Review

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Review

Research Progress on Newton's iterative methods for nonlinear equation

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Abstract: Reviewed the research progress of Newton's iterative methods for nonlinear equation. Convergence with second order, third order, fourth order, fifth order, sixth order, seventh order, eighth order and ninth order Newton's iterative methods are given in turn, and the efficiency index of these iterative methods are analyzed. Numerical experiments show that the convergence process of the three Newton iterative methods, and the results indicate that the convergence of the higher-order Newton's method can be well demonstrated only when the initial point is close to the root.

Keywords: nonlinear equation; Newton's iteration method; convergence order; efficiency index; initial point

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1. Introduction

For the nonlinear equation $f(x)=0$, $f(x)$ is a smooth curve, and has a single root on the open interval I .The classical Newton's method and the modified Newton's method are the most basic methods for solving nonlinear equations. In order to improve the convergence speed, the third, fourth, fifth, sixth, seventh, eighth and ninth order Newton's methods have appeared one after another. For the initial point x_0, the higher the order of convergence, and the faster the convergence speed. If the nonlinear equations are convergent, the computational complexity also needs to be considered. In general, the higher the order of convergence and the larger the computational workload. Therefore, when constructing a higher-order Newton method, it is necessary to consider both the convergence order and the computational efficiency.

Definition1 Let the sequence $\{x_k\}$ converge to a , a is single root of the equation $f(x)=0$,denote $e_k = x_k - a$. If there exist $p \geq 1$ and a constant $C \neq 0$, such that the limit $\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^p} = C$ holds, then the sequence $\{x_k\}$ is convergent of p order. Here C is the convergence factor and $e_{k+1} = Ce_k^p + O(e_k^{p+1})$ is the error equation. When $p=1$,the convergence is linear convergence. When $p>1$,the convergence is superlinear convergence. When $p=2$, the convergence is square convergence.

Definition 2 If the convergence order of an iterative method is $p>1$, and the computational workload for each iteration from x_k to x_{k+1} is n (the computation numbers of function values and derivative function values), then the efficiency index of this iterative method is $E=\sqrt[n]{p}$.

There are some Newton's methods for solving nonlinear equations. Following we analyses the convergence order and efficiency index of these Newton's iterative methods.

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2. Several Newton's iterative methods

2.1. Classical Newton's iterative

The process of classical Newton iteration is as follows:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

The classical Newton's method has a second-order convergence rate, i.e. $p = 2$ .Each iteration from x_k to x_{k+1} requires the calculation of the function value $f(x_k)$ and the derivative function $f'(x_k)$ value. So the computational workload is 2. Therefore, the efficiency index of the classical Newton method is $\sqrt{2} \approx 1.414$. The second-order convergence can be well reflected when the initial point x_0 is very close to the root a in actual computation.

2.2. Newton's iterative method with third order convergence

Common Newton's iteration methods with third order convergence are listed in following.

①Euler iterative method

$$x_{k+1} = x_k - \frac{2}{1+\sqrt{1-2L_f(x_k)}}\frac{f(x_k)}{f'(x_k)},$$

$$L_f(x_k) = \frac{f(x_k)f''(x_k)}{[f'(x_k)]^2}.$$

Each iteration from x_k to x_{k+1} requires the calculation of the function value $f(x_k)$, and the derivative function value $f'(x_k), f''(x_k)$. So the computational workload is 3, thus the efficiency index of the iteration is $\sqrt[3]{3} \approx 1.442$.

