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Leader-Following Output Feedback H_∞ Consensus of Fractional-Order Multi-Agent Systems with Input Saturation

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Abstract: This paper investigates the leader-following H_∞ consensus of fractional-order multi-agent systems (FOMASs) under input saturation via the output feedback. Based on the bounded real lemma for FOSs, the sufficient conditions of H_∞ consensus for FOMASs are provided in $\alpha \in (0,1)$ and $[1,2)$, respectively. Furthermore, the iterative linear matrix inequalities (ILMIs) approaches are applied for solving quadratic matrix inequalities (QMIs). The ILMI algorithms show a method to derive initial values and transform QMIs into LMIs. Mathematical tools are employed to transform the input saturation issue into optimal solution of LMIs for estimating stable regions. The ILMI algorithms avoid the conditional constraints on matrix variables during the LMIs construction and reduce conservatism. The approach does not disassemble the entire MASs by transformations to the Laplacian matrix, instead adopting a holistic analytical perspective to obtain gain matrices. In the end, numerical examples are conducted to validate the efficiency of the approach.

Keywords: fractional-order multi-agent systems; H_∞ control; static output feedback; iterative linear matrix inequality

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1. Introduction

In recent decades, multi-agent systems (MASs) have been the subject of extensive research, resulting in substantial achievements on the field. MASs are utilized in many practical domains, including traffic management [1] and power systems [2,3]. These applications rely significantly on the consensus of MASs, which involves the states of agents converging to a desired state by the neighboring information [4]. Extensive studies are conducted on the consensus for MASs with integer-order differential models. However, integer-order models are inadequate for accurately representing some non-classical phenomena in various physical systems.

As the advancement of fractional calculus theory, fractional-order systems (FOSs) and fractional-order MASs (FOMASs) have emerged as a significant direction. Many achievements have been made in integer-order systems [5–7], but the utilization of fractional derivatives allows for a more comprehensive understanding of the characteristics of materials and systems exhibiting power-law, nonlocal, or long-term memory. FOSs offer enhanced capabilities for modelling and analysing complex systems, e.g., electrical systems [8,9], economic systems [10], motion models [11], and biological models [12,13]. The stability of control systems is fundamental problem, certainly including those involving FOSs. It is not possible to derive the stability criteria of FOSs directly from those of integer-order systems. Fortunately, in [14], authors establish the LMI-based stability criteria and design a method for robust feedback state stabilization control for commensurate FOSs with $\alpha \in (0,1)$ and $\alpha \in [1,2)$. In [15], a necessary and sufficient condition in unified LMIs formulation is provided to ensure the stability of FOSs within $\alpha \in (0,2)$. The aforementioned study makes a contribution to the consensus problem of FOMASs. In [16], the consensus of FOMASs with time delay is studied. Using the Lyapunov method, some consensus criteria are proposed to guarantee the consensus. Focusing on singular systems, the authors in [17] investigate the observer-based consensus problem of singular

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FOMASs. Then, the paper provides the corresponding control protocol and the calculation method of gain matrices. In [18], the distributed fixed-time consensus for FOMASs with a dynamic virtual leader under external disturbances is investigated, and a sliding-mode control protocol is designed. Meanwhile, the singular perturbation FOMASs are modeled and studied in [19], which provides a sufficient condition for consensus. Nevertheless, the consensus problem of FOMASs in these results is all based on state feedback. In most cases, only the measurement output of the agents is available, which indicates that the above method has certain limitations.

In comparison to the state feedback, the control via output feedback is a challenging problem, largely due to the affect of measurement matrices. Furthermore, the consensus of FOMASs via output feedback is a highly intricate issue. The stability criteria of FOSs with $\alpha \in (0,1)$ comprise two matrix variables, whereas in $\alpha \in [1,2)$, the matrix variable is related to the order number. In [20–23], static output feedback control is employed for FOSs, and a matrix exchange condition is utilized to integrate matrix variable with gain matrices. This approach utilizes singular value decomposition (SVD) of the measurement matrices. Nevertheless, this approach imposes constraints on the form of matrix variables and tends to be conservative. Authors in [24] also adopt some strong assumptions to get a feasible solution, which brings conservatism. Similarly, in the output feedback consensus of FOMASs, the limitation often exist. In [20], sufficient conditions are provided for the leader-following consensus of singular FOMASs with $\alpha \in (0,2)$. The authors in [25] also do the analogous study, and the SVD method is always indispensable. To complicate matters, disturbances persist in actual systems, and the output information transmitted between the agents contains the disturbances.

For FOMASs, the H_∞ control method provides substantial benefits in addressing system uncertainty, robustness optimization, and other pertinent issues. The paper [26] derives the bounded real lemma for FOSs and establishes the foundation for H_∞ control. In [27], a proposal is made to extend the application of H_∞ control method from integer-order systems to FOSs. Robust fault-tolerant H_∞ control for FOSs with actuator faults and uncertainties is addressed in [28] through the design of an output feedback controller. For singular FOSs, a state feedback control strategy is presented that guarantees the prescribed H_∞ performance in [29]. These works form the foundation o the H_∞ consensus of FOMASs. In [30], the admissible consensus of fuzzy singular FOMASs is considered, a sufficient condition for the system achieving admissible consensus while satisfying H_∞ performance. The paper [31] investigates the H_∞ consensus problem for discrete-time FOMASs. However, the relatively research is little. The output feedback H_∞ consensus remains a challenging field. Meanwhile, the control input saturation is a common feature of practical engineering systems, due to physical limitations [32–34]. This renders the output feedback consensus for FOMASs a complex process.

The discussion provides the impetus for the ILMI algorithms towards the leader-following H_∞ consensus of FOMASs with input saturation via output feedback. The contributions are as following:

(i) Based on the real bound lemma of FOSs, sufficient conditions for output feedback H_∞ consensus of FOMASs in $\alpha \in (0,1)$ and $[1,2)$ are provided. The proposed method adopts a holistic analytical perspective to the entire system, which differs from the decomposition of the entire MASs.

(ii) For solving the QMIs, the ILMI algorithms are provided, which propose a calculation method for initial values. Based on the stability region of FOSs, the iterative condition are designed to guarantee the consensus condition of FOMASs. This paper delves deeper into the issue of the input saturation, which is reframed as an LMI-based optimisation problem. The ILMI algorithms circumvent the necessity for matrix exchange conditions from the SVD method. No strong assumptions are required for feasible solutions, and it reduces the conservatism.

Notations: Given a matrix A, sym(A) denotes $A^T + A$, where A^T is the transpose of A, and her$(A) = A^* + A$, where A^* is conjugate transpose of matrix A. $A > 0$ and $A < 0$ represent that A is positive definite and negative definite, respectively. Then $A \geqslant 0$ denotes that A is negative semi-definite and symbol $\star$ stands for the symmetric part. The Kronecker product is represented by $\otimes$. arg$(\cdot)$ denotes the argument of complex numbers, and spec(A) indicates the spectrum of matrix

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A. $\mathbb{R}^{n\times m}$ stands for sets of $n \times m$ real matrices. diag$(\cdot)$ represents a diagonal matrix. With $\alpha \in (0,2)$, denote $a = \sin\left(\alpha\frac{\pi}{2}\right)$, $b = \cos\left(\alpha\frac{\pi}{2}\right)$.

2. Problem Formulation and Preliminaries

An MAS consists of $\mathcal{N}$ followers represented by the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, \mathcal{N}\}$ presents a set of $\mathcal{N}$ followers, and $\mathcal{E} = \{(i,j) \colon i, j \in \mathcal{V}\}$ is the edge set. The adjacency matrix is $\mathcal{A} = [a_{ij}]_{\mathcal{N}\times\mathcal{N}}$, and if $(i,j) \in \mathcal{E}$, $a_{ij} > 0$, otherwise $a_{ij} = 0$.

