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Not peer-reviewed version

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Posted Date: 25 September 2024

doi: 10.20944/preprints202409.1972.v1

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Article

Robin's Criterion on Divisibility (II)

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Abstract: Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show that the Robin inequality is true for all natural numbers $n > 5040$ that are not divisible by some prime between 2 and 1771559. We prove that the Robin inequality holds when $\frac{\pi^2}{6} \cdot \log \log n' \leq \log \log n$ for some $n > 5040$ where n' is the square free kernel of the natural number n . The possible smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$, $1 < \frac{(1 + \frac{1.2762}{\log q_m}) \cdot \log(1.006479799241)}{\log \log n} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta_n} < 1.000208229291 \cdot \log(N_m)$ and $n < (1.006479799241)^m \cdot N_m$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta_n = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. By combining these results, we present a proof of the Riemann hypothesis. This work is an expansion and refinement of the article "Robin's criterion on divisibility", published in The Ramanujan Journal.

Keywords: Riemann hypothesis; Robin inequality; Sum-of-divisors function; Prime numbers; Riemann zeta function

1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n :

$$\sum_{d|n} d$$

where $d | n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robin(n) holds provided

$$f(n) < e^\gamma \cdot \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. The following inequality is based on natural logarithms:

Proposition 1. For $t > 0$ [1]:

$$\log\left(1 + \frac{1}{t}\right) < \frac{1}{t}.$$

The Ramanujan's Theorem stated that if the Riemann hypothesis is true, then Robin(n) holds for large enough n [2]. Next, we have the Robin's Theorem:

Proposition 2. Robin(n) holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [3].

It is known that Robin(n) holds for many classes of numbers n . Robin(n) holds for all natural numbers $n > 5040$ that are not divisible by 2 [4]. We extend the indivisibility property on the following result:

Proposition 3. *Robin(n) holds for all natural numbers $n > 5040$ that are not divisible by some prime between 3 and 1771559 [5].*

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Proposition 4. *Robin(n) holds for all natural numbers $n > 5040$ that are square free [4].*

In addition, we show that Robin(n) holds for some $n > 5040$ when $\frac{\pi^2}{6} \cdot \log \log n' \leq \log \log n$ such that n' is the square free kernel of the natural number n [5]. In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [2]. These numbers were also studied by Leonidas Alaoglu and Paul Erdős (1944) [6]. Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [4]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 5. *If n is superabundant, then n is an Hardy-Ramanujan integer [6].*

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \text{ for } (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

Proposition 6. *Every colossally abundant number is superabundant [6].*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis could be false.

Proposition 7. *The smallest counterexample of the Robin inequality greater than 5040 must be a superabundant number [7].*

Suppose that $n > 5040$ is the possible smallest counterexample of the Robin inequality, then we prove that $q_m > e^{31.018189471}$, $1 < \frac{(1 + \frac{1.2762}{\log q_m}) \cdot \log(1.006479799241)}{\log \log n} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta_n} < 1.000208229291 \cdot \log(N_m)$ and $n < (1.006479799241)^m \cdot N_m$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta_n = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ (Refer to preliminary results in Vega's paper [5]).

Proposition 8. *If the Riemann hypothesis is false, then there are infinitely many colossally abundant numbers $n > 5040$ such that Robin(n) fails (i.e. Robin(n) does not hold) [3].*

We can further deduce that

Lemma 1. *If the Riemann hypothesis is false, then there are infinitely many superabundant numbers n such that Robin(n) fails.*

Proof. This is a direct consequence of Propositions 2, 6 and 8. \square

Putting all together yields a proof of the Riemann hypothesis.

2. A Central Lemma

These are known results:

Proposition 9. For $n > 1$ [4]:

$$f(n) < \prod_{q|n} \frac{q}{q-1}.$$

Proposition 10. We have [8]:

$$\prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{q_i^2}} = \zeta(2) = \frac{\pi^2}{6}.$$

The following is a key Lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers n . The bound is too weak to prove $\text{Robin}(n)$ directly, but is critical because it holds for all natural numbers n . Further the bound only uses the primes that divide n and not how many times they divide n .