②Halley iterative method

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2[f'(x_k)]^2 - f(x_k)f''(x_k)}.$$

③Chebyshev iteration method

$$x_{k+1} = x_k - \left[1+\frac{1}{2}L_f(x_k)\right]\frac{f(x_k)}{f'(x_k)},$$

$$L_f(x_k) = \frac{f(x_k)f''(x_k)}{[f'(x_k)]^2}.$$

Moreover,

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$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{2f(x_k)}{f'(x_k)+f'(y_k)}, \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{2}\left(\dfrac{1}{f'(x_k)} + \dfrac{1}{f'(y_k)}\right), \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f(x_k)+f(y_k)}{f'(x_k)}, \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(\frac{x_k+y_k}{2})}, \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{2f(x_k)}{2f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{5f'(x_k)-f'(y_k)}{2f'(x_k)+2f'(y_k)}\dfrac{f(x_k)}{f'(x_k)}, \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{3f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f'(x_k)+f'(y_k)}{4f'(x_k)-2f'(y_k)}\dfrac{f(x_k)}{f'(x_k)}. \end{cases}$$

These are all convergent Newton's iterations with third order, and the efficiency indices of all these iterative methods are $\sqrt[3]{3} \approx 1.442$. There are a variety of third order convergent Newton's iterations in the literature [1]. Among them, what is striking is that

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2x_k}\left[x_k^2 - \left(\frac{f(x_k)}{f'(x_k)}\right)^2\right].$$

The computational workload for each iteration from x_k to x_{k+1} is 2 (just calculate $f(x_k), f'(x_k)$). Therefore, the efficiency index of this iteration is $\sqrt[2]{3} \approx 1.732$, which is the best result for the efficiency index of the current third order convergent Newton iteration method.

2.3. Newton's iterative method with fourth order convergence

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The classical double Newton's iterative method

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(y_k)}. \end{cases}$$

It has fourth order convergence and its efficiency index is $\sqrt[4]{4} \approx 1.414$.

There is the fourth-order convergent Newton iterative method in the literature [2],

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} - 2\dfrac{f(y_k)}{f'(x_k)} + \dfrac{f(y_k)f'(y_k)}{[f'(x_k)]^2}. \end{cases}$$

Its efficiency index is $\sqrt[4]{4} \approx 1.414$. In the literature [3], another fourth-order convergent Newton iterative method is given,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f^2(x_k) + f^2(y_k)}{f'(x_k)\left[f(x_k) - f(y_k)\right]}. \end{cases}$$

The Jarratt iterative method is an iterative format with fourth-order convergence [4]. Where one iteration requires the calculation of two derivative function values and one function value.The specific forms are as follows.

$$\begin{cases} y_k = x_k - \dfrac{2}{3}\dfrac{f(x_k)}{f'(x_k)} \\ x_{k+1} = x_k - \dfrac{3f'(y_k) + f'(x_k)}{6f'(y_k) - 2f'(x_k)} \cdot \dfrac{f(x_k)}{f'(x_k)} \end{cases}$$

In the literature [5], an iterative method with fourth-order convergence is presented.

$$\begin{cases} y_k = x_k - \dfrac{2}{3}\dfrac{f(x_k)}{f'(x_k)} \\ z_k = y_k - \left[\dfrac{1}{28} + \dfrac{\left(f'(x_k)\right)^2}{\dfrac{294}{125}\left(f'(y_k)\right)^2 + \dfrac{126}{125}\left(f'(x_k)\right)^2}\right]\dfrac{f(x_k)}{f'(y_k)}. \end{cases}$$

King gave a class of iterative formats with fourth-order convergence in the literature [6].

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = x_k - \dfrac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta - 2) f(y_k)} \dfrac{f(y_k)}{f'(x_k)}. \end{cases}$$

where $\beta \in R$.

The computational efficiency index of the last four iterations is $\sqrt[3]{4} \approx 1.587$, which is the best result for the efficiency index of the current fourth-order convergent Newton's iteration method.

2.4. Newton's iterative method with fifth order convergence

The variation of the double Newton method

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)} \\ x_{k+1} = y_k - \left[1 + \left(\dfrac{f(y_k)}{f(x_k)}\right)^2\right] \dfrac{f(y_k)}{f'(y_k)}. \end{cases}$$

has fifth-order convergence.

Ham gave a fifth order convergent Newtonian iterative method in [7]

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)} \\ x_{k+1} = x_k - \dfrac{Af'(y_k) + Bf'(x_k)}{Cf'(y_k) + Df'(x_k)} \cdot \dfrac{f(y_k)}{f'(x_k)}. \end{cases}$$

Where the parameters A, B, C, D satisfy $A + B = C + D, C = B + 2A, B = 3A, C + D \neq 0$.