Furthermore, denote the communication graph between the leader and followers as $\widetilde{\mathcal{G}}$. Define diag$(h_1, h_2, \cdots, h_{\mathcal{N}})$ for representing the communication. If follower i receives the information from the leader, then $h_i > 0$, otherwise $h_i = 0$. The Laplacian matrix is defined as $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, where

$$l_{ij} = \begin{cases} -a_{ij}, i = j, \\ \sum_{j=1}^{\mathcal{N}} a_{ij}, i \neq j. \end{cases}$$

Consider the FOMAS under actuator saturation, and the $\mathcal{N}$ followers are described by

$$\begin{cases} D^{\alpha} x_i(t) = A x_i(t) + B\text{sat}(u_i(t)) + D_1 w_i(t), \\ z_i(t) = C x_i(t) + D_2 w_i(t), \\ y_i(t) = C_y x_i(t) + D_3 w_i(t), \end{cases} \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{n_u}$, $y_i(t) \in \mathbb{R}^m$, $z_i(t) \in \mathbb{R}^{n_z}$, and $w_i(t) \in \mathbb{R}^{n_w}$ denote the state, control input, measured output, controlled output, and disturbances, respectively; A, B, D_1, D_2, D_3, C, and C_y are constant real matrices; D^{α} represents the Caputo fractional derivative of $f(t)$ as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(\lceil\alpha\rceil - \alpha)} = \int_0^t \frac{f^{(\lceil\alpha\rceil)}(\tau)}{(t-\tau)^{\alpha+1-\lceil\alpha\rceil}} d\tau,$$

and $\Gamma(\cdot)$ is the Euler Gamma function; the saturation function sat$(\cdot)$ for $u(t)$ is denoted as

$$\begin{aligned} &\text{sat}(u) = [\text{sat}(u_1) \ \cdots \ \text{sat}(u_{n_u})]^T, \\ &\text{sat}(u_s) = \text{sign}(u_s)\min\{|u_s|, 1\}, s = 1, \cdots, n_u. \end{aligned}$$

The leader is described by

$$\begin{cases} D^{\alpha} x_0(t) = A x_0(t), \\ y_0(t) = C_y x_0(t), \end{cases} \tag{2}$$

where $x_0(t) \in \mathbb{R}^n$ is the state, and $y_0(t) \in \mathbb{R}^n$ is the output.

Definition 1 ([35]). *The leader-following consensus of the* FOMAS *in (1) and (2) is achieved if*

$$\lim_{t\to\infty} \|x_i(t) - x_0(t)\| = 0, \ i = 1, 2, \cdots, \mathcal{N}.$$

For achieving consensus of the FOMAS in (1) and (2), a distributed consensus protocol is carried out by

$$u_i(t) = K \sum_{j=1}^{\mathcal{N}} a_{ij}\big(y_i(t) - y_j(t)\big) + h_i\big(y_i(t) - y_0(t)\big) \text{ , } i = 1, 2, \cdots, \mathcal{N}.$$

The initial values are considered as $x_i(0) \in \mathbb{R}^n$. Denote the state trajectory as $T(x_i(0), t)$, and the domain of attraction is

$$\mathbb{D} = \left\{ x_i(0) : \lim_{t\to\infty} T(x_i(0), t) = 0 \right\}.$$

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For $F \in \mathbb{R}^{n_u \times m}$, let

$$\mathcal{L}(F) = \{x(t) \in \mathbb{R}^n : |f_s C_y x(t)| \leqslant 1, s = 1, \cdots, n_u\},$$

where f_s denotes the s-th row of the matrix F; $\mathcal{L}(F)$ denotes the region where $FC_y x(t)$ does not saturate.

Lemma 1 ([17,36]). *Denote K, $F \in \mathbb{R}^{n_u \times m}$, then for any $y(t) \in \mathcal{L}(F)$, there is*

$$\text{sat}(Ky(t)) \in \text{co}\{\Gamma_l Ky(t) + \Gamma_l^- Fy(t),\ l = 1, \cdots, 2^{n_u}\},$$

where co$\{\cdot\}$ indicates a convex hull; Γ_l are diagonal matrices, whose diagonal elements are 0 or 1; Γ_l^- are set as $\Gamma_l^- = 1 - \Gamma_l$. Further, the input saturation is written as

$$\text{sat}(Ky(t)) = \sum_{l=1}^{2^{n_u}} h_l(\Gamma_l K + \Gamma_l^- F)y(t), \tag{3}$$

where $h_l \geqslant 0$ and $\sum_{l=1}^{2^{n_u}} h_l = 1$.

With the consensus protocol, the FOMAS in (1) and (2) is written as

$$\begin{cases} D^\alpha x(t) = (I_\mathcal{N} \otimes A + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes C_y))x(t) + (I_\mathcal{N} \otimes D_1 + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes D_3))w(t), \\ z(t) = (I_\mathcal{N} \otimes C)x(t) + (I_\mathcal{N} \otimes D_2)w(t), \end{cases} \tag{4}$$

where
$x(t) = \begin{bmatrix} x_1^T(t) & \cdots & x_\mathcal{N}^T(t) \end{bmatrix}^T$, $z(t) = \begin{bmatrix} z_1^T(t) & \cdots & z_\mathcal{N}^T(t) \end{bmatrix}^T$, $w(t) = \begin{bmatrix} w_1^T(t) & \cdots & w_\mathcal{N}^T(t) \end{bmatrix}^T$, $\tilde{K} = I_\mathcal{N} \otimes \sum_{l=1}^{2^{n_u}} h_l(\Gamma_l K + \Gamma_l^- F)$, $\mathcal{H} = \mathcal{L} + \text{diag}(h_1, h_2, \cdots, h_\mathcal{N})$.

Lemma 2 ([37]). *Consider the FOS as*

$$\begin{cases} D^\alpha x(t) = Ax(t) + B\text{sat}(u(t)), \\ y(t) = C_y x(t), \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, and $y(t) \in \mathbb{R}^{n_u}$; $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, and $C_y \in \mathbb{R}^{m \times n}$. With $u(t) = 0$, the system in (5) reduces to

$$D^\alpha x(t) = Ax(t). \tag{6}$$

The system in (6) is stable if and only if $|\arg(\text{spec}(A))| > \alpha\frac{\pi}{2}$.

Remark 1. Based on Lemma 2, if the eigenvalues of A are in the region of eigenvalues shown in Figure 1, it is obvious that the system in (6) is stable. Meanwhile, the system in (6) is also stable even if the eigenvalues of A are moved by $\frac{\tau}{2}$ unit in the positive direction of the x-axis, where $\tau > 0$. Thus, the presence of eigenvalues of A in a specific region of Figure 1 serves as a sufficient condition for the system stability.

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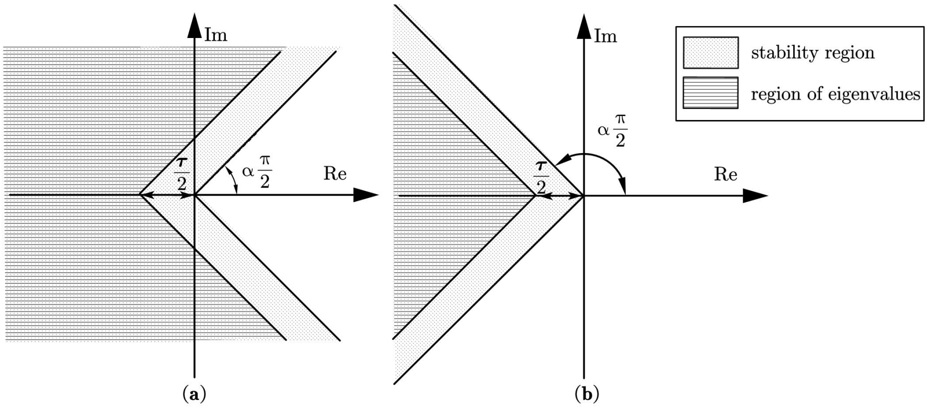

Figure 1. Stability region of the system in (6) and region of spec(A) : **(a)** $\alpha \in (0,1)$ and **(b)** $\alpha \in [1,2)$.

Lemma 3 ([38]). *Via $u(t) = KC_y x(t)$, if the system in (5) satisfies the stability condition in Lemma 2, i.e., $\left|\arg\left(\text{spec}\left(A + B\text{sat}(KC_y)\right)\right)\right| > \alpha\frac{\pi}{2}$, then the system in (5) is asymptotically stable for $x(0) \in \mathcal{B}_\sigma = \{x(t) \in \mathbb{R}^n : x^T x \leqslant \sigma\} \subset \mathcal{L}(F)$.*

Definition 2 ([39]). *Define the H_∞ norm of transfer function $G(s)$ as $\|G(s)\|_\infty = \sup\limits_{\text{Re}(s)\geqslant 0} \sigma_{\max}(G(s))$, where $\sigma_{\max}(\cdot)$ is the maximum singular value of a matrix.*

Lemma 4 ([26]). *Consider the FOS as*

$$\begin{cases} D^\alpha x(t) = Ax(t) + D_1 w(t), \\ z(t) = Cx(t) + D_2 w(t), \end{cases} \tag{7}$$

with the transfer function $G_{wz}(s) = C(s^\alpha I - A)^{-1} D_1 + D_2$, where $x(t) \in \mathbb{R}^n$ and $w(t) \in \mathbb{R}^{n_w}$. Given a scalar $\gamma > 0$, the system in (7) is stable and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if
1) for the case $\alpha \in (0,1)$: There exist $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} > 0, \tag{8}$$

$$\begin{bmatrix} \text{sym}(A(aX + bY)) & (aX + bY)^T C^T & D_1 \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{9}$$

2) for the case $\alpha \in [1,2)$: There exists $X \in \mathbb{R}^{n\times n} > 0$ such that

$$\begin{bmatrix} \text{her}(rAX) & \bar{r}C^T X & D_1 \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{10}$$

where $r = e^{\mathrm{j}(1-\alpha)\frac{\pi}{2}} = a + \mathrm{j}b$.