Lemma 2. Let $n > 1$ and let all its prime divisors be $q_1 < \dots < q_m$. Then,

$$f(n) < \frac{\pi^2}{6} \cdot \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

Proof. Putting together the Propositions 9 and 10 yields the proof:

$$f(n) < \prod_{i=1}^m \left(\frac{q_i}{q_i - 1} \right) = \prod_{i=1}^m \left(\frac{q_i + 1}{q_i} \cdot \frac{1}{1 - \frac{1}{q_i^2}} \right) < \frac{\pi^2}{6} \cdot \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

□

3. Robin on Divisibility

We know the following Propositions:

Proposition 11. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \dots < q_m$, then [9]:

$$\left(\prod_{i=1}^m \frac{q_i}{q_i - 1} \right) < \frac{1771561}{1771560} \cdot e^\gamma \cdot \log \log n.$$

Proposition 12. $\text{Robin}(n)$ holds for all natural numbers $10^{10^{13.11485}} \geq n > 5040$ [10].

Theorem 1. Suppose $n > 5040$. If there exists a prime $q \leq 1771559$ with $q \nmid n$, then $\text{Robin}(n)$ holds.

Proof. We have that $f(n) < \frac{1771561}{1771560} \cdot e^\gamma \cdot \log \log(n)$ for any number $n > 10^{10^{13.11485}}$ since the inequality $10^{10^{13.11485}} > e^{e^{23.762143}}$ is satisfied. Note that $f(n) < \frac{n}{\varphi(n)} = \prod_{q|n} \frac{q}{q-1}$ from the Proposition 9, where

$\varphi(x)$ is the Euler's totient function. Suppose that n is not divisible by some prime $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Then,

$$\begin{aligned} f(n) &< \frac{n}{\varphi(n)} \\ &= \frac{n \cdot q}{\varphi(n \cdot q)} \cdot \frac{q-1}{q} \\ &< \frac{1771561}{1771560} \cdot \frac{q-1}{q} \cdot e^\gamma \cdot \log \log(n \cdot q) \end{aligned}$$

and

$$\begin{aligned} \frac{f(n)}{e^\gamma \cdot \log \log(n)} &< \frac{1771561}{1771560} \cdot \frac{q-1}{q} \cdot \frac{\log \log(n \cdot q)}{\log \log(n)} \\ &= \frac{1771561}{1771560} \cdot \frac{q-1}{q} \cdot \frac{\log \log(n) + \log(1 + \frac{\log(q)}{\log(n)})}{\log \log(n)} \\ &= \frac{1771561}{1771560} \cdot \frac{q-1}{q} \cdot \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log \log(n)} \right) \end{aligned}$$

So

$$\frac{f(n)}{e^\gamma \cdot \log \log(n)} < \frac{1771561}{1771560} \cdot \frac{q-1}{q} \cdot \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log \log(n)} \right)$$

for $n \geq 10^{10^{13.11485}}$. The right hand side is less than 1 for $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Therefore, Robin(n) holds. \square

4. On the Greatest Prime Divisor

We know that

Proposition 13. For $x \geq 2973$ [11]:

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \cdot \left(\log x + \frac{0.2}{\log(x)} \right).$$

Theorem 2. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents a_1, \dots, a_m . If $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $q_m > e^{31.018189471}$.

Proof. According to the Propositions 5 and 7, the primes $q_1 < \dots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the Theorem 1, we know that necessarily $q_m \geq 1771559$. So,

$$e^\gamma \cdot \log \log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^\gamma \cdot \left(\log q_m + \frac{0.2}{\log(q_m)} \right)$$

because of the Propositions 9 and 13. Hence,

$$\log \log n - \frac{0.2}{\log(q_m)} < \log q_m.$$

However, from the Proposition 12 and Theorem 1, we would obtain that

$$\begin{aligned} \log \log n - \frac{0.2}{\log(q_m)} &\geq 13.11485 \cdot \log(10) + \log \log 10 - \frac{0.2}{\log(1771559)} \\ &> 31.018189471. \end{aligned}$$

Since, we have that

$$\log q_m > \log \log n - \frac{0.2}{\log(q_m)} > 31.018189471$$

then, we would obtain that $q_m > e^{31.018189471}$ under the assumption that $n > 5040$ is the smallest integer such that Robin(n) does not hold. \square

5. Some Feasible Cases

We can easily prove that Robin(n) is true for certain kind of numbers:

Lemma 3. Robin(n) holds for $n > 5040$ when $q \leq 7$, where q is the largest prime divisor of n .

