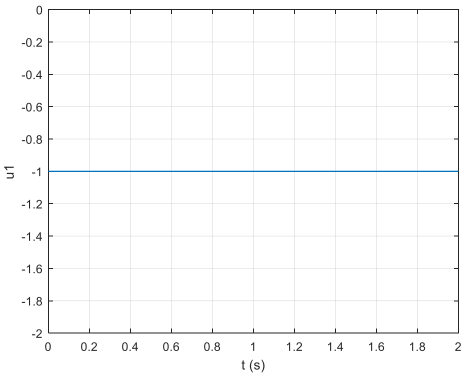
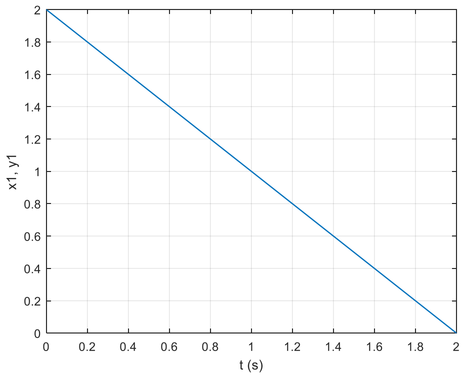
We obtain for the initial state $\boldsymbol{x}^1_{(n-1)0}$ (9) of Problem P$(n-1)$, here $\boldsymbol{x}^1_{10}$ in the one-dimensional state-space of Problem P(1):

$$\boldsymbol{x}^1_{10} = (x^1_{10}) = (2). \tag{70}$$

Figure 1 shows the transition from the above initial state $\boldsymbol{x}^1_{10}$ of the system of Problem P(1) to the one-dimensional state-space origin under the control $u_1(t)$ (68) with a duration t^1_{1f} (69), which is an illustration of Corollary 1 of Construction 1. Since the matrix C_1 for the example according to (46) of [1] is $C_1 = (1)$, the output of the system of Problem P(1) $y^1_{n-1}(t)$ (15) is $y^1_{n-1}(t) = y^1_1(t)$ and coincides with $\boldsymbol{x}^1_{n-1}(t)$, the state $\boldsymbol{x}^1_1(t)$ here.



(a)



(b)

Figure 1. The purpose built function $\boldsymbol{u_1(t)}$ (67) – (69) and the resulting time-optimal process of the system of Problem P(1): (a) the control $\boldsymbol{u_1(t)}$; (b) the state $\boldsymbol{x^1_1(t)}$ and the output $\boldsymbol{y^1_1(t)}$ of the system of Problem P(1).

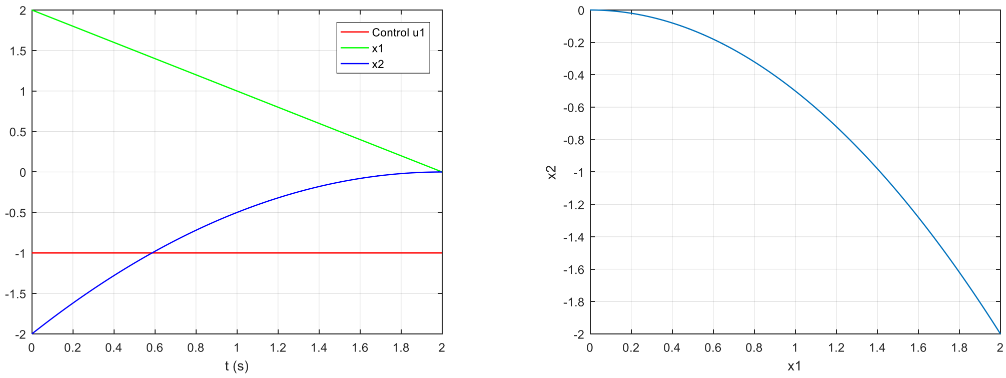
We obtain for the point $\boldsymbol{x}^1_0$ (17) by Construction 2:

$$\boldsymbol{x}^1_0 = \begin{pmatrix} \boldsymbol{x}^1_{(n-1)0} \\ x^1_{n0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^1_{10} \\ x^1_{20} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \tag{71}$$

