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Approximation Properties of Chlodovsky Type Two Dimensional Bernstein Operators Based on (p,q)-integers

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Abstract: In this study, we introduce two dimensional Chlodovsky type Bernstein operators based on (p,q)-integers. We assess the approximation properties of these operators through a Korovkin-type theorem. Additionally, we analyze the local approximation properties and determine the convergence rates using the modulus of continuity and a Lipschitz-type maximal function. A Voronovskaja-type theorem is also derived for these operators. Furthermore, we investigate their weighted approximation properties and estimate the convergence rates in the same function space. Finally, visual illustrations created using Maple demonstrate the convergence rates of these operators for specific functions. The optimization of approximation speeds by operators during system control provides significant improvements in stability and performance. As a result, the control and modeling of dynamic systems become more efficient and effective through innovative methods. These advancements in the fields of modeling fractional differential equations and control theory offer substantial benefits to both modeling and optimization processes, expanding the range of applications in these areas.

Keywords: two dimensional (p,q)- Chlodovsky type Bernstein operators; Voronovskaja type theorem; (p,q)-integer; control theory

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1. Introduction

Approximation theory is rapidly emerging as an essential tool, extending its influence beyond classical domains to other mathematical areas such as differential equations, orthogonal polynomials, and geometric design. Following the introduction of Korovkin's renowned theorem in 1950, the topic of approximating functions using linear positive operators has become an increasingly significant focus within approximation theory. A wealth of literature has been produced on this subject [1,2,10,12,14,15,23,24].

In recent years, particularly over the last twenty years, the role of q-calculus in approximation theory has been thoroughly investigated. The initial work on Bernstein polynomials derived from q-integers was conducted by Lupaş [6]. His findings indicated that q-Bernstein polynomials can provide superior approximations compared to classical methods when an appropriate choice of q is made. This discovery has encouraged numerous researchers to develop q-generalizations of various operators and to explore their approximation properties further. Numerous studies have contributed to this field [3,7,8,13].

In recent years, Mursaleen et al. have concentrated on utilizing (p,q)-calculus for approximations through linear positive operators, introducing the (p,q)-analogues of Bernstein operators [20,21]. They analyzed the uniform convergence of these operators and determined their rates of convergence. For additional recent studies related to (p,q)-operators, readers can refer to [17–19,26,27].

The main motivation behind this study is that, to the authors' knowledge, there have been no investigations into approximating two-variable operators using (p,q)- calculus thus far. In this context,

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we introduce two dimensional Chlodovsky type Bernstein operators based on (p,q)-integers. We investigate the approximation properties of our newly defined operators with the aid of the Korovkin-type theorem. Furthermore, we delve into the local approximation characteristics and determine the rates of convergence through the modulus of continuity and a Lipschitz type maximal function. A Voronovskaja type theorem relevant to these operators is also presented. Another significant aim of this research is to examine the weighted approximation properties of our operators in the first quadrant of $\mathbb{R}^2_+$, specifically within the range of $[0,\infty)\times[0,\infty)$. To achieve these results, we intend to apply a weighted Korovkin-type theorem. We will begin by revisiting some definitions and notations pertinent to the concept of (p, q)-calculus. The (p, q)-integer associated with a given number n is defined as

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \quad n = 1,2,3\ldots, \quad 0 < q < p \le 1.$$

The (p,q)-factorial $[n]_{p,q}!$ and the (p,q)-binomial coefficients are defined as :

$$[n]_{p,q}! := \begin{cases} [n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}, & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}.$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, 0 \le k \le n.$$

Further, the (p,q)-binomial expansions are given as

$$(ax+by)^n_{p,q} = \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} a^{n-k} b^k x^{n-k} y^k.$$

and

$$(x-y)^n_{p,q} = (x-y)(px-qy)(p^2x-q^2y)\cdots(p^{n-1}x-q^{n-1}y).$$

Further information related to (p,q)-calculus can be found in [25,28].

2. Construction of the Operators

Recently, Ansari and Karaisa [16] have defined and studied (p,q)-analogue of Chlodovsky operators as follows:

$$C_{n,p,q}(f;x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k-1}_{p,q} f\left(\frac{[k]_{p,q}}{[n]_{p,q} p^{k-n}} b_n\right), \tag{1}$$

where

$$\left(1 - \frac{x}{b_n}\right)^{n-k-1}_{p,q} = \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{x}{b_n}\right).$$

For $0 < q_1, q_2 < p_1, p_2 \le 1$, we define Chlodovsky type two dimensional Bernstein operator based on (p,q)-integers as follows:

$$C^{(p_1,q_1),(p_1,q_1)}_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} \Phi_{n,k}(p_1,q_1;x)\Phi_{m,j}(p_2,q_2;y) f\left(\frac{[k]_{p_1,q_1}}{[n]_{p_1,q_1} p_1^{k-n}}\alpha_n, \frac{[j]_{p_2,q_2}}{[m]_{p_2,q_2} p_2^{j-m}}\beta_m\right), \tag{2}$$

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for all $n, m \in N$, $f \in C(I_{\alpha_n \beta_m})$ with $I_{\alpha_n \beta_m} = \{(x, y) : 0 \leq \alpha_n \leq x\ , 0 \leq \beta_m \leq y\}$ and $C(I_{\alpha_n \beta_m}) = \{f : I_{\alpha_n \beta_m} \longrightarrow R$ is continuous $\}$. Here (α_n) and (β_m) be increasing unbounded sequences of positive real numbers such that

$$\lim_{n\to\infty} \frac{\alpha_n}{[n]_{p_1,q_1}} = 0, \tag{3}$$

$$\lim_{m\to\infty} \frac{\beta_m}{[m]_{p_2,q_2}} = 0. \tag{4}$$

Also, the basis elements are

$$\begin{aligned} \Phi_{n,k}(p_1, q_1; x) &= p_1^{\frac{k(k-1)-n(n-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p_1,q_1} \left(\frac{x}{\alpha_n}\right)^k \prod_{s=0}^{n-k-1} \left(p_1^s - q_1^s \frac{x}{\alpha_n}\right), \\ \Phi_{m,j}(p_2, q_2; y) &= p_2^{\frac{j(j-1)-m(m-1)}{2}} \begin{bmatrix} m \\ j \end{bmatrix}_{p_2,q_2} \left(\frac{y}{\beta_m}\right)^j \prod_{s=0}^{m-j-1} \left(p_2^s - q_2^s \frac{y}{\beta_m}\right). \end{aligned}$$

We require the following lemmas to establish our main results.