Lemma 5 ([40]). *$X + Y\mathrm{j} > 0$ is equivalent to (8).*

3. Main Results

This section introduces the iterative algorithms for solving H_∞ consensus of FOMASs via output feedback.

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3.1. ILMI Algorithm for Output Feedback H_∞ Consensus with $\alpha \in (0,1)$

In this subsection , it is essential to reference the following lemmas, which form the basis of the ILMI algorithms. To facilitate the derivation process, the matrices are expressed as follows:

$$P_\alpha = \begin{bmatrix} aX+bY & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \tilde{A} = \begin{bmatrix} I_{\mathcal{N}} \otimes A & I_{\mathcal{N}} \otimes D_1 & 0 \\ 0 & -\frac{1}{2}\gamma I & 0 \\ I_{\mathcal{N}} \otimes C & I_{\mathcal{N}} \otimes D_2 & -\frac{1}{2}\gamma I \end{bmatrix}, \tilde{B} = \begin{bmatrix} \mathcal{H} \otimes B \\ 0 \\ 0 \end{bmatrix}, \tag{11}$$
$$\tilde{C} = \begin{bmatrix} I_{\mathcal{N}} \otimes C_y & I_{\mathcal{N}} \otimes D_3 & 0 \end{bmatrix}.$$

Lemma 6. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $K \in \mathbb{R}^{n_u \times m}$, and $F \in \mathbb{R}^{n_u \times m}$ such that (8) and*

$$(\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T P_\alpha + P_\alpha^T (\tilde{A} + \tilde{B}\tilde{K}\tilde{C}) < 0, \tag{12}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11) and $\tilde{K}$ is from (4).

Proof. In the case of $\alpha \in (0,1)$ of Lemma 4, consider matrices $X_0 > 0$, $Y_0 = -Y_0^T$ such that

$$\begin{bmatrix} \mathrm{sym}(A(aX_0+bY_0)) & (aX_0+bY_0)^T C^T & D_1 \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{13}$$

Then introduce a congruence transformation. Pre- and post- multiplying (13) by

$$\begin{bmatrix} (aX_0+bY_0)^{-T} & 0 & 0 \\ \star & 0 & I \\ \star & \star & 0 \end{bmatrix},$$

and its transpose, one obtains

$$\begin{bmatrix} \mathrm{sym}(A^T(aX_0+bY_0)^{-1}) & (aX_0+bY_0)^{-T} D_1 & C^T \\ \star & -\gamma I & D_2^T \\ \star & \star & -\gamma I \end{bmatrix} < 0.$$

Meanwhile, there exist matrices $X = X^T$ and $Y = -Y^T$ satisfy

$$(aX_0+bY_0)^{-1} = aX + bY. \tag{14}$$

To verify (14), only set X and Y as

$$X = \frac{(aX_0+bY_0)^{-1} + (aX_0-bY_0)^{-1}}{2a}, \; Y = \frac{(aX_0+bY_0)^{-1} - (aX_0-bY_0)^{-1}}{2b},$$

which also satisfy $X = X^T$ and $Y = -Y^T$. It is easy to see $(aX_0+bY_0)(aX+bY) = I$, which proves (14). Then (13) is transformed into

$$\begin{bmatrix} \mathrm{sym}(A^T(aX+bY)) & (aX+bY)^T D_1 & C^T \\ \star & -\gamma I & D_2^T \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{15}$$

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And the positive definite property of $\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$ is proved by pre- and post- multiplying it by $I_2 \otimes (aX_0 + bY_0)$.

Apply (15) in the consensus problem of the FOMAS in (4) via output feedback. Then, the consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed with $\alpha \in (0,1)$, if there exist X, Y, K, and F such that (8) and

$$\begin{bmatrix} \mathrm{sym}(A_{\tilde{K}}{}^{T(aX+bY)}) & (aX+bY)^T D_{\tilde{K}} & (I_\mathcal{N} \otimes C_y)^T \\ \star & -\gamma I & (I_\mathcal{N} \otimes D_2)^T \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{16}$$

where

$$A_{\tilde{K}} = I_\mathcal{N} \otimes A + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes C_y),\ D_{\tilde{K}} = I_\mathcal{N} \otimes D_1 + (\mathcal{H} \otimes B)\tilde{K}(I_\mathcal{N} \otimes D_3). \tag{17}$$

Then, using basic matrix operations, one simplifies inequality (16) into (12). □

Lemma 7. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $K \in \mathbb{R}^{n_u \times m}$, and $F \in \mathbb{R}^{n_u \times m}$ such that (8) and*

$$\mathrm{sym}(\tilde{A}^T P_\alpha) - P_\alpha^T \tilde{B}\tilde{B}^T P_\alpha + (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C}) < 0, \tag{18}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11) and $\tilde{K}$ is from (4).

Proof. With matrix transformation, (18) is rewritten as

$$(\tilde{A} + \tilde{B}\tilde{K}\tilde{C})P_\alpha + P_\alpha^T(\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T + \tilde{C}^T\tilde{K}^T\tilde{K}\tilde{C} < 0.$$

As $\tilde{C}^T\tilde{K}^T\tilde{K}\tilde{C} \geqslant 0$ is obvious, it is readily apparent that (12) holds. Based on Lemma 7, the H_∞ consensus is achieved without input saturation. □

Lemma 8. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n}$, $K \in \mathbb{R}^{n_u \times m}$, $F \in \mathbb{R}^{n_u \times m}$, and S with appropriate dimension such that (8) and*

$$\begin{bmatrix} \mathrm{sym}(\tilde{A}^T P_\alpha - S^T\tilde{B}\tilde{B}^T P_\alpha) + S^T\tilde{B}\tilde{B}^T S & (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T \\ \star & -I \end{bmatrix} < 0, \tag{19}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11) and $\tilde{K}$ is from (4).

Proof. For matrices S and P_α, $(S - P_\alpha)^T\tilde{B}\tilde{B}^T(S - P_\alpha) \geqslant 0$ always holds, and this inequality is written as

$$S^T\tilde{B}\tilde{B}^T P_\alpha + P_\alpha^T\tilde{B}\tilde{B}^T S - S^T\tilde{B}\tilde{B}^T S - P_\alpha^T\tilde{B}\tilde{B}^T P_\alpha \leqslant 0. \tag{20}$$

Using the Schur complement on (19), one obtains

$$\mathrm{sym}(\tilde{A}^T P_\alpha - S^T\tilde{B}\tilde{B}^T P_\alpha) + S^T\tilde{B}\tilde{B}^T S + (\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T(\tilde{B}^T P_\alpha + \tilde{K}\tilde{C}) < 0. \tag{21}$$

Combining (20) and (21) yields (18). With the sufficient condition from Lemma 7, this completes the proof. □

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Lemma 9. *With $\alpha \in (0,1)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed for any initial value $x_i(0) \in B_\varepsilon \subset \mathcal{L}(F)$, if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n}$, $Y \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n}$, $K \in \mathbb{R}^{n_u\times m}$, $F \in \mathbb{R}^{n_u\times m}$, S with appropriate dimension, and a constant $\varepsilon > 0$ such that (8) and*

$$\begin{bmatrix} \mathrm{sym}(\tilde{A}^T P_\alpha - S^T\tilde{B}\tilde{B}^T P_\alpha) + S^T\tilde{B}\tilde{B}^T S & (\tilde{B}^T P_\alpha + I_{\mathcal{N}} \otimes (\Gamma_l K + \Gamma_l^- F)\tilde{C})^T \\ \star & -I \end{bmatrix} < 0,\ l = 1,\dots,2^{n_u}, \tag{22}$$

$$\begin{bmatrix} \frac{1}{\varepsilon} I & C_y^T f_s^T \\ f_s C_y & 1 \end{bmatrix} \geqslant 0,\ s = 1,\cdots,n_u. \tag{23}$$

where P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11).

Proof. Since $h_l \geqslant 0$, (22) holds, then (19) is guaranteed. Based on Lemma 8, if (19), (8), and (23) hold, the FOMAS in (4) achieves consensus without input saturation.

Then consider that input is limited. According to the statement in Lemma 3, the FOMAS achieves consensus for $x(0) \in \mathcal{B}_\sigma \subset \mathcal{L}(F)$. From (23), using Schur complement, one obtains

$$\frac{1}{\varepsilon} I \geqslant C_y^T f_s^T f_s C_y,\ s = 1,\cdots,n_u.$$

So $1 \geqslant \frac{1}{\varepsilon} x_i(t)^T x_i(t) \geqslant x_i(t)^T C_y^T f_s^T f_s C_y x_i(t)$ holds, for $s = 1,2,\cdots,n_u$, $x_i(t) \in \mathcal{B}_\varepsilon$, and $\mathcal{B}_\varepsilon \subset \mathcal{L}(F)$. Therefore, $\mathcal{L}(F)$ is estimated by B_ε. □

According to Lemma 9, the stable region is approximated by utilizing optimization theory:

$$\begin{aligned} &\text{Maximize } \varepsilon \\ &X, Y, K, F, \text{subject to } (8),\ (22),\ \text{and } (23). \end{aligned}$$

It is evident that the LMIs from (22) display the nonlinear characteristics, particularly due to the incorporation of product terms. Nevertheless, if the matrix S is predetermined and holds constant, the QMIs (22) are simplied to into LMIs, which inherently possess the convexity. This transformation allows the LMIs to contain feasible solutions for the gain matrices K and F. Consequently, this paper presents an iterative algorithm aimed at solving (22) through the following steps.