Proof. This is an immediate consequence of Theorem 1. \square

The next Theorem implies that Robin(n) holds for a wide range of natural numbers $n > 5040$.

Theorem 3. Let $\frac{\pi^2}{6} \cdot \log \log n' \leq \log \log n$ for some $n > 5040$ such that n' is the square free kernel of the natural number n . Then Robin(n) holds.

Proof. Let n' be the square free kernel of the natural number n , that is the product of the distinct primes q_1, \dots, q_m . By assumption we have that

$$\frac{\pi^2}{6} \cdot \log \log n' \leq \log \log n.$$

For all square free $n' \leq 5040$, Robin(n') holds if and only if $n' \notin \{2, 3, 5, 6, 10, 30\}$ [4]. However, Robin(n) holds for all $n > 5040$ when $n' \in \{2, 3, 5, 6, 10, 15, 30\}$ due to Lemma 3. When $n' > 5040$, we know that Robin(n') holds and so

$$f(n') < e^\gamma \cdot \log \log n'$$

because of the Proposition 4. By the previous Lemma 2:

$$f(n) < \frac{\pi^2}{6} \cdot \prod_{i=1}^m \frac{q_i + 1}{q_i}.$$

So,

$$\begin{aligned} f(n) &< \frac{\pi^2}{6} \cdot \prod_{i=1}^m \frac{q_i + 1}{q_i} \\ &= \frac{\pi^2}{6} \cdot f(n') \\ &< \frac{\pi^2}{6} \cdot e^\gamma \cdot \log \log n' \\ &\leq e^\gamma \cdot \log \log n \end{aligned}$$

according to the formula $f(x)$ for the square free numbers [4]. \square

6. On Possible Counterexample

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{0.2}{\log^2(p_n)}}}{\left(1 - \frac{0.01}{\log^3(p_n)}\right)}$.

Lemma 4. *As the prime number p_n increases, the sequence Y_n is strictly decreasing.*

Proof. This Lemma is obvious. \square

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x . We know that

Proposition 14. *For $x \geq 7232121212$ [12]:*

$$\theta(x) \geq \left(1 - \frac{0.01}{\log^3(x)}\right) \cdot x.$$

Proposition 15. *For $x \geq 2278382$ [12]:*

$$\prod_{q \leq x} \frac{q}{q-1} \leq e^\gamma \cdot \left(\log x + \frac{0.2}{\log^2(x)}\right).$$

We will prove another important inequality:

Lemma 5. *Let q_1, q_2, \dots, q_m denote the first m consecutive primes such that $q_1 < q_2 < \dots < q_m$ and $q_m > 7232121212$. Then*

$$\prod_{i=1}^m \frac{q_i}{q_i-1} \leq e^\gamma \cdot \log(Y_m \cdot \theta(q_m)).$$

Proof. From the Proposition 14, we know that

$$\theta(q_m) \geq \left(1 - \frac{0.01}{\log^3(q_m)}\right) \cdot q_m.$$

In this way, we can show that

$$\begin{aligned} \log(Y_m \cdot \theta(q_m)) &\geq \log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right) \cdot q_m\right) \\ &= \log q_m + \log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right). \end{aligned}$$

We know that

$$\begin{aligned}\log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right) &= \log\left(\frac{e^{\frac{0.2}{\log^2(q_m)}}}{\left(1 - \frac{0.01}{\log^3(q_m)}\right)} \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right) \\ &= \log\left(e^{\frac{0.2}{\log^2(q_m)}}\right) \\ &= \frac{0.2}{\log^2(q_m)}.\end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right) \geq \left(\log q_m + \frac{0.2}{\log^2(q_m)}\right).$$

Due to the Proposition 15, we prove that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \cdot \left(\log q_m + \frac{0.2}{\log^2(q_m)}\right) \leq e^\gamma \cdot \log(Y_m \cdot \theta(q_m))$$

when $q_m > 7232121212$. \square

We use the following Proposition:

Proposition 16. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents a_1, \dots, a_m . Then [9]:

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

The following Theorems have a great significance, because these mean that the possible smallest counterexample of the Robin inequality greater than 5040 must be very close to its square free kernel.