Lemma 1 ([16]).

$$\begin{aligned} C_{n,p,q}(1; x) &= 1, \\ C_{n,p,q}(e_1; x) &= x, \\ C_{n,p,q}(e_2; x) &= \frac{p^{n-1} b_n}{[n]_{p,q}} x + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2 \\ C_{n,p,q}(e_3; x) &= \frac{b_n^2 x}{[n]_{p,q}^2} p^{2n-2} + \frac{(2p+q)q[n-1]_{p,q} x^2 b_n}{[n]_{p,q}^2} p^{n-1} + \frac{q^3 [n-1]_{p,q} [n-2]_{p,q} x^3}{[n]_{p,q}^2}, \\ C_{n,p,q}(e_4; x) &= \frac{b_n^3 x}{[n]_{p,q}^3} p^{3n-3} + \frac{q(3p^2 + 3qp + q^3)[n-1]_{p,q} b_n^2 x^2}{[n]_{p,q}^3} p^{2n-4} \\ &\quad + \frac{q^3(3p^2 + 2pq + q^2)[n-1]_{p,q}[n-2]_{p,q} b_n x^3}{[n]_{p,q}^3} p^{n-3} + \frac{q^6 [n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q} x^4}{[n]_{p,q}^3}. \end{aligned}$$

From Lemma 1, we have following:

Lemma 2.

$$\begin{aligned} C_{n,m}^{(p_1,q_1),(p_2,q_2)}(1; x, y) &= 1, \\ C_{n,m}^{(p_1,q_1),(p_2,q_2)}(s; x, y) &= x, \\ C_{n,m}^{(p_1,q_1),(p_2,q_2)}(t; x, y) &= y, \\ C_{n,m}^{(p_1,q_1),(p_2,q_2)}(st; x, y) &= xy, \\ C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(s^2; x, y\right) &= \frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}} x + \frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} x^2, \\ C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(t^2; x, y\right) &= \frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}} y + \frac{q_2[m-1]_{p_2,q_2}}{[m]_{p_2,q_2}} y^2. \end{aligned}$$

Using Lemma 2 and by linearity of $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$, we have

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Remark 1.

$$C_{n,m}^{(p_1,q_1),(p_2,q_2)}\Big((t-x)^2;x,y\Big) = \frac{-p_1^{n-1}x^2}{[n]_{p_1,q_1}} + \frac{xp_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}, \quad (5)$$

$$C_{n,m}^{(p_1,q_1),(p_2,q_2)}\Big((s-y)^2;x,y\Big) = \frac{-p_2^{m-1}y^2}{[m]_{p_2,q_2}} + \frac{yp_2^{m-1}\beta_m}{[m]_{p_2,q_2}}. \quad (6)$$

Theorem 1. *Let $q_1 := (q_{1,n})$, $p_1 := (p_{1,n})$, $q_2 := (q_{2,m})$, $p_2 := (p_{2,m})$ such that $0 < q_{1,n}, q_{2,m} < p_{1,n}, p_{2,m} \leq 1$. If*

$$\lim_n p_{1,n} = 1,\ \lim_n q_{1,n} = 1,\ \lim_m p_{2,m} = 1,\ \lim_m q_{2,m} = 1,\ \lim_n p_{1,n}^n = a_1 \text{ and } \lim_m p_{1,m}^m = a_2, \quad (7)$$

the sequence $C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)$ convergence uniformly to $f(x,y)$, on $[0,a]\times[0,b] = I_{ab}$ for each $f \in C(I_{ab})$, where a, b be reel numbers such that $a \leq \alpha_n$, $b \leq \beta_m$ and $C(I_{ab})$ be the space of all real valued continuous function on I_{ab} with the norm

$$\| f \|_{C(I_{ab})} = \sup_{(x,y)\in I_{ab}} |f(x,y)|.$$

Proof. Assume that the equities (7), (3) and (4) are holds. Then, we have

$$\frac{p_{1,n}^{n-1}\alpha_n}{[n]_{p_{1,n},q_{1,n}}} \to 0,\ \frac{p_{2,m}^{m-1}\beta_m}{[m]_{p_{2,m},q_{2,m}}} \to 0,\ \frac{q_{1,n}[n]_{p_{1,n},q_{1,n}}}{[n]_{p_{1,n},q_{1,n}}} \to 1 \text{ and } \frac{q_{2,m}[m-1]_{p_{2,m},q_{2,m}}}{[m]_{p_{2,m},q_{2,m}}} \to 1.$$

as $n, m \longrightarrow \infty$. From Lemma 2, we obtain $\lim_{n,m\to\infty} C_{n,m}^{(p_1,q_1),(p_2,q_2)}(e_{ij};x,y) = e_{ij}(x,y)$ uniformly on I_{ab}, where $e_{ij}(x,y) = x^i y^j, 0 \leq i+j \leq 2$ are the test functions. By Korovkin's theorem for functions of two variables was presented by Volkov [29], it follows that $\lim_{n,m\to\infty} C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) = f(x,y)$, uniformly on I_{ab}, for each $f \in C(I_{ab})$. □

3. Rate of Convergence

In this section, we analyze the convergence rates of the operators $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ to the function $f(x,y)$ using the modulus of continuity. Furthermore, we will present a summary of the relevant notations and definitions concerning the modulus of continuity and Peetre's K-functional for bivariate real-valued functions.