Prior to commencing the ILMIs algorithm, it is essential to define the following notation:

$$P_{\alpha p} = \begin{bmatrix} aX_p + bY_p & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix},\ \text{and } \tilde{P}_{\alpha p} = \begin{bmatrix} aX_p & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}. \tag{24}$$

Algorithm 1.
Step 1: Set $p = 1$ and $Y_{\triangle,p} = 0$. Select $Q_{\alpha 01} \in \mathbb{R}^{n\times n} > 0$ and solve the Riccati equation:

$$a(I_{\mathcal{N}} \otimes A)^T X_{\triangle,p} + aX_{\triangle,p}(I_{\mathcal{N}} \otimes A) - a^2 X_{\triangle,p}(\mathcal{H} \otimes B)(\mathcal{H} \otimes B)^T X_{\triangle,p} + Q_{\alpha 01} = 0. \tag{25}$$

Step 2: Set

$$S_p = \begin{bmatrix} aX_{\triangle,p} + bY_{\triangle,p} & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}.$$

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Maximize τ_p subject to the following LMIs:

$$\begin{bmatrix} \text{sym}(\tilde{A}^T P_{\alpha p} - S_p^T \tilde{B}\tilde{B}^T P_{\alpha p}) + S_p^T \tilde{B}\tilde{B}^T S_p + \tau_p \tilde{P}_{\alpha p} & (\tilde{B}^T P_{\alpha p} + I_{\mathcal{N}} \otimes (\Gamma_l K + \Gamma_l^- F)\tilde{C})^T \\ \star & -I \end{bmatrix} < 0,$$

$$l = 1, \ldots, 2^{n_u}, \tag{26}$$

$$\begin{bmatrix} X_p & Y_p \\ -Y_p & X_p \end{bmatrix} > 0. \tag{27}$$

Step 3: Denote $\tau_{\max,p}$ as the maximum value of τ_p. If $\tau_{\max,p} \geqslant 0$ holds, go to Step 7.
Step 4: Minimize trace(X_p) subject to the LMIs (26) and (27) with $\tau_{\max,p}$ until the minimized trace $X_{\min,p}$ and the corresponding $Y_{\min,p}$ is obtained.
Step 5: Give a small tolerance $\delta > 0$. If $\|X_{\triangle,p} - X_{\min,p}\| < \delta$ holds, then go to Step 6, else set $p = p+1$, $X_{\triangle,p} = X_{\min,p-1}$, $Y_{\triangle,p} = Y_{\min,p-1}$, and return to Step 2.
Step 6: The leader-following consensus of the FOMAS in (4) may not be achieved by static output feedback, stop.
Step 7: Maximize ε subject to (8), (22), and (23) with S_p and obtain gain matrices K and F for stabilizing the system, stop.
Remark 2. For the simplification, the inequality are still not presented in detail. P_α, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11), and $P_{\alpha p}$, $\tilde{P}_{\alpha p}$ are in (24). p only denotes the iterative times.
Remark 3. To verity the necessity of Step 1, one considers inequality (20), where equality holds with $S = P_\alpha$. S is closed to P_α, then the solutions of (19) is also close to those of (18). For obtaining the appropriate initial value S_1, consider the following equation based on (18):

$$\tilde{A}^T P_\alpha + P_\alpha^T \tilde{A} - P_\alpha^T \tilde{B}\tilde{B}^T P_\alpha + \tilde{Q} = 0,$$

where $\tilde{Q}$ represents the product term $(\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})^T(\tilde{B}^T P_\alpha + \tilde{K}\tilde{C})$. However, if $\tilde{Q}$ is set to a certain positive definite matrix directly, the solution of P_α is difficult to obtain and may not be formally consistent with that in (11). For convenient calculation, setting $Y = 0$, it yields

$$\begin{bmatrix} \text{sym}(a(I_{\mathcal{N}} \otimes A)^T X) - a^2 X(\mathcal{H} \otimes B)(\mathcal{H} \otimes B)^T X & aX(I_{\mathcal{N}} \otimes D_1) & (I_{\mathcal{N}} \otimes C)^T \\ \star & -\gamma I & (I_{\mathcal{N}} \otimes D_2)^T \\ \star & \star & -\gamma I \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix} = 0,$$

where $Q_{11} = Q_{\alpha 01}$ is the sub-block of $\tilde{Q}$. After setting Q_{11} and obtaining the solution of X, it is obvious that the results of other sub-blocks are obtained and $\tilde{Q}$ is automatically generated. Therefore, it is only necessary to set Q_{11} in Algorithm 1.
Remark 4. Considering (26) from Step 2, the system matrix is substituted with $I_{\mathcal{N}} \otimes A + \frac{\tau}{2} I$, whose eigenvalues are translated $\frac{\tau}{2}$ unit towards the x-axis relative to the eigenvalues of $I_{\mathcal{N}} \otimes A$ with $\tau > 0$.

In Step 3, if $\tau_{\max,p} \geqslant 0$ holds, with $\tau = \tau_{\max,p}$, (26) guarantees that the eigenvalues of $I_{\mathcal{N}} \otimes A$ still are in the stability region after they move forward by $\frac{\tau_{\max,p}}{2}$ unit towards the x-axis, e.g., in Figure 1. And with $\tau = \tau_{\max,p}$, (26) is seen as a sufficient condition for the consensus.
Remark 5. In Step 4, Y_p satisfy $Y_p^T = -Y_p$ and its trace is 0. Thus, the minimization problem only involves the trace of X_p.
Remark 6. After multiple iterations, S_p has been obtained to ensure (8) and (22) have feasible solutions. To obtain a large domain of attraction $\mathcal{L}(F)$ for the initial value $x_i(0)$, thus in Step 7, an optimization problem is solved subject to (8), (22), and (23) to acquire gain matrices.

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3.2. ILMI Algorithm for Output Feedback Consensus with $\alpha \in [1,2)$

In the section, the case of $\alpha \in [1,2)$ is considered. Complex numbers exist in (10) of Lemma 4 and bring difficulties to the solution. Thus, similar to the case in $\alpha \in (0,1)$, the following lemmas are given for obtaining the gain matrices. Then the parameter matrices are set as

$$\tilde{X}_a = \begin{bmatrix} aX & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \tilde{X}_b = \begin{bmatrix} bX & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}, \check{X} = \begin{bmatrix} \tilde{X}_a & \tilde{X}_b \\ -\tilde{X}_b & \tilde{X}_a \end{bmatrix}, \tag{28}$$

$$\check{A} = I_2 \otimes \tilde{A},\ \check{B} = I_2 \otimes \tilde{B},\ \check{C} = I_2 \otimes \tilde{C},\ \check{K} = I_2 \otimes \tilde{K} = I_{2\mathcal{N}} \otimes \sum_{l=1}^{2^{n_u}} h_l(\Gamma_l K + \Gamma_l^- F),$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are shown in (11).

Lemma 10. *With $\alpha \in [1,2)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u \times m}$, and $F \in \mathbb{R}^{n_u \times m}$ such that*

$$(\check{A} + \check{B}\check{K}\check{C})^T \check{X} + \check{X}^T(\check{A} + \check{B}\check{K}\check{C}) < 0, \tag{29}$$

where $\check{A}$, $\check{B}$, $\check{K}$, $\check{C}$, and $\check{X}$ are shown in (28).

Proof. Based on the proof from (13) to (15), the system in (7) is stable and $\|G_{wz}(s)\|_\infty < \gamma$ holds, if there exist $X > 0$ such that

$$\begin{bmatrix} \mathrm{her}(rA^T X) & \bar{r}XD_1 & C^T \\ \star & -\gamma I & D_2 \\ \star & \star & -\gamma I \end{bmatrix} < 0. \tag{30}$$

Considering the FOMAS in (4) without input saturation, one obtains

$$\mathrm{her}\left(\begin{bmatrix} A_{\tilde{K}} & D_{\tilde{K}} & 0 \\ 0 & -\frac{1}{2}\gamma I & 0 \\ I_{\mathcal{N}} \otimes C & I_{\mathcal{N}} \otimes D_2 & -\frac{1}{2}\gamma I \end{bmatrix}\begin{bmatrix} (a+b\mathrm{j})X & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}\right) < 0, \tag{31}$$

where $A_{\tilde{K}}$ and $D_{\tilde{K}}$ are shown in (17).