where the coordinate x^1_{n0} (18), in this case x^1_{20}, is

$$x^1_{n0} = x^1_{20} = -2. \tag{72}$$

Figure 2 illustrates Corollary 2 of Construction 2. Figure 2a depicts the process of system of Problem P(2) with an initial state at the point $\boldsymbol{x}^1_0$ (71) under the control $u_1(t)$ (68), (69). The process in the phase plane x_1x_2 of the system of Problem P(2) is depicted in Figure 2b, and this phase trajectory is a part of the manifold M^-_2.



(a) (b)

Figure 2. The movement in the manifold $\boldsymbol{M_2^-}$ of Problem P$(\mathbf{2})$ from the obtained initial state $\boldsymbol{x_0^1} = (\mathbf{2} \quad \mathbf{-2})^{\boldsymbol{T}}$ (71), for which the control $\boldsymbol{u_1(t)}$ (67) – (69) is the time-optimal one: (a) the process and the control $\boldsymbol{u_1(t)}$; (b) the phase trajectory in $\boldsymbol{M_2^-}$.

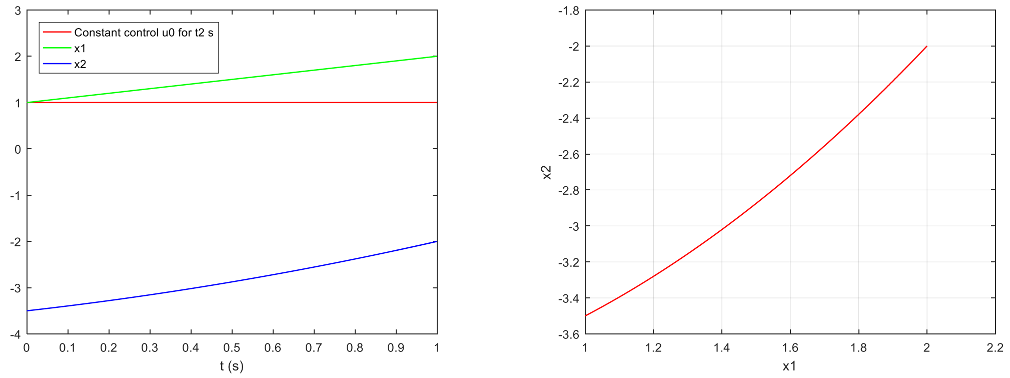
Let us proceed to Construction 3. We now choose t_n, in our case t_2, as an arbitrary finite positive number according to (30). Let, for example,

$$t_2 = 1. \tag{73}$$

For the point $\boldsymbol{x}_0$ (31), we obtain

$$\boldsymbol{x}_0 = \begin{pmatrix} 1 \\ -3.5 \end{pmatrix}. \tag{74}$$

As an illustration of Corollary 3.1 of Construction 3 Figure 3a shows the process of the system of Problem P(2) with an initial point $\boldsymbol{x}_0$ (74) under the constant control $+u_0$ for $t \in [0,\ t_2]$ transferring the system at the point $\boldsymbol{x}_0^1$ (71). The phase trajectory is depicted in Figure 3b. This trajectory without the final point $\boldsymbol{x}_0^1$ (71) is a part of the manifold M_1^+, while the final point $\boldsymbol{x}_0^1$ (71) belongs to the manifold M_2^-.