For a function $f \in C(I_{ab})$, the complete modulus of continuity in the bivariate context is defined as follows:

$$\omega(f,\delta) = \sup\Big\{ |f(t,s) - f(x,y)| : \sqrt{(t-x)^2 + (s-y)^2} \leq \delta \Big\}.$$

for every $(t,s), (x,y) \in I_{ab}$. Additionally, the partial moduli of continuity concerning x and y are defined as follows:

$$\omega^1(f,\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : y \in [0,b] \text{ and } |x_1 - x_2| \leq \delta\}$$
$$\omega^2(f,\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : x \in [0,a] \text{ and } |y_1 - y_2| \leq \delta\},$$

It is evident that they fulfill the properties of the standard modulus of continuity [11].

For $\delta > 0$, the Peetre-K functional [22] is defined as follows:

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$$K(f,\delta)=\inf_{g\in C^2(I_{ab})}\left\{\|f-g\|_{C(I_{ab})}+\delta\|g\|_{C^2(I_{ab})}\right\},$$

where $C^2(I_{ab})$ is the space of functions of f such that f, $\frac{\partial^j f}{\partial x^j}$ and $\frac{\partial^j f}{\partial y^j}$ $(j=1,2)$ in $C(I_{ab})$. The norm $\|.\|$ on the space $C^2(I_{ab})$ is defined by

$$\|f\|_{C^2 I_{ab}}=\|f\|_{C(I_{ab})}+\sum_{j=1}^{2}\left(\left\|\frac{\partial^j f}{\partial y^j}\right\|_{C(I_{ab})}+\left\|\frac{\partial^j f}{\partial y^j}\right\|_{C(I_{ab})}\right).$$

We now provide an estimate for the rate of convergence of the operators $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$.

Theorem 2. *Let $f\in C(I_{ab})$. For all $x\in I_{ab}$, we have*

$$\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}-f(x,y)\right|\leq 2\omega(f;\delta_{n,m}),$$

where

$$\delta_{n,m}^2 \quad = \quad \frac{a\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}+\frac{b\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}.$$

Proof. By definition, the complete modulus of continuity of $f(x,y)$, along with the linearity and positivity of our operator, allows us to express:

$$\begin{aligned}|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)| &\leq C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s)-f(x,y)|;x,y)\\ &\leq C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\omega\left(f;\sqrt{(t-x)^2+(s-y)^2}\right);x,y\right)\\ &\leq \omega(f,\delta_{n,m})\left[\frac{1}{\delta_{n,m}}C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\sqrt{(t-x)^2+(s-y)^2};x,y\right)\right].\end{aligned}$$

Using Cauchy-Scwartz inequality, from (5) and (6), one can write following

$$\begin{aligned}&|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)|\\ \leq\ &\omega(f,\delta_{n,m})\left[1+\frac{1}{\delta_{n,m}}\left\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2+(s-y)^2;x,y\right)\right\}^{1/2}\right]\\ =\ &\omega(f,\delta_{n,m})\left[1+\frac{1}{\delta_{n,m}}\left\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right)+C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y\right)\right\}^{1/2}\right]\\ \leq\ &\omega(f,\delta_{n,m})\left[1+\frac{1}{\delta_{n,m}}\left(\frac{a\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}+\frac{b\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}\right)^{1/2}\right].\end{aligned}$$

Choosing $\delta_{n,m}=\left(\frac{a\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}+\frac{b\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}\right)^{1/2}$, for all $(x,y)\in I_{ab}$, we get desired the result.
□

Theorem 3. *Let $f\in C(I_{ab})$, then the following inequalities satisfy*

$$\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}-f(x,y)\right|\leq \omega^1(f;\delta_n)+\omega^2(f;\delta_m),$$

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where

$$\delta_n^2 = \frac{a\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}, \tag{8}$$

$$\delta_m^2 = \frac{b\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}. \tag{9}$$

Proof. By definition, the partial moduli of continuity of $f(x,y)$ and the application of the Cauchy-Schwarz inequality imply that:

$$\begin{aligned}
|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)| &\le C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s) - f(x,y)|;x,y) \\
&\le C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s) - f(x,s)|;x,y) + C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(x,s) - f(x,y)|;x,y) \\
&\le C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(|\omega^1(f;|t-x|)|;x,y\right) + C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(|\omega^2(f;|s-y|)|;x,y\right) \\
&\le \omega^1(f,\delta_n)\left[1 + \frac{1}{\delta_n} C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t-x|;x,y)\right] \\
&\quad + \omega^2(f,\delta_m)\left[1 + \frac{1}{\delta_m} C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s-y|;x,y)\right] \\
&\le \omega^1(f,\delta_n)\left[1 + \frac{1}{\delta_n}\left(C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right)\right)^{1/2}\right] \\
&\quad + \omega^2(f,\delta_m)\left[1 + \frac{1}{\delta_m}\left(C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y\right)\right)^{1/2}\right].
\end{aligned}$$

Consider (5), (6) and choosing

$$\delta_n^2 = \frac{a\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}},$$

$$\delta_m^2 = \frac{b\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}.$$

we reach the result. □

For $\widehat{\alpha}_1, \widehat{\alpha}_2 \in (0,1]$ and $(s,t),(x,y) \in I_{ab}$, we define the Lipschitz class $LipM(\widehat{\alpha}_1,\widehat{\alpha}_2)$ for the bivariate case as follows:

$$|f(s,t) - f(x,y)| \le M|s-x|^{\widehat{\alpha}_1}|t-y|^{\widehat{\alpha}_2}.$$

Theorem 4. *Let $f \in Lip_M(\widehat{\alpha}_1,\widehat{\alpha}_2)$. Then, for all $(x,y) \in I_{ab}$, we have*

$$|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)| \le M\delta_n^{\widehat{\alpha}_1/2}\delta_m^{\widehat{\alpha}_2/2},$$

where δ_n and δ_m defined in (8) and (9), respectively.