Based on Lemma 5, an equivalent condition of (31) is given as

$$\begin{bmatrix} \mathrm{sym}\big((\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T \tilde{X}_a\big) & (\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T \tilde{X}_b - \tilde{X}_b^T(\tilde{A} + \tilde{B}\tilde{K}\tilde{C}) \\ \star & \mathrm{sym}\big((\tilde{A} + \tilde{B}\tilde{K}\tilde{C})^T \tilde{X}_a\big) \end{bmatrix} < 0. \tag{32}$$

Simplify (32) and this completes the proof. □

Due to the similarity in the proof process with Lemma 7, 8, and 9, the following lemmas do not provide the relevant proof.

Lemma 11. *With $\alpha \in [1,2)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if there exist $X \in \mathbb{R}^{\mathcal{N}n \times \mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u \times m}$, and $F \in \mathbb{R}^{n_u \times m}$ such that*

$$\check{A}^T \check{X} + \check{X}^T \check{A} - \check{X}^T \check{B}\check{B}^T \check{X} + (\check{B}^T \check{X} + \check{K}\check{C})^T(\check{B}^T \check{X} + \check{K}\check{C}) < 0, \tag{33}$$

where $\check{A}$, $\check{B}$, $\check{K}$, $\check{C}$, and $\check{X}$ are shown in (28).

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Lemma 12. *With $\alpha \in [1,2)$ and a given scalar $\gamma > 0$, the leader-following consensus of the FOMAS in (4) without input saturation is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed, if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u\times m}$, $F \in \mathbb{R}^{n_u\times m}$, and S with appropriate dimension such that*

$$\begin{bmatrix} \check{A}^T\check{X} + \check{X}^T\check{A} - S^T\check{B}\check{B}^T\check{X} - \check{X}^T\check{B}\check{B}^T S + S^T\check{B}\check{B}^T S & (\check{B}^T\check{X} + \check{K}\check{C})^T \\ \star & -I \end{bmatrix} < 0, \tag{34}$$

where $\check{A}$, $\check{B}$, $\check{K}$, $\check{C}$, and $\check{X}$ are shown in (28).

Lemma 13. *With $\alpha \in [1,2)$, the leader-following consensus of the FOMAS in (4) is achieved and $\|G_{wz}(s)\|_\infty < \gamma$ is guaranteed for any $x_i(0) \in B_\varepsilon \subset \mathcal{L}(F)$, if there exist $X \in \mathbb{R}^{\mathcal{N}n\times\mathcal{N}n} > 0$, $K \in \mathbb{R}^{n_u\times m}$, $F \in \mathbb{R}^{n_u\times m}$, S with appropriate dimension, and a constant $\varepsilon > 0$ such that (23) and*

$$\begin{bmatrix} \check{A}^T\check{X} + \check{X}^T\check{A} - S^T\check{B}\check{B}^T\check{X} - \check{X}^T\check{B}\check{B}^T S + S^T\check{B}\check{B}^T S & (\check{B}^T\check{X} + \check{K}_l\check{C})^T \\ \star & -I \end{bmatrix} < 0, \tag{35}$$
$$l = 1, \ldots, 2^{n_u},$$

where $\check{A}$, $\check{B}$, $\check{C}$, and $\check{X}$ are shown in (28) and $\check{K}_l = I_{2\mathcal{N}} \otimes (\Gamma_l K + \Gamma_l^- F)$.

Similarly, as with the issue pertaining to $\alpha \in (0,1)$, the domain of attraction is estimated by the following method:

$$\begin{aligned} &\text{Maximize } \varepsilon \\ &X, Y, K, F, \text{subject to } X > 0,\ (23),\ \text{and } (35). \end{aligned}$$

In addition, (35) are not LMIs and difficult to solve with the LMI toolbox in MATLAB. Accordingly, the following iterative algorithm is provided for obtain matrices K and F. To simplify writing, one denotes

$$\check{P}_{\alpha p} = I_2 \otimes \begin{bmatrix} aX_p & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}. \tag{36}$$

Algorithm 2.
Step 1: Set $p = 1$ and select $Q_{\alpha 12} > 0$, then solve the Riccati equation:

$$a(I_\mathcal{N} \otimes A)^T X_{\triangle,p} + aX_{\triangle,p}(I_\mathcal{N} \otimes A) - a^2 X_{\triangle,p}(\mathcal{H} \otimes B)(\mathcal{H} \otimes B)^T X_{\triangle,p} + Q_{\alpha 12} = 0. \tag{37}$$

Step 2: Set $S_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} aX_{\triangle,p} & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} bX_{\triangle,p} & 0 & 0 \\ \star & I & 0 \\ \star & \star & I \end{bmatrix}$.
Maximize τ_p subject to the following LMIs:

$$X_p > 0, \tag{38}$$

$$\begin{bmatrix} \check{A}^T\check{X}_p + \check{X}_p^T\check{A} - S_p^T\check{B}\check{B}^T\check{X}_p - \check{X}_p^T\check{B}\check{B}^T S_p + S_p^T\check{B}\check{B}^T S_p - \tau_p\check{P}_{\alpha p} & (\check{B}^T\check{X}_p + \check{K}_l\check{C})^T \\ \star & -I \end{bmatrix} < 0,$$
$$l = 1, \ldots, 2^{n_u}, \tag{39}$$

Step 3: Denote $\tau_{\max,p}$ as the maximum value of τ_p, if $\tau_{\max,p} \geqslant 0$, go to Step 7.
Step 4: Minimize trace(X_p) subject to the LMIs (39) and (38) with $\tau_{\max,p}$ until the minimized trace $X_{\min,p}$ is obtained.

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Step 5: Give a small tolerance $\delta > 0$, if $\|X_{\triangle,p} - X_{\min,p}\| < \delta$, then go to Step 6, else set $p = p + 1$ and $X_{\triangle,p} = X_{\min,p-1}$, and go back to Step 2.
Step 6: The leader-following consensus of the FOMAS in (4) may not be achieved by static output feedback, stop.
Step 7: Maximize ε subject to (35), (23) with S_p and obtain K and F for achieving consensus, stop.
Remark 8. In order to provide greater clarity, the inequalities are not presented in full detail. $\check{A}$, $\check{B}$, $\check{C}$, and $\check{X}_p$ are shown in (28), and p is the iterative times. $\check{K}_l$ is from Lemma 13.
Remark 9. The iterative process is similar to that in the Algorithm 1. Due to the difficulty in solving

$$\check{A}^T\check{X} + \check{X}^T\check{A} - \check{X}^T\check{B}\check{B}^T\check{X} + \tilde{Q} = 0,$$

$X_{\triangle,1}$ is solved by (37), which is the sub-block in the first row and first column in the consolidation of the left matrices $\check{A}^T\check{X} + \check{X}^T\check{A} - \check{X}^T\check{B}\check{B}^T\check{X}$. In Step 2, S_p is constructed and close to $\check{X}_p$.
Remark 10. After certain experiments, the initial value $X_{\triangle,1}$ is related to the convergence of the algorithm. Then, the select of $Q_{\alpha 01}$ and $Q_{\alpha 12}$ in Step 1 indirectly influences the convergence property. In some cases, $Q_{\alpha 01}$ and $Q_{\alpha 12}$ are set as W^TW, where W is nonsingular. $Q_{\alpha 12} = I$ is also a choice, which leads to a convergent result.
Remark 11. In most existing study, the consensus problem of MAS relies on transformation towards to $\mathcal{H}$, such as $\mathcal{H} = T^{-1}JT$, where J is a diagonal matrix containing eigenvalues λ_i of $\mathcal{H}$. If each subsystem matrices $A + \lambda_i BKC_y$ satisfy the stability condition, i.e., $\left|\arg\left(\text{spec}\left(A + \lambda_i BKC_y\right)\right)\right| > \alpha\frac{\pi}{2}$, the consensus of the MAS is achieved. However, when an MAS contains disturbances $w(t)$ or nonlinear terms, the analysis process becomes complex after multiplying the parameter matrix with the transformation matrix T. In contrast, ILMI algorithms contribute the holistic analysis of the MAS, which facilitates performance assessment under disturbances.
Remark 12. From Algorithm 1 and 2, it is evident that no constraint is imposed on the matrix variables containing X and Y.

The existing main method involves SVD towards to C_y, such as $C_y = U\begin{bmatrix}\tilde{C}_y & 0\end{bmatrix}V^T$, where U and V are the unitary matrices. Then, the structure of X is restricted as

$$X = V\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}V^T,$$

or a upper triangular matrix in special cases for satisfying exchange condition $C_yX = X_eC_y$, where X_e is another matrix variable. Note that, for satisfying exchange condition, the stability criterion used is similar to that of integer-order system, which is independent of the order α. Thus, the method contains the conservatism.
Remark 13. From Table 1, it is shown that this paper takes into account multiple scenarios in the study of FOMASs with $\alpha \in (0,1)$ or $[1,2)$. In existing literature, there is relatively little research on H_∞ consensus for FOMASs, e.g., [30]. With the further consideration of output feedback, the limitation to the matrix variables is a common feature of many articles and an essential condition. The ILMI algorithms presented in this article avoid this limitation and are used in cases where there are disturbances from the output information and the control input saturation. Then, the ILMI algorithms have a broad scope of applicability.