Theorem 4. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents a_1, \dots, a_m . If $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $(\log n)^{\beta_n} \leq Y_m \cdot \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta_n = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$.

Proof. According to the Propositions 5 and 7, the primes $q_1 < \dots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the Theorem 2, we know that necessarily $q_m > e^{31.018189471}$. From the Proposition 16, we note that

$$f(n) = \left(\prod_{i=1}^m \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

However, we know that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \cdot \log(Y_m \cdot \log(N_m))$$

because of the Lemma 5 when $q_m > 7232121212$. If we multiply by $\prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right)$ the both sides of the previous inequality, then we obtain that

$$f(n) \leq e^\gamma \cdot \log(Y_m \cdot \log(N_m)) \cdot \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

If n is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^\gamma \cdot \log \log n \leq e^\gamma \cdot \log(Y_m \cdot \log(N_m)) \cdot \prod_{i=1}^m \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

because of

$$e^\gamma \cdot \log \log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \cdot \log \log n \leq \log(Y_m \cdot \log(N_m)) \quad (1)$$

which is equivalent to

$$(\log n)^{\beta_n} \leq Y_m \cdot \log(N_m)$$

where $\beta_n = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$. Therefore, the proof is done. \square

Theorem 5. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents a_1, \dots, a_m . If $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $(\log n)^{\beta_n} < 1.000208229291 \cdot \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta_n = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$.

Proof. From the Theorem 2, we know that necessarily $q_m > e^{31.018189471}$. Using the Theorem 4, we obtain that

$$(\log n)^{\beta_n} < 1.000208229291 \cdot \log(N_m)$$

due to Lemma 4 since $Y_m < 1.000208229291$ whenever $q_m > e^{31.018189471}$. \square

Theorem 6. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents a_1, \dots, a_m . If $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $n < (1.006479799241)^m \cdot N_m$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m .

Proof. According to the Propositions 5 and 7, the primes $q_1 < \dots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the Lemma 5, we know that

$$\prod_{i=1}^m \frac{q_i}{q_i - 1} \leq e^\gamma \cdot \log(Y_m \cdot \theta(q_m)) = e^\gamma \cdot \log \log(N_m^{Y_m})$$

for $q_m > 7232121212$. In this way, if $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $n < N_m^{Y_m}$ since by the Proposition 9 we have that

$$e^\gamma \cdot \log \log n \leq f(n) < \prod_{i=1}^m \frac{q_i}{q_i - 1}.$$

That is the same as $n < N_m^{Y_m-1} \cdot N_m$. We can check that $q_m^{Y_m-1}$ is monotonically decreasing for all primes $q_m > e^{31.018189471}$. Certainly, the derivative of the function

$$g(x) = x \left(\frac{\frac{0.2}{e^{\log^2(x)}}}{\left(1 - \frac{0.01}{\log^3(x)}\right) - 1} \right)$$

is less than zero for all real numbers $x \geq e^{31.018189471}$. Consequently, we would have that

$$q_m^{Y_m-1} < g(e^{31.018189471}) < 1.006479799241$$

for all primes $q_m > e^{31.018189471}$. Moreover, we would obtain that

$$q_m^{Y_m-1} > q_j^{Y_m-1}$$

for every integer $1 \leq j < m$. Finally, we can state that $n < (1.006479799241)^m \cdot N_m$ since $N_m^{Y_m-1} < (1.006479799241)^m$ when $n > 5040$ is the smallest integer such that Robin(n) does not hold. \square

We know the following results:

Proposition 17. For $x > 1$ [13]:

$$\pi(x) \leq \left(1 + \frac{1.2762}{\log x}\right) \cdot \frac{x}{\log x}$$

where $\pi(x)$ is the prime counting function.

Proposition 18. If $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $p < \log n$ where p is the largest prime divisor of n [4].

Theorem 7. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_m$ with natural numbers as exponents a_1, \dots, a_m . If $n > 5040$ is the smallest integer such that Robin(n) does not hold, then $1 < \frac{(1 + \frac{1.2762}{\log q_m}) \cdot \log(1.006479799241)}{\log \log n} + \frac{\log N_m}{\log n}$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m .