Xue gave a fifth order convergent Newton iterative method in [8]

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \quad z_k = x_k - \dfrac{f(x_k)}{f'\left(\dfrac{x_k + y_k}{2}\right)}, \\ x_{k+1} = z_k - \dfrac{f(z_k) f'\left(\dfrac{x_k + y_k}{2}\right)}{3f'(x_k) f'\left(\dfrac{x_k + y_k}{2}\right) - 2\left[f'(x_k)\right]^2}. \end{cases}$$

Liu gave a class of Newton iterative methods with fifth order convergence in [9]

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{f(y_k)}{f'(y_k)} - \dfrac{f^2(y_k)\left(\left(f'(x_k)\right)^3 - f'(y_k)\left(f'(x_k)\right)^2 - \mu\left(f(x_k)\right)^2\right)}{2\left(f'(y_k)\right)^3 f'(x_k) f(x_k)}. \end{cases}$$

Wu gave two Newton iterative methods with fifth order convergence in [10]

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$$\begin{cases} y_k = x_k - \dfrac{2f(x_k)}{3f'(x_k)}, \\ z_k = x_k - \dfrac{5f'(x_k) - f'(y_k)}{2f'(x_k) + 2f'(y_k)} \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \dfrac{4f(z_k)\left[f'(x_k) + f'(y_k)\right]}{22f'(y_k)f'(x_k) - 11\left[f'(x_k)\right]^2 - 3\left[f'(y_k)\right]^2}. \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{3f'(x_k)}, \\ z_k = x_k - \dfrac{f'(x_k) + f'(y_k)}{4f'(y_k) - 2f'(y_k)} \dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \dfrac{2f(z_k)\left[2f'(y_k) - f'(x_k)\right]}{4f'(y_k)f'(x_k) - 5\left[f'(x_k)\right]^2 + 3\left[f'(y_k)\right]^2}. \end{cases}$$

More fifth-order convergent Newton iteration methods are given in the literature [11-17].

In recent years, a new fifth order convergent Newton iteration method is presented in the literature [18],

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{2f'(x_k)}, \\ z_k = x_k - \dfrac{f(x_k)}{f'(y_k)}, \\ x_{k+1} = z_k - \left[\dfrac{2}{f'(y_k)} - \dfrac{1}{f'(x_k)}\right] f(z_k). \end{cases}$$

It is successfully applied to solve linear programming and portfolio optimization.

The computational efficiency index of all these fifth order Newton iterations above is $\sqrt[4]{5} \approx 1.495$, which is the best result of the efficiency index of the current fifth order convergent Newton's iteration method.

2.5. Newton's iterative method for sixth order convergence

In order to improve the convergence order, Kou [4] et al. constructed a new three-step iterative method as follows.

$$\begin{cases} y_k = x_k - \dfrac{2}{3} \dfrac{f(x_k)}{f'(x_k)} \\ z_k = x_k - J_f(x_k) \cdot \dfrac{f(x_k)}{f'(x_k)}, \quad J_f(x_k) = \dfrac{3f'(y_k) + f'(x_k)}{6f'(y_k) - 2f'(x_k)} \\ x_{k+1} = z_k - \dfrac{2f(z_k)}{\dfrac{3}{2} J_f(x_k) f'(y_k) + \left(1 - \dfrac{3}{2} J_f(x_k)\right) f'(x_k)}. \end{cases}$$

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and proved that the iteration has sixth order convergence.