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Table 1. Comparison of existing methods.

Ref.	Range of α	Output Feedback	H_∞	MASs	Input Saturation	No Limitation to Matrix Variables
[41]	$(0,1)$	✓	×	×	×	×
[25]	$(0,1)$	✓	×	✓	✓	×
[42]	$(0,1)$	✓	×	×	×	×
[43]	$(0,1)$	✓	×	×	×	×
[44]	$(0,1)$	✓	×	×	×	×
[30]	$(0,1)$	✓	✓	✓	×	×
[45]	1	×	✓	✓	×	-
[24]	$(0,2)$	✓	✓	×	×	×
[17]	$(0,2)$	×	×	✓	✓	-
ours	$(0,1)$, $[1,2)$	✓	✓	✓	✓	✓

4. Numerical Examples

This section presents two numerical examples to verify the effectiveness of the proposed ILMI algorithms with $\alpha \in (0,1)$ and $[1,2)$, respectively.

Example 1. Consider the FOMAS in (4) with $\alpha = 0.7$ and set $\gamma = 10$. The undirected graph is shown in Figure 2 and

$$\mathcal{H} = \begin{bmatrix} 4 & -0.7 & 0 & -0.7 \\ -0.7 & 3 & -0.7 & 0 \\ 0 & -0.7 & 2 & -0.7 \\ -0.7 & 0 & -0.7 & 2 \end{bmatrix}.$$

System matrices of each agent are

$$A = \begin{bmatrix} -1 & 6 & 1 \\ 1 & -0.5 & -2 \\ 1 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0.5 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_y = \begin{bmatrix} 1 & 0.5 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of A are -4.0840 , $0.7920 + 0.9175$j, and $0.7920 - 0.9175$j. It is easy to see that the consensus is not achieved.

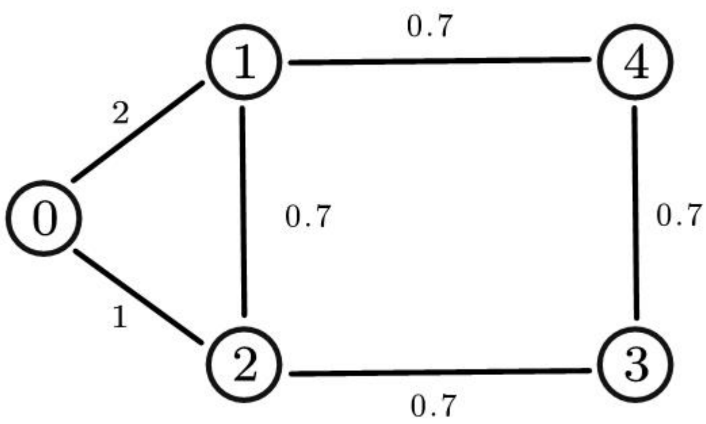

Figure 2. The weighted undirected graph in Example 1.

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Based on Algorithm 1, select $Q_{\alpha 01} = I$ in Step 1. Solve the equation (25) and obtain

$$X_{\triangle,1} = \begin{bmatrix} 0.1727 & 0.0308 & 0.0419 & 0.0226 & 0.0268 & 0.0190 & 0.0207 & 0.0295 & 0.0167 & 0.0343 & 0.0430 & 0.0286 \\ 0.0308 & 0.2243 & -0.1982 & 0.0268 & 0.0793 & -0.0494 & 0.0295 & 0.0892 & -0.0426 & 0.0430 & 0.1281 & -0.0730 \\ 0.0419 & -0.1982 & 0.5661 & 0.0190 & -0.0494 & 0.1370 & 0.0167 & -0.0426 & 0.1102 & 0.0286 & -0.0730 & 0.1994 \\ 0.0226 & 0.0268 & 0.0190 & 0.2005 & 0.0623 & 0.0658 & 0.0459 & 0.0593 & 0.0383 & 0.0251 & 0.0363 & 0.0200 \\ 0.0268 & 0.0793 & -0.0494 & 0.0623 & 0.3167 & -0.2592 & 0.0593 & 0.1769 & -0.0967 & 0.0363 & 0.1100 & -0.0521 \\ 0.0190 & -0.0494 & 0.1370 & 0.0658 & -0.2592 & 0.7389 & 0.0383 & -0.0967 & 0.2618 & 0.0200 & -0.0521 & 0.1331 \\ 0.0207 & 0.0295 & 0.0167 & 0.0459 & 0.0593 & 0.0383 & 0.2705 & 0.1537 & 0.1237 & 0.0701 & 0.0948 & 0.0572 \\ 0.0295 & 0.0892 & -0.0426 & 0.0593 & 0.1769 & -0.0967 & 0.1537 & 0.5903 & -0.4068 & 0.0948 & 0.2852 & -0.1475 \\ 0.0167 & -0.0426 & 0.1102 & 0.0383 & -0.0967 & 0.2618 & 0.1237 & -0.4068 & 1.1357 & 0.0572 & -0.1475 & 0.3895 \\ 0.0343 & 0.0430 & 0.0286 & 0.0251 & 0.0363 & 0.0200 & 0.0701 & 0.0948 & 0.0572 & 0.2662 & 0.1470 & 0.1205 \\ 0.0430 & 0.1281 & -0.0730 & 0.0363 & 0.1100 & -0.0521 & 0.0948 & 0.2852 & -0.1475 & 0.1470 & 0.5694 & -0.3973 \\ 0.0286 & -0.0730 & 0.1994 & 0.0200 & -0.0521 & 0.1331 & 0.0572 & -0.1475 & 0.3895 & 0.1205 & -0.3973 & 1.1129 \end{bmatrix}.$$

Then S_1 is obtained from Step 2.

Maximize τ subject to (39) and (27) and the result is $\tau_{\max,1} = 0.5064 > 0$. Go to Step 7 and maximize ε subject to (8), (37), and (23) with S_p. The following feasible solutions are obtained:

$$\varepsilon = 11.7647,$$

$$K = \begin{bmatrix} -0.9646 & 0.4148 \\ 0.2009 & -0.2746 \end{bmatrix},$$

$$F = \begin{bmatrix} -0.1064 & -0.0626 \\ 0.2009 & -0.2746 \end{bmatrix}.$$

Consider the case as $\text{sat}(Ky(t)) = FC_y x(t)$. Calculating $\text{eig}(I_4 \otimes A + (\mathcal{H} \otimes B)(I_4 \otimes F)(I_4 \otimes C_y))$, one obtains $-1.7916 + 3.7520\text{j}$, $-1.7916 - 3.7520\text{j}$, $0.2397 + 1.6734\text{j}$, $0.2397 - 1.6734\text{j}$, $-1.0159 + 2.7754\text{j}$, $-1.0159 - 2.7754\text{j}$, $-0.6200 + 2.4083\text{j}$, $-0.6200 - 2.4083\text{j}$, -3.5668, -1.5913, -2.2096, and -2.6417, which satisfy the stability condition in Lemma 2.

The state of each agent and the control input $u(t)$ are shown as Figure 4. At the beginning, the input is saturated. From Figure 4, it is shown that the FOMAS has achieved consensus.