Proof. Note that $n < (1.006479799241)^m \cdot N_m$ when n is the smallest integer such that Robin(n) does not hold. If we apply the logarithm to the both sides, then

$$\log n < m \cdot \log(1.006479799241) + \log N_m.$$

According to the Proposition 17, we have that

$$\log n < \left(1 + \frac{1.2762}{\log q_m}\right) \cdot \frac{q_m}{\log q_m} \cdot \log(1.006479799241) + \log N_m.$$

From the Proposition 18, we would have

$$\log n < \left(1 + \frac{1.2762}{\log q_m}\right) \cdot \frac{\log n}{\log \log n} \cdot \log(1.006479799241) + \log N_m.$$

which is the same as

$$1 < \frac{\left(1 + \frac{1.2762}{\log q_m}\right) \cdot \log(1.006479799241)}{\log \log n} + \frac{\log N_m}{\log n}$$

after of dividing by $\log n$. \square

7. A Conclusive Approach

We use the following results:

Lemma 6. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of a superabundant number $n > 5040$ as the product of the first m consecutive primes $q_1 < \dots < q_m$ with the natural numbers $a_1 \geq a_2 \geq \dots \geq a_m \geq 1$ as exponents. Suppose that $\text{Robin}(n)$ fails. Then,

$$\beta_n \leq \frac{\log \log(N_m)^{Y_m}}{\log \log n},$$

where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta_n = \prod_{i=1}^m \left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \right)$.

Proof. Using the inequality (1) and Lemma 1, then this result will be a generalization of Theorem 4 for every possible counterexample $n > 5040$ of the Robin's inequality. \square

Proposition 19. Let $x \geq 11$. For $y > x$ [14]:

$$\frac{\log \log y}{\log \log x} < \sqrt{\frac{y}{x}}.$$

This is the main insight.

Lemma 7. Let $\prod_{i=1}^m q_i^{a_i}$ be the representation of a superabundant number $n > 5040$ as the product of the first m consecutive primes $q_1 < \dots < q_m$ with the natural numbers $a_1 \geq a_2 \geq \dots \geq a_m \geq 1$ as exponents. Suppose that $\text{Robin}(n)$ fails. Then,

$$\beta_n < \sqrt{\frac{(N_m)^{Y_m}}{n}},$$

where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and $\beta_n = \prod_{i=1}^m \left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \right)$.

Proof. When $n > 5040$ is a superabundant number and $\text{Robin}(n)$ fails, then we have

$$\beta_n \leq \frac{\log \log(N_m)^{Y_m}}{\log \log n}$$

by Lemma 6. We assume that $(N_m)^{Y_m} > n > 5040 > 11$ since $\beta_n > 1$. Consequently,

$$\frac{\log \log(N_m)^{Y_m}}{\log \log n} < \sqrt{\frac{(N_m)^{Y_m}}{n}}$$

by Proposition 19. As result, we obtain that

$$\beta_n < \sqrt{\frac{(N_m)^{Y_m}}{n}}.$$

\square

In number theory, the p -adic order of an integer n is the exponent of the highest power of the prime number p that divides n . It is denoted $v_p(n)$. Equivalently, $v_p(n)$ is the exponent to which p appears in the prime factorization of n . This is the main Theorem.

Theorem 8. The Riemann hypothesis is true.

Proof. Under the assumption that the Riemann hypothesis is false, then there would exist infinitely many superabundant numbers n such that $\text{Robin}(n)$ fails according to Lemma 1. Let $n_m > 5040$ be a large enough superabundant number (as larger as we want) such that q_m is the largest prime factor of n_m . Suppose that $\text{Robin}(n_m)$ fails. This implies that $q_m > e^{31.018189471}$ by Theorem 2. Let $n_{m'} > 5040$ be another large enough superabundant number (as larger as we want) such that $n_{m'} > n_m$. Suppose that $\text{Robin}(n_{m'})$ fails too. By Lemma 7, we have

$$\beta_{n_m} < \sqrt{\frac{(N_m)^{Y_m}}{n_m}}, \quad \beta_{n_{m'}} < \sqrt{\frac{(N_{m'})^{Y_{m'}}}{n_{m'}}}.$$