Chun proposed the Newton iterative methods with sixth order convergence based on the literature [19],

$$\begin{cases} y_k = x_k - \dfrac{2}{3}\dfrac{f(x_k)}{f'(x_k)} \\ z_k = x_k - J_f(x_k)\cdot\dfrac{f(x_k)}{f'(x_k)}, \quad J_f(x_k) = \dfrac{3f'(y_k)+f'(x_k)}{6f'(y_k)-2f'(x_k)} \\ x_{k+1} = z_k - \dfrac{2f(z_n)}{\lambda(z_k-x_k)(z_k-y_k)+\dfrac{3}{2}J_f(x_k)f'(y_k)+\left(1-\dfrac{3}{2}J_f(x_k)\right)f'(x_k)}. \end{cases}$$

where $\lambda \in R$, and it is the sixth-order convergence. When $\lambda = 0$, it is the method of the Kou's method. In the literature [10], Wu gave two Newton's iterative methods with sixth order convergence in the following,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = x_k - \dfrac{2f(x_k)}{f'(x_k)+f'(y_k)}, \\ x_{k+1} = z_k - \dfrac{f(z_k)\left[f'(x_k)+f'(y_k)\right]}{3f'(y_k)f'(x_k)-\left[f'(x_k)\right]^2}. \end{cases}$$

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = x_k - \dfrac{\left[f'(x_k)\right]^2+\left[f'(y_k)\right]^2}{f'(x_k)f'(y_k)+\left[f'(y_k)\right]^2}\dfrac{f(x_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \dfrac{f(z_k)f'(y_k)\left[f'(x_k)+f'(y_k)\right]}{2\left[f'(x_k)\right]^2 f'(y_k)-\left[f'(x_k)\right]^3+\left[f'(y_k)\right]^3}. \end{cases}$$

another sixth order convergent Newton iterative method is proposed in the literature [20].

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \dfrac{(x_k-y_k)}{f(x_k)-2f(y_k)}f(y_k), \\ x_{k+1} = z_k - \dfrac{(x_k-y_k)}{f(x_k)-2f(y_k)}f(z_k). \end{cases}$$

The efficiency index of these sixth order Newton iterations above are $\sqrt[4]{6} \approx 1.565$.

2.6. Newton's iterative method with seventh order convergence

Two Newton iterative methods with seventh-order convergence are given by Zheng in the literature [21], one of them is

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$$\begin{cases} z_k = x_k - \left(1 + \dfrac{1}{2}\dfrac{f''(x_k)f(x_k)}{[f'(x_k)]^2}\right)\dfrac{f(x_k)}{f'(x_k)}, \\ y_k = z_k - \dfrac{f(z_k)}{f'(x_k) + f''(x_k)(z_k - x_k)}, \\ x_{k+1} = y_k - \dfrac{2f(y_k)}{f'(z_k) + f'(y_k)}. \end{cases}$$

Its computational efficiency index is $\sqrt[7]{7} \approx 1.320$. Another Newton iterative method with seventh order convergence is

$$\begin{cases} z_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ y_k = z_k - \dfrac{f'(z_k) + 3f'(x_k)}{5f'(z_k) - f'(x_k)}\dfrac{f(z_k)}{f'(x_k)}, \\ x_{k+1} = y_k - \dfrac{2f(y_k)}{f'(z_k) + f'(y_k)}. \end{cases}$$

Its computational efficiency index is $\sqrt[6]{7} \approx 1.383$.

Shno [22] constructed a new three-step iterative method based on the Jarratt iterative method,

$$\begin{cases} y_k = x_k - \dfrac{2}{3}\dfrac{f(x_k)}{f'(x_k)} \\ z_k = x_k - J_f(x_k)\cdot\dfrac{f(x_k)}{f'(x_k)}, \quad J_f(x_k) = \dfrac{3f'(y_k) + f'(x_k)}{6f'(y_k) - 2f'(x_k)} \\ x_{k+1} = z_k - \dfrac{f(z_k)}{3\dfrac{f(z_k) - f(x_k)}{(z_k - x_k)} - 2f'(x_k) + \dfrac{1}{2}f''(x_k)(z_k - x_k)} \end{cases}$$

Its computational efficiency index is $\sqrt[5]{7} \approx 1.476$.