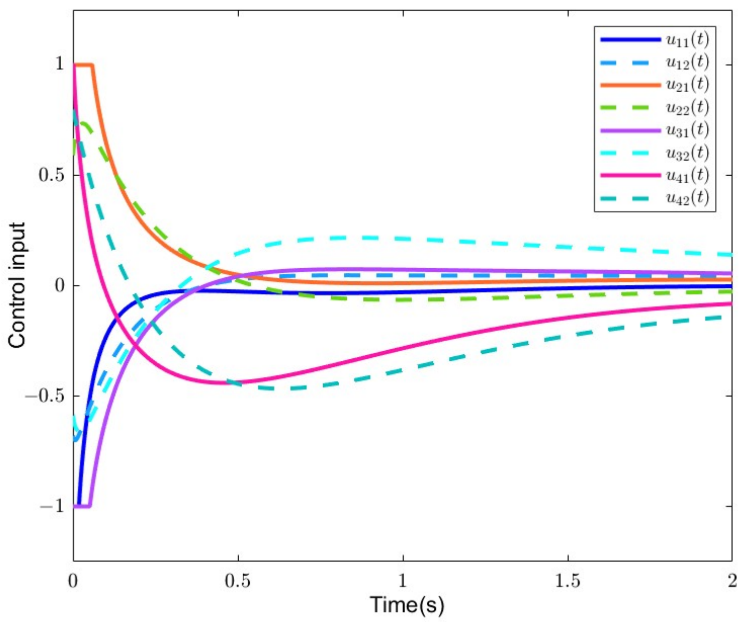

Figure 3. The control input of each agent in Example 1

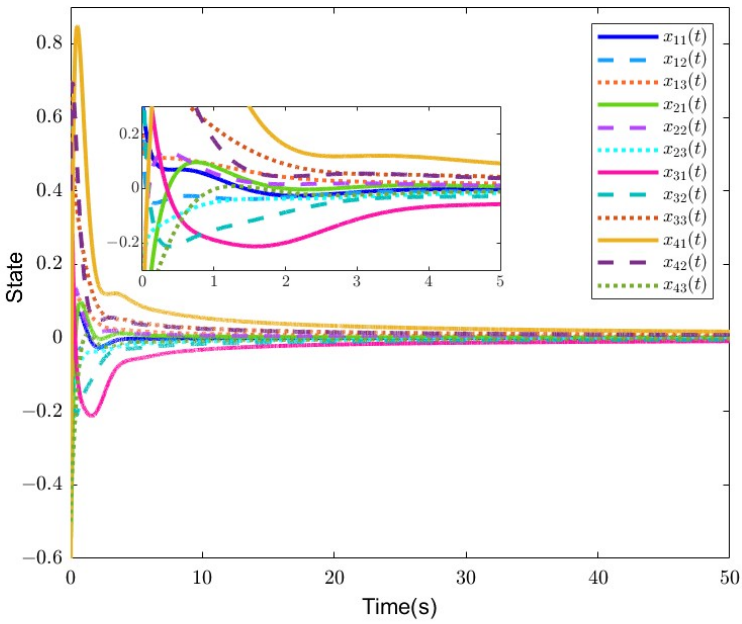

Figure 4. The state of each agent in Example 1

Remark 14. From the eigenvalues in the results, it is obvious that 0.2397 ± 1.6734j contain positive real parts. As mentioned in Remark 12, the stability criterion is not related to α for the exchange condition $C_y X = X_e C_y$. Thus, the eigenvalues of matrix must have negative real part. In other words, there exist gain matrices for the consensus of FOMASs, but they may not be obtained by using the SVD method. Thus, the ILMI algorithms in this paper are less conservative than the SVD method.

Example 2. Consider the FOMAS in (4) with $\alpha = 1.3$ and set $\gamma = 20$. The undirected graph is shown in Figure 5 and parameter matrices are as follow:

$$\mathcal{H} = \begin{bmatrix} 3 & -0.3 & 0 & -0.3 \\ -0.3 & 2 & -0.3 & 0 \\ 0 & -0.3 & 1 & 0 \\ -0.3 & 0 & 0 & 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 1 & -1 & -2 \\ 2 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},\ D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix},\ D_3 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},\ C_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are -4.1117, $0.0558 + 1.2067$j, and $0.0558 - 1.2067$j, the latter two of which have positive real parts and satisfy

$$|\arg(0.0558 \pm 1.2067\mathrm{j})| < \alpha \frac{\pi}{2}.$$

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Then the consensus of FOMAS is not achieved. Given $Q_{\alpha 12} = I$, the equation (37) has a feasible solution as

$$X_{\triangle,1} = \begin{bmatrix} 0.2475 & -0.0833 & 0.2211 & 0.0179 & 0.0088 & 0.0194 & 0.0055 & 0.0038 & 0.0044 & 0.0346 & 0.0203 & 0.0329 \\ -0.0833 & 0.2366 & -0.2457 & 0.0088 & 0.0129 & -0.0024 & 0.0038 & 0.0047 & 0.0008 & 0.0203 & 0.0266 & 0.0004 \\ 0.2211 & -0.2457 & 0.671 & 0.0194 & -0.0024 & 0.0399 & 0.0044 & 0.0008 & 0.0059 & 0.0329 & 0.0004 & 0.0578 \\ 0.0179 & 0.0088 & 0.0194 & 0.3073 & -0.0538 & 0.2859 & 0.053 & 0.0332 & 0.0475 & 0.0055 & 0.0038 & 0.0044 \\ 0.0088 & 0.0129 & -0.0024 & -0.0538 & 0.2797 & -0.2539 & 0.0332 & 0.0421 & 0.0031 & 0.0038 & 0.0047 & 0.0008 \\ 0.0194 & -0.0024 & 0.0399 & 0.2859 & -0.2539 & 0.804 & 0.0475 & 0.0031 & 0.0774 & 0.0044 & 0.0008 & 0.0059 \\ 0.0055 & 0.0038 & 0.0044 & 0.053 & 0.0332 & 0.0475 & 0.4785 & 0.0529 & 0.4399 & 0.0017 & 0.0013 & 0.0011 \\ 0.0038 & 0.0047 & 0.0008 & 0.0332 & 0.0421 & 0.0031 & 0.0529 & 0.4154 & -0.2443 & 0.0013 & 0.0018 & -0.0000 \\ 0.0044 & 0.0008 & 0.0059 & 0.0475 & 0.0031 & 0.0774 & 0.4399 & -0.2443 & 1.056 & 0.0011 & -0.000 & 0.0018 \\ 0.0346 & 0.0203 & 0.0329 & 0.0055 & 0.0038 & 0.0044 & 0.0017 & 0.0013 & 0.0011 & 0.473 & 0.0485 & 0.4362 \\ 0.0203 & 0.0266 & 0.0004 & 0.0038 & 0.0047 & 0.0008 & 0.0013 & 0.0018 & -0.000 & 0.0485 & 0.4093 & -0.2442 \\ 0.0329 & 0.0004 & 0.0578 & 0.0044 & 0.0008 & 0.0059 & 0.0011 & -0.000 & 0.0018 & 0.4362 & -0.2442 & 1.0502 \end{bmatrix}.$$

From Step 2, S_1 is set immediately.

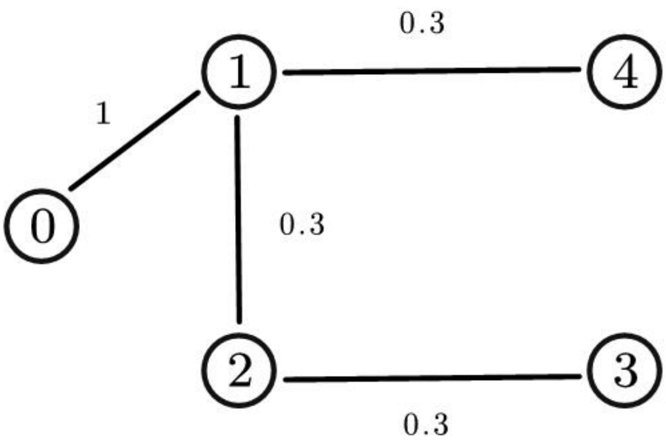

Figure 5. The weighted undirected graph in Example 2.

Maximize τ subject to (39) and (38), then one obtains $\tau_{\max,1} = -0.0061 < 0$. Thus, go to Step 4 and minimize trace(X_1) subject to the LMIs (39) and (38) with $\tau_{\max,1} = -0.0061 < 0$. The result is

$$X_{\min,1} = \begin{bmatrix} 0.3316 & 0.0708 & 0.1426 & 0.0342 & 0.0257 & 0.0322 & 0.025 & 0.0112 & 0.0149 & 0.0818 & 0.0541 & 0.0659 \\ 0.0708 & 0.1683 & -0.0769 & 0.0257 & 0.0259 & 0.0004 & 0.0112 & 0.0224 & -0.0024 & 0.0541 & 0.0685 & -0.0002 \\ 0.1426 & -0.0769 & 0.463 & 0.0322 & 0.0004 & 0.0388 & 0.0149 & -0.0024 & 0.0369 & 0.0659 & -0.0002 & 0.1009 \\ 0.0342 & 0.0257 & 0.0322 & 0.4458 & 0.1565 & 0.2498 & 0.1652 & 0.0916 & 0.1154 & 0.025 & 0.0112 & 0.0149 \\ 0.0257 & 0.0259 & 0.0004 & 0.1565 & 0.2546 & -0.0757 & 0.0916 & 0.1431 & -0.0082 & 0.0112 & 0.0224 & -0.0024 \\ 0.0322 & 0.0004 & 0.0388 & 0.2498 & -0.0757 & 0.5925 & 0.1154 & -0.0082 & 0.2238 & 0.0149 & -0.0024 & 0.0369 \\ 0.025 & 0.0112 & 0.0149 & 0.1652 & 0.0916 & 0.1154 & 0.9715 & 0.4505 & 0.6197 & 0.0358 & 0.0091 & 0.0158 \\ 0.0112 & 0.0224 & -0.0024 & 0.0916 & 0.1431 & -0.0082 & 0.4505 & 0.7093 & -0.1008 & 0.0091 & 0.032 & -0.0075 \\ 0.0149 & -0.0024 & 0.0369 & 0.1154 & -0.0082 & 0.2238 & 0.6197 & -0.1008 & 1.3018 & 0.0158 & -0.0075 & 0.0608 \\ 0.0818 & 0.0541 & 0.0659 & 0.025 & 0.0112 & 0.0149 & 0.0358 & 0.0091 & 0.0158 & 0.852 & 0.4202 & 0.5672 \\ 0.0541 & 0.0685 & -0.0002 & 0.0112 & 0.0224 & -0.0024 & 0.0091 & 0.032 & -0.0075 & 0.4202 & 0.6025 & -0.0758 \end{bmatrix}.$$

Set $X_{\triangle,2} = X_{\min,1}$ and obtain S_2. Then, maximize τ subject to (39) and (38), then one obtains $\tau_{\max,2} = 0.3955 > 0$. Jump to Step 7, and maximize ε subject to the LMIs (23), $X > 0$, and (35) with S_2. The following feasible solutions are as

$$\varepsilon = 1.6886,$$

$$K = \begin{bmatrix} -0.8223 & -1.3950 \\ -1.3286 & -0.2252 \end{bmatrix},$$

$$F = \begin{bmatrix} -0.6009 & -0.3399 \\ -0.5739 & -0.3625 \end{bmatrix}.$$

There exists the input saturation in FOMAS from Figure 6. The state of each agent is shown in Figure 7. It is shown that the FOMAS has achieved consensus.