Proof. As $f \in Lip_M(\widehat{\alpha}_1,\widehat{\alpha}_2)$, it follows

$$\begin{aligned}
|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)| &\le C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|f(t,s) - f(x,y)|, q_n; x,y) \\
&\le MC_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t-x|^{\widehat{\alpha}_1}|s-y|^{\widehat{\alpha}_2};x,y) \\
&= MC_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t-x|^{\widehat{\alpha}_1};x)C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s-y|^{\widehat{\alpha}_2};y).
\end{aligned}$$

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For $\widehat{p}=\frac{1}{\widehat{\alpha}_1},\widehat{q}=\frac{\widehat{\alpha}_1}{2-\widehat{\alpha}_1}$ and $\widehat{p}=\frac{1}{\widehat{\alpha}_2},\widehat{q}=\frac{\widehat{\alpha}_2}{2-\widehat{\alpha}_2}$ applying the Hölder's inequality, we get

$$\begin{aligned}|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)| &\leq M\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t-x|^2;x)\}^{\widehat{\alpha}_1/2}\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x)\}^{\widehat{\alpha}_1/2}\\ &\quad\times\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s-y|^2;y)\}^{\widehat{\alpha}_{12}/2}\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;y)\}^{\widehat{\alpha}_2/2}\\ &= M\delta_n^{\widehat{\alpha}_1/2}\delta_m^{\widehat{\alpha}_2/2}.\end{aligned}$$

Hence, we obtain the desired result. □

Theorem 5. *Let $f\in C^1(I_{ab})$ and $0<q_{1,n},q_{2,m}<p_{1,n},p_{2,m}\leq 1$. Then, we have*

$$|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)| \leq \parallel f_x' \parallel_{C(I_{ab})}\delta_n+\parallel f_y'\parallel_{C(I_{ab})}\delta_m.$$

Proof. For $(t,s)\in I_{ab}$, we obtain

$$f(t)-f(s)=\int_x^t f_u'(u,s)du+\int_y^s f_v'(x,v)du$$

By applying our operator to both sides of the above equation, we deduce:

$$\begin{aligned}|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)| &\leq C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\left|\int_x^t f_u'(u,s)du\right|;x,y\right)\\ &\quad+C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\left|\int_y^s f_v'(x,v)du\right|;x,y\right).\end{aligned}$$

As

$$\left|\int_x^t f_u'(u,s)du\right|\leq\parallel f_x'\parallel_{C(I_{ab})}|t-x| \text{ and } \left|\int_y^s f_v'(x,v)du\right|\leq f_y'\parallel_{C(I_{ab})}|s-y|,$$

we have

$$\begin{aligned}|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)| &\leq f_x'\parallel_{C(I_{ab})}C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|t-x|;x,y)\\ &\quad+f_y'\parallel_{C(I_{ab})}C_{n,m}^{(p_1,q_1),(p_2,q_2)}(|s-y|;x,y).\end{aligned}$$

Using the Cauchy-Schwarz inequality, we can write following

$$\begin{aligned}|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)| &\leq \parallel f_x'\parallel_{C(I_{ab})}\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right)\}^{1/2}\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y)\}^{1/2}\\ &\quad+\parallel f_y'\parallel_{C(I_{ab})}\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y\right)\}^{1/2}\{C_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y)\}^{1/2}.\end{aligned}$$

Form (5) and (6), we get desired the result. □

Using Maple, illustrative graphics demonstrate the rate of convergence of the operators $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ to certain functions:

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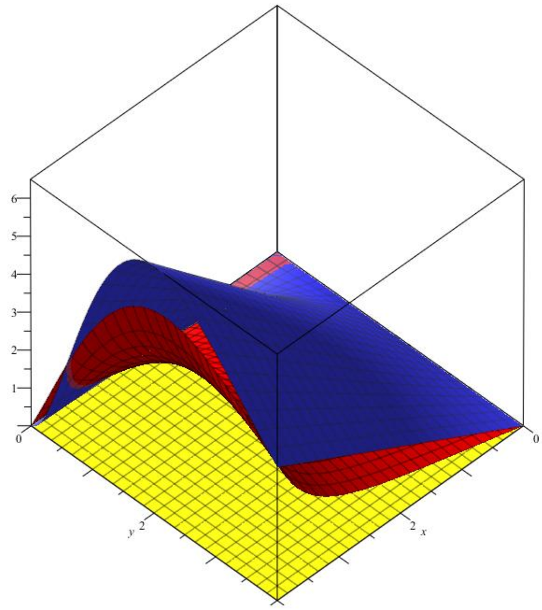

Figure 1. The comparison convergence of $C_{20,20}^{(0.999,0.9),(0.999,0.9)}(f;x,y)$(red), $C_{20,20}^{(0.90,0.86),(0.996,0.89)}(f;x,y)$(yellow) with $\alpha_n = \ln(n)$, $\beta_m = \sqrt{m}$ and $f(x,y) = 3xy^2e^{-y}$(blue)

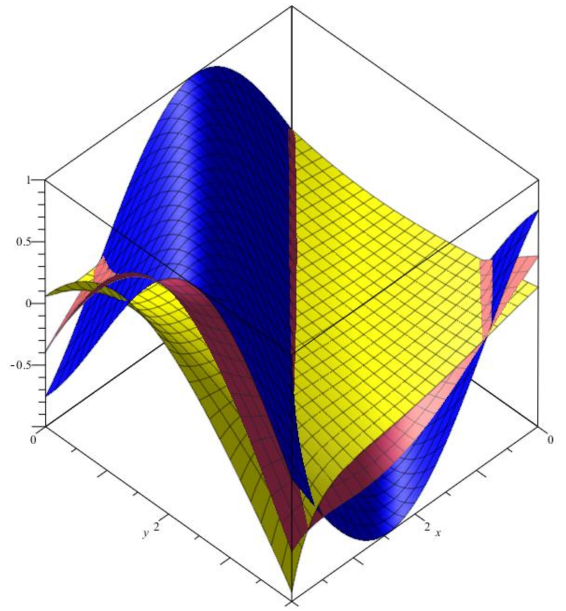

Figure 2. The comparison convergence of $C_{20,20}^{(0.999,0.9),(0.99,0.9)}(f;x,y)$(red), $C_{20,20}^{(0.990,0.86),(0.996,0.89)}(f;x,y)$(yellow) with $\alpha_n = \ln(n)$, $\beta_m = \sqrt{m}$ and $f(x,y) = \sin(x-y)$(blue).