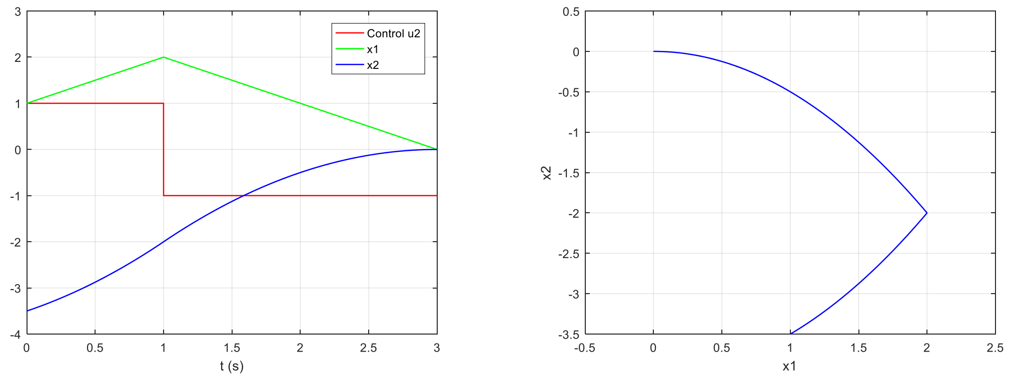
(a) (b)

Figure 3. The trajectory of the system of Problem $\mathrm{P}(\mathbf{2})$ with a duration $\boldsymbol{t_2} = \mathbf{1}$ s from the initial state $\boldsymbol{x_0} = (\mathbf{1} \quad -\mathbf{3.5})^{\boldsymbol{T}}$ (74) in the manifold $\boldsymbol{M_1^+}$ untill entering the manifold $\boldsymbol{M_2^-}$ at the state $\boldsymbol{x_0^1} = (\mathbf{2} \quad -\mathbf{2})^{\boldsymbol{T}}$ (71): (a) the process itself; (b) the phase trajectory in $\boldsymbol{M_1^+}$.

Let the piecewise constant function $u_2(t)$ with 2 intervals of constancy be formed according to Construction 3 in the way (38):

$$u_2(t) = \begin{cases} u_0 & \text{при} \quad t \in [0, t_2) \\ -u_0 & \text{при} \quad t \in [t_2, t_2 + t_1) \end{cases} \tag{75}$$

Figure 4a shows the process of the system of Problem P(2) with an initial point $\boldsymbol{x}_0 = (1 \quad -3.5)^T$ (74) under the piecewise constant function $u_2(t)$ (75) while in Figure 4b the respective trajectory in the phase plane $x_1 x_2$ is depicted. The first section of the trajectory from the point $\boldsymbol{x}_0$ (74) until the state $\boldsymbol{x}_0^1 = (2 \quad -2)^T$ (71), but without the final point $\boldsymbol{x}_0^1$ itself, is in the manifold M_1^+, while the second section starting at $\boldsymbol{x}_0^1$ (71) till the state-space origin is a movement lying in the manifold M_2^- or on the part of the switching hyper-surface formed under the control $-u_0$. According to Corollary 3.2 of Construction 3, the piecewise constant function $u_2(t)$, $t \in [0, t_2 + t_1]$, (75) with two non-zero intervals of constancy is the time-optimal control of Problem P(2) with an initial state at the point $\boldsymbol{x}_0$ (74).



(a) (b)

Figure 4. The optimal process of the system of Problem $\mathrm{P}(\mathbf{2})$ with the derived initial state $\boldsymbol{x_0} = (\mathbf{1} \quad -\mathbf{3.5})^{\boldsymbol{T}}$ (74) for which the time-optimal control is the specially constructed function $\boldsymbol{u_2}(\boldsymbol{t})$ (67), (73), (75): (a) the process; (b) the trajectory in the phase plane $\boldsymbol{x_1 x_2}$.

Let us proceed to Construction 4. We form the sub-problem Problem P(1) of Problem P(2) with an initial state $\boldsymbol{x}_0$ (31), which in this case is the obtained point $\boldsymbol{x}_0$ (74). The equations of the system of Problem P(1) are (48) of [1] (p. 10), and the initial state of Problem P(1) derived from $\boldsymbol{x}_0$ (74) is:

$$x_{10} = 1. \tag{76}$$

The solution of Problem P(1) is presented by the formulas (49) – (51) of [1] (p. 10). We obtain:

$$s_{11}^o = -sign(b_1 x_{10}) = -1, \tag{77}$$
$$t_{11}^o = \frac{|x_{10}|}{|b_1|u_0} = 1,$$
$$t_{1f}^o = t_{11}^o = 1.$$