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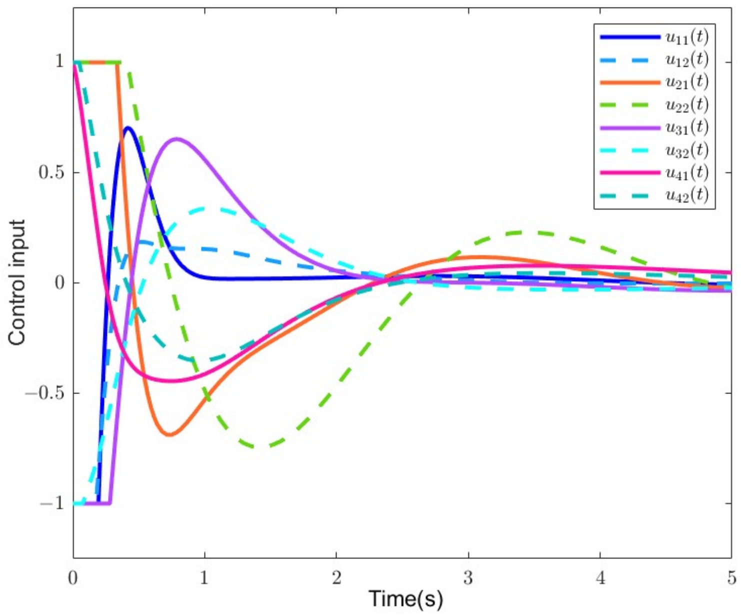

Figure 6. The control input of each agent in Example 2

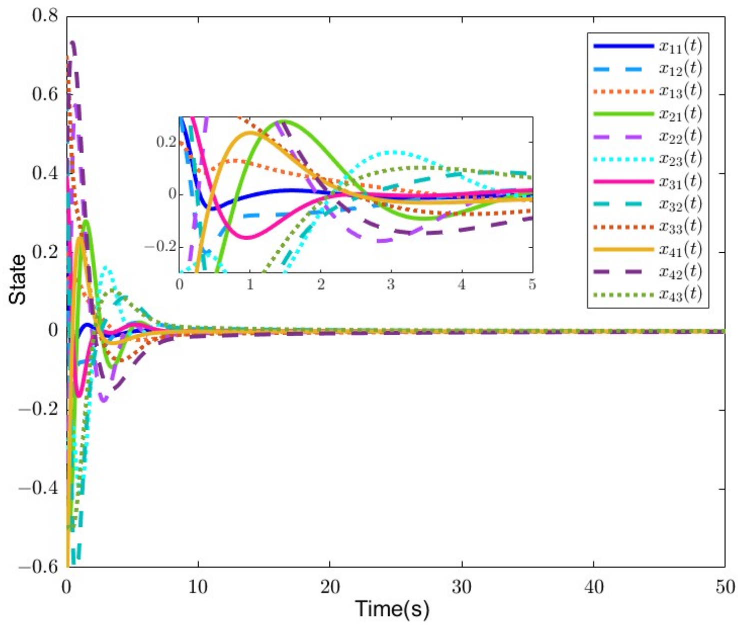

Figure 7. The state of each agent in Example 2

One also chooses other $Q_{\alpha 12}$ for different initial values $X_{\triangle,1}$. In the start of algorithm, the positive definite matrix $Q_{\alpha 12}$ is selected as $I_4 \otimes W^T W$, where

$$W = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Solve equation (37) and obtain

$$X_{\triangle,1} = \begin{bmatrix} 0.3542 & -0.0026 & 0.5094 & 0.0378 & 0.0290 & 0.0452 & 0.0124 & 0.0103 & 0.0121 & 0.0751 & 0.0602 & 0.0821 \\ -0.0026 & 0.4036 & -0.0967 & 0.0290 & 0.0323 & 0.0267 & 0.0103 & 0.0109 & 0.0073 & 0.0602 & 0.0648 & 0.0495 \\ 0.5094 & -0.0967 & 0.8312 & 0.0452 & 0.0267 & 0.0648 & 0.0121 & 0.0073 & 0.0160 & 0.0821 & 0.0495 & 0.1134 \\ 0.0378 & 0.0290 & 0.0452 & 0.4801 & 0.0941 & 0.6601 & 0.1165 & 0.0947 & 0.1225 & 0.0124 & 0.0103 & 0.0121 \\ 0.0290 & 0.0323 & 0.0267 & 0.0941 & 0.5113 & -0.0078 & 0.0947 & 0.1013 & 0.0740 & 0.0103 & 0.0109 & 0.0073 \\ 0.0452 & 0.0267 & 0.0648 & 0.6601 & -0.0078 & 1.0470 & 0.1225 & 0.0740 & 0.1667 & 0.0121 & 0.0073 & 0.0160 \\ 0.0124 & 0.0103 & 0.0121 & 0.1165 & 0.0947 & 0.1225 & 0.8562 & 0.3993 & 1.0565 & 0.0040 & 0.0034 & 0.0035 \\ 0.0103 & 0.0109 & 0.0073 & 0.0947 & 0.1013 & 0.0740 & 0.3993 & 0.8379 & 0.2315 & 0.0034 & 0.0039 & 0.0016 \\ 0.0121 & 0.0073 & 0.0160 & 0.1225 & 0.0740 & 0.1667 & 1.0565 & 0.2315 & 1.5868 & 0.0035 & 0.0016 & 0.0047 \\ 0.0751 & 0.0602 & 0.0821 & 0.0124 & 0.0103 & 0.0121 & 0.0040 & 0.0034 & 0.0035 & 0.8427 & 0.3881 & 1.0448 \\ 0.0602 & 0.0648 & 0.0495 & 0.0103 & 0.0109 & 0.0073 & 0.0034 & 0.0039 & 0.0016 & 0.3881 & 0.8249 & 0.2262 \\ 0.0821 & 0.0495 & 0.1134 & 0.0121 & 0.0073 & 0.0160 & 0.0035 & 0.0016 & 0.0047 & 1.0448 & 0.2262 & 1.5712 \end{bmatrix}.$$

Then, maximize τ subject to (39) and (38), and the result is $\tau_{\max,1} = 0.3999 > 0$. Go to Step 7. The subsequent solving process is omitted.

Remark 15. It is seen that the number of iterations in $Q_{\alpha 12} = I_4 \otimes W^T W$ is less than that in $Q_{\alpha 12} = I$. $Q_{\alpha 12} = I$ is convenient to set, but I is a diagonal matrix, which differs from the $(a(\mathcal{H} \otimes B)^T X + \tilde{K}(I_4 \otimes C_y))^T (a(\mathcal{H} \otimes B)^T X + \tilde{K}(I_4 \otimes C_y))$ under general conditions. Thus, the choice of $Q_{\alpha 01}$ and $Q_{\alpha 12}$ affects the number of iterations. When the algorithm cannot converge, it may be considered to replace the initial value.

5. Conclusion

The leader-following H_∞ consensus of FOMASs under input saturation via the output feedback has been investigated. Lemmas 8 and 12 have provided sufficient conditions of H_∞ consensus for FOMAS in (4) with $\alpha \in (0,1)$ and $[1,2)$, respectively. Additionally, based on the ILMI approach, the Algorithms 1 and 2 have been presented to compute the gain matrices. In both ILMI algorithms, the methods to derive initial values are also given and ILMI algorithms have solved the H_∞ consensus problem for FOMASs under actuator saturation. The algorithms are effective and convergent, avoiding matrix exchange condition in the SVD method and strong assumption. The method also provides a holistic approach for calculating the gain matrices of MASs. Eventually, numerical examples have been conducted to validate the efficiency of the proposed approach.

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