Wang [3] gave a Newton iterative method with seventh order convergence using difference quotient in the following form,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = x_k - \dfrac{[f(x_k)]^2 + [f(y_k)]^2}{f'(x_k)[f(x_k) - f(y_k)]}, \\ x_{k+1} = z_k - \dfrac{f(z_k)}{2f[z_k, y_k] + f[z_k, x_k, x_k](z_k - y_k)}. \end{cases}$$

Where $f[z_k, y_k], f[z_k, x_k, x_k]$ is the first-order difference quotient and second-order difference quotient, respectively. Its computational efficiency index is $\sqrt[4]{7} \approx 1.627$.

The Newton iterative method with seventh order convergence is given by Kou [23] in the following form,

 doi:10.20944/preprints202410.1642.v1

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \dfrac{f(y_k)}{f(x_k) - 2f(y_k)}(x_k - y_k), \\ x_{k+1} = z_k - \left[\left(1 + \dfrac{f(y_k)}{f(x_k) - 2f(y_k)}\right)^2 + \dfrac{f(z_k)}{f(y_k)}\right]\dfrac{f(z_k)}{f(x_k)}. \end{cases}$$

Whose efficiency index is $\sqrt[4]{7} \approx 1.627$,which is the best result of the efficiency index of the current seventh order convergent Newton's iteration method.

2.7. Newton's iterative method with eighth order convergence

The literature [3] gave the Newton's iterative method with eighth order convergence in the following form,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \dfrac{[f(x_k)]^2 + [f(y_k)]^2}{f'(x_k)[f(x_k) - f(y_k)]}, \\ x_{k+1} = z_k - \dfrac{f(z_k)}{f'(z_k)}. \end{cases}$$

Whose efficiency index is $\sqrt[5]{8} = 1.516$.

The literature [24] gave the Newton iterative method with eighth order convergence in the following form,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)} \\ z_k = y_k - \dfrac{f(x_k) + \beta f(y_k)}{f(x_k) + (\beta - 2)f(y_k)} \dfrac{f(y_k)}{f'(x_k)} \\ x_{k+1} = z_k - H(\mu_k)\dfrac{f(z_k)}{f[z_k, y_k] + f[z_k, x_k, x_k](z_k - y_k)}. \end{cases}$$

Where $H(t)$ is a real-valued function and satisfies $H(0) = 1, H'(0) = 2, |H''(0)| < \infty, \beta = -0.5$, $\mu_k = \dfrac{f(z_k)}{f(x_k)}$, $f[z_k, y_k], f[z_k, x_k, x_k]$ are the first-order difference quotient and the second-order difference quotient, respectively. The function value needs to be evaluated three times at each iteration and its first-order derivative is evaluated once, so the efficiency index of this method is $\sqrt[4]{8} = 1.682$.

A class of Newton's iterative methods with eighth order convergent is given in the literature [25] with efficiency index of $\sqrt[4]{8} = 1.682$,which is the best result for the efficiency index of the current eighth order convergent Newton's iterative method. More Newton iterative methods with eighth order convergence can be found in the literature [26-28].

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2.8. Newton's iterative method with ninth order convergence

A Newton's iteration method with ninth order convergence is in the literature [29] as follows,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \left[1 + \dfrac{f^2(y_k)}{f^2(x_k)}\right] \dfrac{f(y_k)}{f'(y_k)}, \\ x_{k+1} = z_k - \left[1 + \dfrac{2f^2(y_k)}{f^2(x_k)} + \dfrac{2f(z_k)}{f(y_k)}\right] \dfrac{f(z_k)}{f'(y_k)}. \end{cases}$$

Whose efficiency index is $\sqrt[5]{9} = 1.552$.

The literature [30] constructed a Newton iterative method with ninth order convergence using the difference quotient,

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)} \\ z_k = y_k - \dfrac{2f(y_k)f'(y_k)}{2\left(f'(y_k)\right)^2 - f(y_k)P_f(x_k, y_k)} \\ x_{k+1} = z_k - \dfrac{f(z_k)}{f[z_k, y_k] + f[z_k, x_k, x_k](z_k - y_k)} \end{cases}$$

Here $P_f(x_k, y_k) = \dfrac{2}{(y_k - x_k)}\left[2f'(y_k) + f'(x_k) - 3\dfrac{f(y_k) - f(x_k)}{y_k - x_k}\right]$, $f[z_k, y_k], f[z_k, x_k, x_k]$ are the first-order difference quotient and the second-order difference quotient respectively, and the efficiency index of this method is $\sqrt[5]{9} = 1.552$.