So, we would have

$$\beta_{n_{m'}} \cdot \beta_{n_m} < \sqrt{\frac{(N_{m'})^{Y_{m'}}}{n_{m'}}} \cdot \beta_{n_m}.$$

Consequently, we get:

$$\beta_{n_{m'}} \cdot \beta_{n_m} < \sqrt{\frac{(N_{m'})^{Y_{m'}}}{n_{m'}}} \cdot \sqrt{\frac{(N_m)^{Y_m}}{n_m}}.$$

We arrive at:

$$(\beta_{n_{m'}} \cdot \beta_{n_m})^2 < \frac{(N_{m'})^{Y_{m'}}}{n_{m'}} \cdot \frac{(N_m)^{Y_m}}{n_m}.$$

We can see that

$$(\beta_{n_{m'}} \cdot \beta_{n_m})^2 > 1.$$

However, we claim that

$$\frac{(N_{m'})^{Y_{m'}}}{n_{m'}} \cdot \frac{(N_m)^{Y_m}}{n_m} \leq 1$$

which is

$$\log Y_m \leq \log \left(\frac{\log(n_{m'} \cdot n_m)}{\log((N_{m'})^{\frac{Y_{m'}}}{Y_m} \cdot N_m)} \right).$$

By Proposition 1, we obtain that

$$\begin{aligned} \log Y_m &= \frac{0.2}{\log^2(q_m)} + \log \left(\frac{\log^3(q_m)}{\log^3(q_m) - 0.01} \right) \\ &= \frac{0.2}{\log^2(q_m)} + \log \left(1 + \frac{0.01}{\log^3(q_m) - 0.01} \right) \\ &< \frac{0.2}{\log^2(q_m)} + \frac{0.01}{\log^3(q_m) - 0.01} \end{aligned}$$

for all $q_m > e^{31.018189471}$. Hence, it is enough to show that

$$\frac{0.2}{\log^2(q_m)} + \frac{0.01}{\log^3(q_m) - 0.01} \leq \log \left(\frac{\log(n_{m'} \cdot n_m)}{\log((N_{m'})^{\frac{Y_{m'}}}{Y_m} \cdot N_m)} \right)$$

which is trivially true under the assumption that

$$\log^2(q_m) \cdot \log \left(\frac{\log(n_{m'} \cdot n_m)}{\log((N_{m'})^{\frac{Y_{m'}}}{Y_m} \cdot N_m)} \right) \geq 0.200322393$$

as a consequence of

$$0.200322393 > 0.2 + \frac{0.01 \cdot \log^2(q_m)}{\log^3(q_m) - 0.01} > \log^2(q_m) \cdot \log Y_m$$

for all $q_m > e^{31.018189471}$. In this way, we reach the contradiction $1 < 1$ under the assumption that Robin(n_m) fails. This is supported by the fact that Y_m is strictly decreasing (i.e. $\frac{Y_{m'}}{Y_m} < 1$ if $m' > m$), $\lim_{m' \rightarrow \infty} Y_{m'} = 1$, and we can always be able to take both superabundant numbers n_m and $n_{m'} > n_m$ as larger as we want. Furthermore, for every fixed prime q , $\nu_q(n)$ goes to infinity as long as n goes to infinity whenever n is superabundant [6,14]. For that reason, we can definitely assure that the inequalities

$$n_{m'} > n_m, \quad \log^2(q_m) \cdot \log \left(\frac{\log(n_{m'} \cdot n_m)}{\log((N_{m'})^{\frac{Y_{m'}}{Y_m}} \cdot N_m)} \right) \geq 0.200322393$$

can simultaneously hold for every superabundant number $n_{m'}$ greater than some threshold. Accordingly, Robin(n) holds for all large enough superabundant numbers $n > 5040$. By Lemma 1, this contradicts the fact that there exist infinitely many superabundant numbers n , such that Robin(n) fails if the Riemann hypothesis were false. By reductio ad absurdum, we prove that the Riemann hypothesis is true. \square

Acknowledgments: Many thanks to Michel Planat, Patrick Solé and Richard J. Lipton for their support.

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