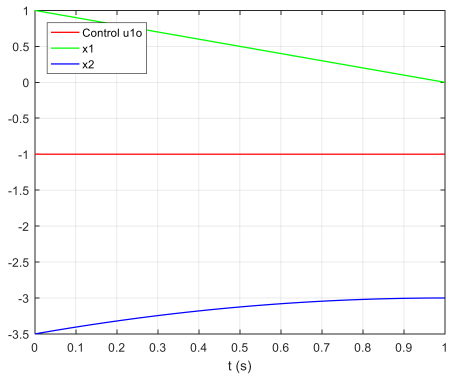
Figure 5a presents the process of the system of Problem P(2) with an initial state at the point $\boldsymbol{x}_0$ (74) under the piecewise constant function $u_1^o(t)$ with one interval of constancy $t_{11}^o = 1$ and a value $s_{11}^o u_0 = -1$, which is the solution of Problem P(1). The process in this case corresponds to the process $\boldsymbol{x}(t)$ (40) or (42) of Construction 4. The black trajectory in Figure 5b shows the respective trajectory in the phase plane x_1x_2 of Problem P(2), while the blue one outlines the trajectory from the same initial point $\boldsymbol{x}_0 = (1 \quad -3.5)^T$ (74) but obtained under the piecewise constant function $u_2(t)$, $t \in [0, t_2 + t_1]$, (75) and also shown in Figure 4b. The final point of this process $\boldsymbol{x}(t_{1f}^o)$, which corresponds to the final point $\boldsymbol{x}(t_{(n-1)f}^o)$ (44) of $\boldsymbol{x}(t)$ (40) or (42), is:

$$\boldsymbol{x}\left(t_{(n-1)f}^o\right) = \boldsymbol{x}\left(t_{1f}^o\right) = \begin{pmatrix} 0 \\ -3 \end{pmatrix} \in M_1^+. \tag{78}$$

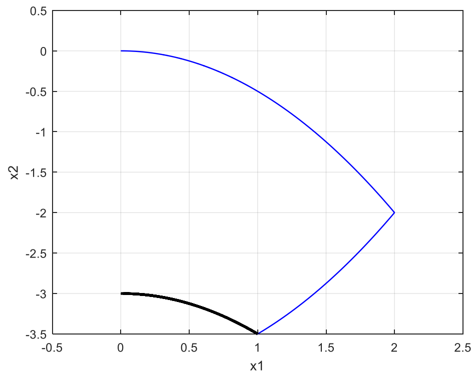
That is a point on the axis Ox_n, in this case Ox_2, different from the state-space origin. The sign of the value of the last coordinate, the second coordinate here, of this point indicates which part of this axis belongs to the manifold M_1^+. Following (46), we obtain:

$$x_{2+} = sign(-3) = -1. \tag{79}$$

Thus, according to Theorem 1 for $x_{2+} = -1$ the negative part of the axis Ox_2 belongs to the manifold M_1^+, i.e. is located above the switching hyper-surface, while the positive part of the axis Ox_2 is below the switching hyper-surface or is a part of the manifold M_1^-.



(a)



(b)

Figure 5. Trajectories of the system of Problem P(**2**) with an initial state $\boldsymbol{x_0} = (\mathbf{1} \quad \mathbf{-3.5})^{\boldsymbol{T}}$ (74): (a) the process under the optimal control $\boldsymbol{u_1^o(t)}$ of the sub-problem Problem P(**1**); (b) marked with black the phase trajectory corresponding to the process in (a), and the optimal phase trajectory for the point $\boldsymbol{x_0}$ marked with blue.