3. Efficiency index of the Newton iterative method

Table 1 gives the values of the corresponding functions $\sqrt[n]{p}$ for p taken from 2 to 9 and n taken from 2 to 9.

Table 1. Values of the functions $\sqrt[n]{p}$ (p taken from 2 to 9 and n taken from 2 to 9).

P \ n	2	3	4	5	6	7	8	9
2	1.414	1.260	1.189	1.149	1.122	1.104	1.091	1.080
3	1.732	1.442	1.316	1.246	1.201	1.170	1.147	1.130
4	2.000	1.587	1.414	1.320	1.260	1.219	1.189	1.167
5	2.236	1.710	1.495	1.380	1.308	1.258	1.223	1.196
6	2.449	1.817	1.565	1.431	1.348	1.292	1.251	1.220
7	2.646	1.913	1.627	1.476	1.383	1.320	1.275	1.241
8	2.828	2.000	1.682	1.516	1.414	1.346	1.297	1.260
9	3.000	2.080	1.732	1.552	1.442	1.369	1.316	1.277

As pointed out earlier, the higher the order of convergence and the greater the computational cost required. Therefore, both the convergence order and the computational complexity need to be considered when constructing a higher order Newton's iterative method for solving nonlinear

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equations. The underlined numbers in Table 1 indicate the best efficiency index that can be achieved when the convergence order is taken from 2 to 9. Whether the efficiency index can theoretically reach 2 needs to be further investigated.

4. Numerical experiments

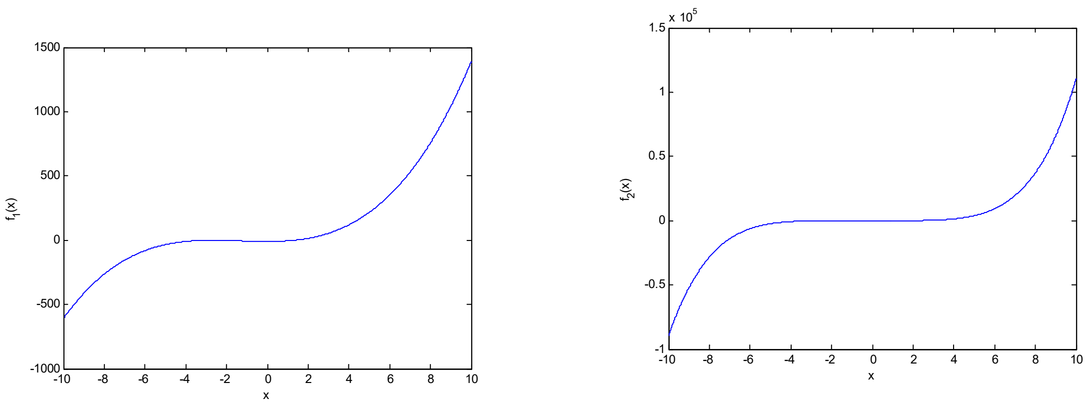
Following we give two numerical examples to verify the convergence process of Newton's iterative method.

$$f_1(x) = x^3 + 4x^2 - 10 = 0,$$

$$f_2(x) = x^5 + x^4 + 4x^2 - 20 = 0.$$

The root of equation $f_1(x) = 0$ is $a = 1.365230013414096887913 73$.

The root of equation $f_2(x) = 0$ is $a = 1.466279073864722670705 87$.



(a) image of function $f_1(x)$ (b) image of function $f_2(x)$

Figure 1. Image of the function $f_1(x), f_2(x)$.