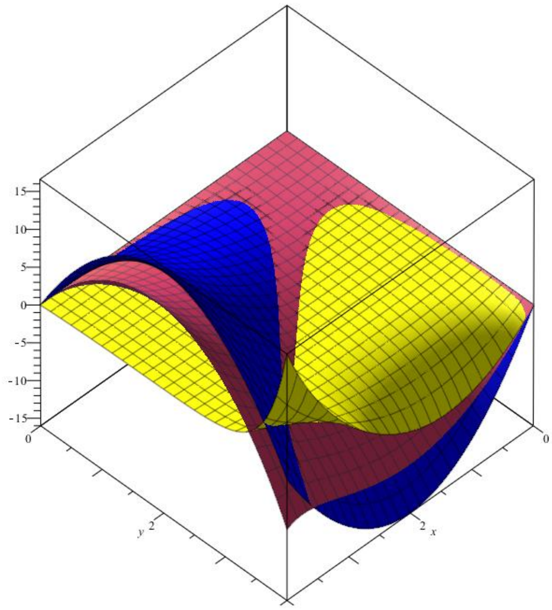

Figure 3. The comparison convergence of $C_{20,20}^{(0.99,0.9),(0.999,0.96)}(f;x,y)$(red), $C_{20,20}^{(0.99,0.9),(0.990,0.90)}(f;x,y)$(yellow) with $\alpha_n = \ln(n)$, $\beta_m = \ln(m)$ and $f(x,y) = x^2y - xy^2$)(blue).

Theorem 6. *Let $f \in C(I_{ab})$, then we have*

$$\left\| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y) \right\|_{C(I_{ab})} \leq 2M(f;\delta_{n,m}(x,y)/2),$$

where

$$\delta_{n,m}(x,y) = \frac{1}{2}\max\left(\frac{a\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}, \frac{b\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}\right).$$

Proof. Let $g \in C^2(I_{ab})$. Utilizing Taylor's formula, we derive:

$$\begin{aligned} g(s_1,s_2) - g(x,y) &= g(s_1,y) - g(x,y) + g(s_1,s_2) - g(s_1,y) \\ &= \frac{\partial g(x,y)}{\partial x}(s_1 - x) + \int_x^{s_1}(s_1 - u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &\quad + \frac{\partial g(x,y)}{\partial x}(s_2 - y) + \int_y^{s_2}(s_2 - v)\frac{\partial^2 g(x,v)}{\partial v^2}dv \\ &= \frac{\partial g(x,y)}{\partial x}(s_1 - x) + \int_0^{s_1 - x}(s_1 - x - u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &\quad + \frac{\partial g(x,y)}{\partial x}(s_2 - y) + \int_0^{s_2 - y}(s_2 - y - v)\frac{\partial^2 g(x,v)}{\partial v^2}dv \end{aligned}$$

By applying $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ to both sides of the above equation, we obtain:

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$$
\begin{aligned}
\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(g;x,y)-g(x,y)\right| &\leq \left|\frac{\partial g(x,y)}{\partial x}\right|\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}((s_1-x);x,y)\right| \\
&+\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\int_0^{s_1-x}(s_1-x-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\right)\right| \\
&+\left|\frac{\partial g(x,y)}{\partial y}\right|\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}((s_2-y);x,y)\right| \\
&+\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left(\int_0^{s_2-y}(s_2-y-v)\frac{\partial^2 g(v,x)}{\partial v^2}dv;x,y\right)\right|
\end{aligned}
$$

As $C_{n,m}^{(p_1,q_1),(p_2,q_2)}((s_1-x);x,y)=0$ and $C_{n,m}^{(p_1,q_1),(p_2,q_2)}((s_2-y);x,y)=0$, one can write following

$$
\begin{aligned}
\left\|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right\|_{C(I_{ab})} &\leq \frac{1}{2}\left\|\frac{\partial g(x,y)}{\partial x}\right\|_{C(I_{ab})}\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s_1-x)^2;x,y\right)\right| \\
&+\frac{1}{2}\left\|\frac{\partial g(x,y)}{\partial y}\right\|_{C(I_{ab})}\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s_2-y)^2;x,y\right)\right|.
\end{aligned}
$$

By (5), (6), we deduce,

$$
\begin{aligned}
\left\|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right\|_{C(I_{ab})} &\leq \frac{1}{2}\max\left(\frac{-p_1^{n-1}x^2}{[n]_{p_1,q_1}}+\frac{xp_1^{n-1}\alpha_n}{[n]_{p_1,q_1}},\frac{-p_2^{m-1}y^2}{[m]_{p_2,q_2}}+\frac{yp_2^{m-1}\beta_m}{[m]_{p_2,q_2}}\right) \\
&\times\left[\left\|\frac{\partial g(x,y)}{\partial x}\right\|_{C(I_{ab})}+\left\|\frac{\partial g(x,y)}{\partial x}\right\|_{C(I_{ab})}\right] \\
&\leq \; \| g \|_{C(I_{ab})}\,\delta_{n,m}. \qquad (10)
\end{aligned}
$$

By the linearity $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$, we obtain

$$
\begin{aligned}
\left\|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right\|_{C(I_{ab})} &\leq \left\|C_{n,m}^{(p_1,q_1),(p_2,q_2)}f-C_{n,m}^{(p_1,q_1),(p_2,q_2)}g\right\|_{C(I_{ab})} \\
&+\left\|C_{n,m}^{(p_1,q_1),(p_2,q_2)}g-g\right\|_{C(I_{ab})}+\|f-g\|_{C(I_{ab})}. \qquad (11)
\end{aligned}
$$

By (10) and (11), one can see that

$$
\left\|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right\|_{C(I_{ab})}\leq 2M(f;\delta_{n,m}(x,y)/2).
$$

This step completes the proof. □

Initially, we need to establish the auxiliary result found in the subsequent lemma.