Let us now illustrate the synthesis of the time-optimal control and Consequence 2 of Theorem 2. For this purpose, let the initial state of Problem P(2) be:

$$\boldsymbol{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{80}$$

The sub-problem Problem P(1) in this case is with an initial state

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$$x_{10} = 1. \tag{81}$$

We use the already known solution (77) of Problem $\mathrm{P}(1)$ with this initial state (81) or (76). For the variable $x_{n,v}$ (57), in our case $x_{2,v}$, employing (52) – (55) of [1] (p. 10), and based on (55) of [1], we obtain

$$x_{2,v} = 1.5. \tag{82}$$

For the optimal control in the initial state of Problem $\mathrm{P}(2)$ according to Theorem 2 by (58), taking into account $x_{2+} = -1$ (79), we obtain:

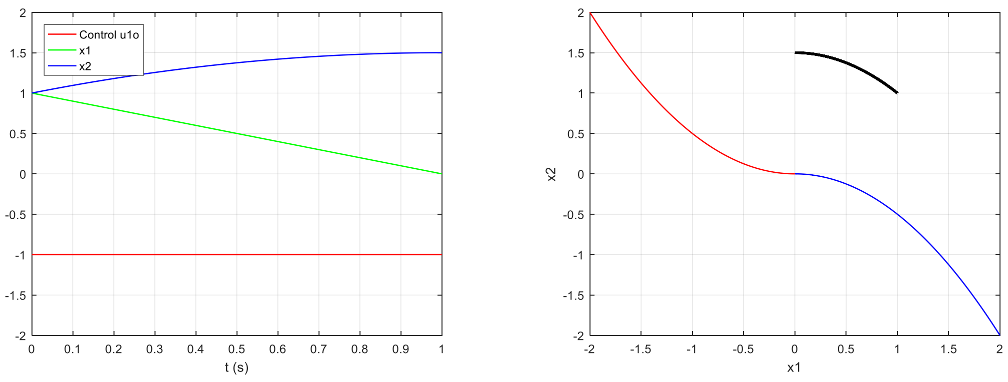
$$u_2^o(0) = u^o(1,1) = -u_0 = -1. \tag{83}$$

It follows from the above result and the solution (77) of Problem $\mathrm{P}(1)$ that:

$$u_2^o(0) = u_1^o(0) = -u_0 = -1. \tag{84}$$

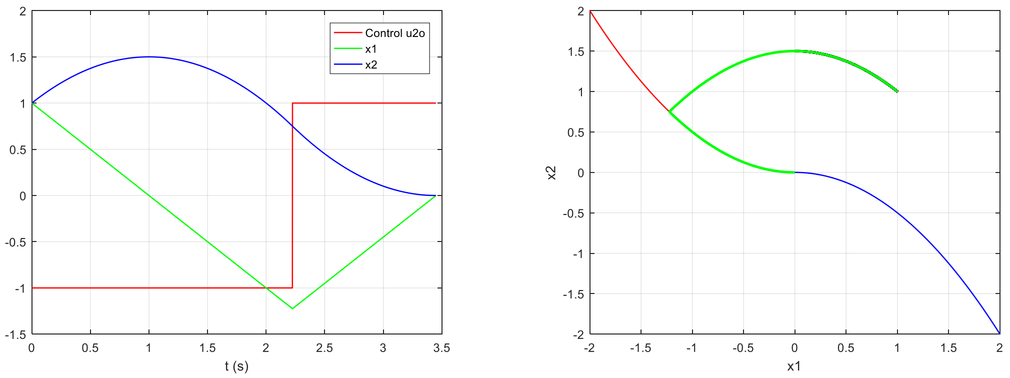
Thus, according to Consequence 2 of Theorem 2 the first section of the trajectory in the phase plane x_1x_2 of Problem $\mathrm{P}(2)$ with an initial state $\boldsymbol{x}_0 = (1 \quad 1)^T$ (80) obtained under the optimal control $u_1^o(t), t \in [0, t_{1f}^o]$ (77) of Problem $\mathrm{P}(1)$, in addition to being outside, above or below, nowhere intersecting the switching curve – the manifold M_2 of Problem $\mathrm{P}(2)$, is also a part of the optimal trajectory of Problem $\mathrm{P}(2)$ located in the manifold M_1^-. This relation is also shown in Figure 6b, where the black trajectory concerns the phase trajectory under the optimal control $u_1^o(t), t \in [0, t_{1f}^o]$ (77) of Problem $\mathrm{P}(1)$, the manifold M_2^- is marked with blue, the manifold M_2^+ is marked with red, and the switching curve is $M_2 = M_2^- \cup M_2^+$. The process of the system of Problem $\mathrm{P}(2)$ with an initial state $\boldsymbol{x}_0 = (1 \quad 1)^T$ formed under the control $u_1^o(t), t \in [0, t_{1f}^o]$ (77) is depicted in Figure 6a.