Three Newton's iterative methods are used for solving, and the program is written by MATLAB R2009a. In the classical Newton iterative method, we set $\left|x^{(k+1)} - x^{(k)}\right| + \left|f(x^{(k)})\right| < \varepsilon$ is the termination condition. In the fifth order Newton method [16] and the ninth-order Newton method [29], we set

$$\left|x^{(k+1)} - x^{(k)}\right| + \left|f(x^{(k)})\right| + \left|f(y^{(k)})\right| + \left|f(z^{(k)})\right| < \varepsilon$$

is the termination condition, where $\varepsilon = 1 \times 10^{-10}$.

Table 2 gives the calculation results of the three Newton's iterative methods by taking the different initial points.

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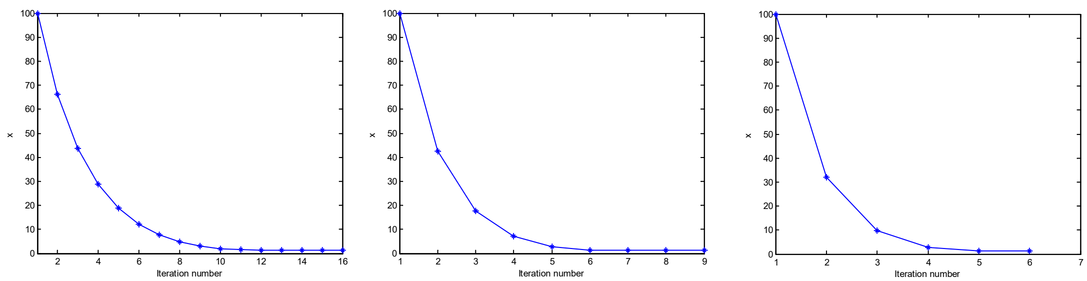
Table 2. Calculation results of three Newton's iterative methods.

$f_i(x)$	Initial point	Number of iterations		
		Classical Newton iteration	Fifth-order Newton iteration	Ninth-order Newton iteration
$f_1(x)$	1	5	3	2
	10	8	5	3
	100	16	9	6
$f_2(x)$	1	6	4	3
	10	12	7	4
	100	25	13	9

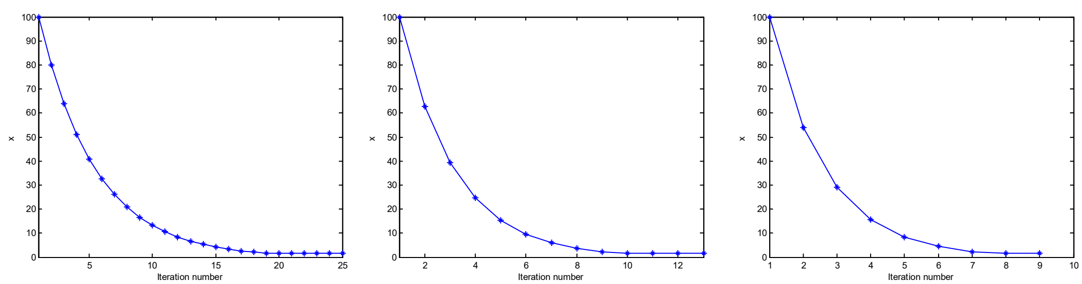
As can be seen from Table 2, although the initial points are sometimes far from the roots, three iterative methods are all eventually convergent.

Given the same initial point, the number of ninth order Newton's iteration is the smallest and the number of classical Newton's iterations is the largest. Figure 2-3 shows the Newton's iterative process of functions $f_1(x)$ and $f_2(x)$ (the horizontal axis indicates the number of iterations, and the vertical axis indicates the value of x during the iterative process).



(a) Classical Newton method (b) Fifth-order Newton iteration (c) Ninth-order Newton iteration

Figure 2. Iteration process $f_1(x)=0$ under the initial iteration point $x_0=100$.



(a) Classical Newton method (b) Fifth-order Newton iteration (c) Ninth-order Newton iteration

Figure 3. Iteration process $f_2(x)=0$ under the initial iteration point $x_0=100$.

More experimental results show that the convergence and convergence speed of various Newton's iterative methods are related to the selection of the initial point. The higher the order of convergence, the greater the computational cost required. Even for the higher-order Newton's iterative method, the higher-order convergence can be well reflected only when the initial point is close to the root.

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