Lemma 3. *Let $0<q_n<p_n\leq 1$ be sequences such that $p_n,q_n\longrightarrow 1$ and $p_n^n\longrightarrow a_1$ as $n\longrightarrow\infty$. Then, we have the following limits:*

(i) $\lim_{n\to\infty}\frac{[n]_{p_n,q_n}}{\alpha_n}C_{n,n}^{(p_n,q_n)}((t-x)^2;x)=a_1x$
(ii) $\lim_{n\to\infty}\frac{[n]^2_{p_n,q_n}}{\alpha_n^2}C_{n,n}^{(p_n,q_n)}((t-x)^4;x)=3a_1x^2.$

Proof. (i) Using Lemma 1, we have

$$
C_{n,n}^{(p_n,q_n)}((t-x)^2;x)=\frac{-p_n^{n-1}x^2}{[n]_{p_n,q_n}}+\frac{xp_n^{n-1}\alpha_n}{[n]_{p_n,q_n}} \qquad (12)
$$

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Then, we get

$$\frac{[n]_{p_n,q_n}}{\alpha_n} C_{n,n}^{(p_n,q_n)}((t-x)^2;x) = \frac{-p_n^{n-1}x^2}{\alpha_n} + xp_n^{n-1}.$$

Taking the limit of both sides of the above equality as $n \to \infty$, we can write:

$$\begin{aligned}\lim_{n\to\infty} \frac{[n]_{p_n,q_n}}{\alpha_n} \left\{ C_{n,n}^{(p_n,q_n)}((t-x)^2,x) \right\} &= \lim_{n\to\infty} \left\{ \frac{-p_n^{n-1}x^2}{\alpha_n} + xp_n^{n-1} \right\} \\ &= a_1 x.\end{aligned}$$

(ii) Utilizing Lemma 1 along with the linearity of the operators $C_{n,n}^{(p_n,q_n)}$, we arrive at:

$$C_{n,n}^{(p_n,q_n)}((t-x)^4;x) = A_{1,n}x^4 + A_{2,n}x^3 + A_{3,n}x^2 + A_{4,n}x \tag{13}$$

where

$$\begin{aligned}
A_{1,n} &= \frac{p_n^{n-3}[n]_{p_n,q_n}^2(-p_n^2+2p_nq_n-q_n^2) + p_n^{n-5}[n]_{p_n,q_n}(-p_n^3+3p_nq_n^2+q_n^3) - p_n^{3n-6}(p_n^2+p_n^3+2p_nq_n^2+q_n^3)}{[n]_{p_n,q_n}^3} \\
A_{2,n} &= \frac{p_n^{n-3}[n]_{p_n,q_n}^2(p_n^2-2p_nq_n+q_n^2)}{[n]_{p_n,q_n}^3}\alpha_n \\
&+ \frac{p^{2n-5}[n]_{p_n,q_n}(-q_n^3-4p_nq_n^2-3p_n^2q_n+2p_n^3) - p_n^{3n-6}(3p_n^3+3p_nq_n^2+5p_n^2q_n+q_n^3)}{[n]_{p_n,q_n}^3}\alpha_n \\
A_{3,n} &= \frac{p_n^{2n-4}[n]_{p_n,q_n}(-p_n^2+3p_nq_n+q_n^2) - p_n^{3n-5}(3p_n^2+q_n^2+3p_nq_n)}{[n]_{p_n,q_n}^3}\alpha_n^2 \\
A_{4,n} &= \frac{p^{3n-3}\alpha_n^3}{[n]_{p_n,q_n}^3},
\end{aligned}$$

It is clear that

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}^2}{\alpha_n^2} \{A_{4,n}x\} = 0. \tag{14}$$

Taking the limit of both sides of $A_{1,n}$, we arrive at:

$$\begin{aligned}
\lim_{n\to\infty} \frac{[n]_{p_n,q_n}^2}{\alpha_n^2}\{A_{1,n}\} &= \lim_{n\to\infty}\left\{ \frac{-p_n^{n-3}[n]_{p_n,q_n}(p_n-q_n)^2}{\alpha_n^2} + \frac{p_n^{n-5}(-p_n^3+3p_nq_n^2+q_n^3)}{\alpha_n^2} - \frac{p_n^{3n-6}(p_n^2+p_n^3+2p_nq_n^2+q_n^3)}{[n]_{p_n,q_n}\alpha_n^2} \right\} \\
&= \lim_{n\to\infty}\left\{ \frac{-p_n^{n-3}(p_n^n-q_n^n)(p_n-q_n)}{\alpha_n^2} + \frac{p^{n-5}(-p_n^3+3p_nq_n^2+q_n^3)}{\alpha_n^2} - \frac{p_n^{3n-6}(p_n^2+p_n^3+2p_nq_n^2+q_n^3)}{[n]_{p_n,q_n}\alpha_n^2} \right\} \\
&= 0.
\end{aligned} \tag{15}$$

Similarly, we can show that;

$$\lim_{n\to\infty} \frac{[n]_{p_n,q_n}^2}{\alpha_n^2}\{A_{2,n}\} = 0 \text{ and } \lim_{n\to\infty} \frac{[n]_{p_n,q_n}^2}{\alpha_n^2}\{A_{3,n}\} = 3a_1x^2 \tag{16}$$

By combining (14)–(16), we reach the desired the result. □

Now, we ready present a Voronovskaja type theorem for $C_{n,n}^{(p_n,q_n)}(f;x,y)$.

Theorem 7. *Let $f \in C^2(I_{ab})$. Then, we have*

$$\lim_{n\to\infty} [n]_{p_n,q_n} C_{n,n}^{(p_n,q_n)}(f;x,y) - f(x,y)) = \frac{a_1 x f''_{x^2}(x,y)}{2} + \frac{a_1 y f''_{y^2}(x,y)}{2}.$$

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Proof. Let $(x,y) \in I_{ab}$. Then, write Taylor's formula of f as follows:

$$f(s,t) = f(x,y) + f'_x(s-x) + f'_y(t-y) + \frac{1}{2}\left\{ f''_x(t-x)^2 + 2f'_{xy}(s-x)(t-y) + f''_y(t-y)^2 \right\} + \varepsilon(s,t)\Big((s-x)^2 + (t-y)^2\Big) \quad (17)$$

where $(s,t) \in I_{ab}$ and $\varepsilon(s,t) \longrightarrow 0$ as $(s,t) \longrightarrow (x,y)$.