Figure 7a shows the near time-optimal process of Problem $\mathrm{P}(2)$ with an accuracy $\varepsilon_r = 1.0e-03$. The green trajectory in Figure 7b outlines the corresponding trajectory in the phase plane x_1x_2 of Problem $\mathrm{P}(2)$. It covers the black phase trajectory obtained under the optimal control $u_1^o(t), t \in [0, t_{1f}^o]$ (77) of the sub-problem Problem $\mathrm{P}(1)$, which is also shown in the previous Figure 6b. The manifold M_2^-, presented in blue, and the manifold M_2^+, presented in red, are also shown in this Figure 7b. The switching curve is $M_2 = M_2^- \cup M_2^+$.



(a) (b)

Figure 6. The trajectory of the system of Problem $\mathrm{P}(\mathbf{2})$ with an initial point $\boldsymbol{x}_0 = (\mathbf{1} \quad \mathbf{1})^T$ obtained under the control $\boldsymbol{u}_1^o(\boldsymbol{t}), \boldsymbol{t} \in [\mathbf{0}, \boldsymbol{t}_{1f}^o]$ (77): (a) the process itself; (b) the trajectory in the phase plane $\boldsymbol{x}_1\boldsymbol{x}_2$ of Problem $\mathrm{P}(\mathbf{2})$.



(a) (b)

Figure 7. The near time-optimal trajectory of the system of Problem P$(\mathbf{2})$ referring to the initial state $\boldsymbol{x_0} = (\mathbf{1} \quad \mathbf{1})^{\boldsymbol{T}}$ with an accuracy of $\boldsymbol{\varepsilon_r} = \mathbf{1.0}\boldsymbol{e} - \mathbf{03}$: (a) the near time-optimal process; (b) the trajectory marked with green in the phase plane $\boldsymbol{x_1 x_2}$ of Problem P$(\mathbf{2})$ alongside with the manifold $\boldsymbol{M_2}$ marked with red and blue.

3. Conclusions

The solution of the time-optimal control problem based on the considered method when the order of the system is higher than two includes generating from the original Problem P(n) the sub-problems Problem P$(n-1)$, …, Problem P(2), Problem P(1). Every single problem of the set "Problem P(n) … Problem P(2)" requires an „axes initialization", in order to determine the respective value of x_{i+} for the considered Problem P(i) and obtaining the set x_{n+}, $x_{(n-1)+}$, …, x_{2+}. A feature here is to obtain this set starting from x_{2+} and by successive ascent reaching the original highest order x_{n+}.

Obtaining by Theorem 2 the optimal control for a given initial state is based on the solution of the corresponding sub-problem. Corollaries 1 and 2 are the theoretical basis for an accelerated solution of all sub-problems of the original Problem P(n) by jumping in certain cases along the optimal trajectories of these sub-problems without the need for internal movement along them. This further reduces the computational load which has already been reduced based on the preliminary knowledge of the corresponding state spaces of Problem P(n) to Problem P(2) through the axes initialization and the subsequent synthesis based on the solution of the generated lower order lighter sub-problem relative to the original one.

References [14–17] of [1] develop a synthesis method for the case of simple non-positive eigenvalues of the system. The new geometric state-space properties obtained here for the classical single-input linear time-optimal control problem with non-positive eigenvalues of the system expand the underlying ideas and fully cover the case of the theorem about the number of switchings or the number of intervals of constancy [9] (Chapter 3, §17, Theorem 10), [13] (Chapter 2, §6, Theorem 2.11, p. 116), [11].

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Conflict Of Interest: The authors declare that they have no conflicts of interest.

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