If we apply the operator $C_{n,n}^{(p_n,q_n)}(f;.)$ on (17), we obtain

$$\begin{aligned} C_{n,n}^{(p_n,q_n)}(f;s,t) - f(x,y) &= f'_x(x,y)C_{n,n}^{(p_n,q_n)}((s-x);x,y) + f'_y(x,y)C_{n,n}^{(p_n,q_n)}((t-y);x,y) \\ &+ \frac{1}{2}\Big\{ f''_{x^2}C_{n,n}^{(p_n,q_n)}((t-x_0)^2;x,y) + 2f'_{xy}C_{n,n}^{(p_n,q_n)}((s-x)(t-y);x,y) \\ &+ f''_{y^2}C_{n,n}^{(p_n,q_n)}((t-y)^2;x,y) \Big\} + C_{n,n}^{(p_n,q_n)}\Big(\varepsilon(s,t)\Big((s-x)^2 + (t-y)^2\Big);x,y\Big). \end{aligned}$$

Applying the limit of both sides of the above equality, we get $n \longrightarrow \infty$, $\square$

$$\begin{aligned} \lim_{n\to\infty}[n]_{p_n,q_n}C_{n,n}^{(p_n,q_n)}(f;s,t) - f(x,y)) &= \lim_{n\to\infty}[n]_{p_n,q_n}\frac{1}{2}\Big\{ f''_{x^2}C_{n,n}^{(p_n,q_n)}((t-x)^2;x,y) \\ &+ 2f'_{xy}C_{n,n}^{(p_n,q_n)}((s-x)(t-y);x,y) + f''_{y^2}C_{n,n}^{(p_n,q_n)}((t-y)^2;x,y) \Big\} \\ &+ \lim_{n\to\infty}[n]_{p_n,q_n}C_{n,n}^{(p_n,q_n)}\Big(\varepsilon(s,t)\Big((s-x)^2 + (t-y)^2\Big);x,y\Big). \end{aligned}$$

By Cauchy-Schwartz inequality, we can write the following

$$\begin{aligned} C_{n,n}^{(p_n,q_n)}\Big(\varepsilon(s,t)\Big((s-x)^2 + (t-y)^2\Big);x,y\Big) &\le \sqrt{\lim_{n\to\infty} C_{n,n}^{(p_n,q_n)}(\varepsilon^2(s,t);x,y)} \\ &\times \sqrt{2\lim_{n\to\infty}[n]^2_{p_n,q_n}C_{n,n}^{(p_n,q_n)}(\varepsilon(s,t)((s-x)^4 + (t-y)^4);x,y)}. \end{aligned}$$

As $\lim_{n\to\infty} C_{n,n}^{(p_n,q_n)}(\varepsilon^2(s,t);x,y) = \varepsilon^2(x,y) = 0$ and from Lemma 3(ii)

$\lim_{n\to\infty}[n]^2_{p_n,q_n}C_{n,n}^{(p_n,q_n)}((s-x)^4 + (t-y)^4);x,y)$ is finite, then we have

$$\lim_{n\to\infty}[n]^2_{p_n,q_n}C_{n,n}^{(p_n,q_n)}\Big(\varepsilon(s,t)\Big((s-x)^4 + (t-y)^4\Big);x,y\Big) = 0.$$

Hence, we deduce

$$\lim_{n\to\infty}[n]_{p_n,q_n}C_{n,n}^{(p_n,q_n)}(f;x,y) - f(x,y)) = \frac{a_1 x f''_{x^2}(x,y)}{2} + \frac{a_1 y f''_{y^2}(x,y)}{2}.$$

This step completes the proof.

4. weighted Approximation Properties of two variable function

In this section, we investigate the convergence of the sequence of linear positive operators $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ to a function of two variables defined within a weighted space. We also compute the rate of convergence using the weighted modulus of continuity.

Let $\rho(x,y) = x^2 + y^2 + 1$, and define B_ρ as the space of all functions f defined on the real axis that satisfy $|f(x,y)| \le M_f\rho(x,y)$, where M_f is a positive constant dependent solely on f. The subspace C_ρ of B_ρ consists of all continuous functions and is equipped with the norm:

$$\|f\|_\rho = \sup_{(x,y)\in\mathbb{R}^2_+} \frac{|f(x,y)|}{\rho(x,y)}.$$

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Let C_ρ^0 represent the subspace of all functions $f \in C_\rho$ for which $\lim_{x\to\infty} \frac{f(x,y)}{\rho(x,y)}$ exists and is finite. For every $f \in C_\rho^0$, the weighted modulus of continuity is defined as

$$\Omega_f(f;\delta_1,\delta_2) = \sup_{(x,y)\in\mathbb{R}_+^2} \sup_{|h_1|\le\delta_1, |h_2|\le\delta_2} \frac{|f(x+h_1,y+h_2) - f(x,y)|}{\rho(x,y)\,\rho(h_1,h_2)}. \tag{18}$$

Lemma 4. *The operators $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ defined (2) act from $C_\rho(\mathbb{R}_+^2)$ to $B_\rho(\mathbb{R}_+^2)$ if and only if the inequality*

$$\| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(\rho;x,y) \|_{x^2} \le c.$$

holds for some positive constant c.

Theorem 8. *Consider the sequence of linear positive operators $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ defined in (2). For any function $f \in C_\rho^0$ and for all points $(x,y) \in I_{\alpha_n\beta_m}$, it follows that*

$$\lim_{n\to\infty} \|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\|_\rho = 0.$$

Proof. From Lemma 2, we obtain

$$\begin{aligned}
\| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(1;x,y) - 1 \|_\rho &= 0, \\
\| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(s;x,y) - x \|_\rho &= 0 \\
\| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(t;x,y) - y \|_\rho &= 0.
\end{aligned}$$

Again by Lemma 2, we can write following

$$\begin{aligned}
&\| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(s^2+t^2;x,y) - (x^2+y^2) \|_\rho \\
= &\sup_{(x,y)\in\mathbb{R}_+^2} \left\{ \frac{p_1^{n-1}\alpha_n x}{[n]_{p_1,q_1}(x^2+y^2+1)} + \frac{p_1^{n-1}x^2}{[n]_{p_1,q_1}(x^2+y^2+1)} + \frac{p_2^{m-1}\beta_m y}{[m]_{p_2,q_2}(x^2+y^2+1)} + \frac{p_2^{m-1}y^2}{[m]_{p_2,q_2}(x^2+y^2+1)} \right\} \\
\le &\frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}} + \frac{p_1^{n-1}}{[n]_{p_1,q_1}} + \frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}} + \frac{p_2^{m-1}}{[m]_{p_2,q_2}}
\end{aligned}$$

Considering the limit of both sides of the preceding inequality as $n,m \to \infty$ and applying (3) and (4), we derive

$$\lim_{m,n\to\infty} \| C_{n,m}^{(p_1,q_1),(p_2,q_2)}(s^2+t^2;x,y) - (x^2+y^2) \|_\rho = 0.$$

Applying weighted Korovkin theorem for two variable which presented by Gadzhiev [4,5], we get desired the results. □

For estimate rate of convergence we need the following lemma.

Lemma 5. *For all $(x;y) \in I_{\alpha_n\beta_m}$, by (5), (6) and (13), one can write the following*

$$C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right) = O\left(\frac{\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}\right)\left(x^2+x\right), \tag{19}$$

$$C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^4;x,y\right) = O\left(\frac{\alpha_n p_1^{n-1}}{[n]_{p_1,q_1}}\right)\left(x^4+x^3+x^2+x\right) \tag{20}$$

and

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$$C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y\right) = O\left(\frac{\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}\right)\left(y^2+y+1\right), \tag{21}$$

$$C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^4;x,y\right) = O\left(\frac{\beta_m p_2^{m-1}}{[m]_{p_2,q_2}}\right)\left(y^4+y^3+y^2+y+1\right). \tag{22}$$

Now, compute rate of convergence the operator $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ in weighted spaces .

Theorem 9. *If $f \in C_\rho^0$, then we have*

$$\sup_{(x,y)\in\mathbb{R}_+^2} \frac{\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\right|}{\rho(x,y)^3} \leq \mathcal{C}_2\omega_\rho(f;\delta_n,\delta_m)$$

, where $\mathcal{C}_2$ is a constant independent of n,m and $\delta_n = \frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}$, $\delta_n = \frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}$.

Proof. Taking into account the following inequality given in [9], we deduce

$$|f(t,s) - f(x,y)| \leq 8\left(1+x^2+y^2\right)\omega_\rho(f;\delta_n,\delta_m)$$

$$\times\left(1+\frac{|t-x|}{\delta_n}\right)\left(1+\frac{|s-y|}{\delta_m}\right)\left(1+(t-x)^2\right)\left(1+(s-y)^2\right).$$

Applying $C_{n,m}^{(p_1,q_1),(p_2,q_2)}$ both side above inequality and using Cauchy-Schwarz inequality, one can write following

$$\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\right| \leq 8\left(1+x^2+y^2\right)\omega_\rho(f;\delta_n,\delta_m)$$

$$\times\left[1 + C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right) + \frac{1}{\delta_n}\sqrt{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right)}\right.$$

$$\left.\frac{1}{\delta_n}\sqrt{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^2;x,y\right)C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((t-x)^4;x,y\right)}\right]$$

$$\times\left[1 + C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y,a\right) + \frac{1}{\delta_m}\sqrt{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y\right)}\right.$$

$$\left.\times\frac{1}{\delta_m}\sqrt{C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^2;x,y\right)C_{n,m}^{(p_1,q_1),(p_2,q_2)}\left((s-y)^4;x,y\right)}\right].$$

By (19)-(22), we obtain

$$\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y) - f(x,y)\right| \leq 8\left(1+x^2+y^2\right)\omega_\rho(f;\delta_n,\delta_m)$$

$$\times\left[1 + O\left(\frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}\right)\left(x^2+x\right) + \frac{1}{\delta_n}\sqrt{O\left(\frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}\right)(x^2+x)}\right.$$

$$\left.+\frac{1}{\delta_n}\sqrt{O\left(\frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}\right)(x^2+x)(x^4+x^3+x^2+x)}\right]$$

$$\times\left[1+\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}\left(y^2+y\right)+\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}\sqrt{\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}}\right.$$
$$\left.+\frac{1}{\delta_m}\sqrt{\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}(y^2+y)\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}(y^4+y^3+y^2+y)}\right].$$

Taking $\delta_n=\left(\frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}\right)^{1/2},\delta_m=\left(\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}\right)^{1/2}$, one write the following:

$$\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right|\leq C_2\left(1+x^2+y^2\right)\omega_\rho(f;\delta_n,\delta_m)$$
$$\times\left[1+\delta_n^2\left(x^2+x\right)+\sqrt{x^2+x}+\sqrt{(x^2+x)(x^4+x^3+x^2+x)}\right]$$
$$\times\left[1+\delta_m^2\left(y^2+y\right)+\sqrt{(y^2+y)}+\sqrt{(y^2+y)(y^4+y^3+y^2+y)}\right],$$

where C_2 is a constant independent of n,m. Since $\delta_n^2<1,\delta_m^2<1$, for sufficiently large n,m, we get

$$\sup_{(x,y)\in\mathbb{R}_+^2}\frac{\left|C_{n,m}^{(p_1,q_1),(p_2,q_2)}(f;x,y)-f(x,y)\right|}{(1+x^2+y^2)^3}\leq C_2\omega_\rho\left(f;\sqrt{\frac{p_1^{n-1}\alpha_n}{[n]_{p_1,q_1}}},\sqrt{\frac{p_2^{m-1}\beta_m}{[m]_{p_2,q_2}}}\right).$$

This step completes the proof. □

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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