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Characterization of Lattices in Terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Ideals

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Abstract: In this paper, (α, β)-fuzzy sublattices and (α, β)-fuzzy ideals of a lattice, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$, $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, are introduced and studied. In the special case, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattices and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of a lattice are presented and their characterizations by three level sublattices and ideals are given.

Keywords: lattice; fuzzy ideal; (α, β)-fuzzy sublattice

MSC: 06D72

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1. Introduction

Zadeh introduced the concept of fuzzy sets for the first time in [1]. After that different fuzzy structures have been introduced and investigated. Rosenfeld introduced fuzzy groups and subgroups in [2]. Wang [3] introduced the notion of fuzzy ring, and Kuroki studied fuzzy semigroups in [4]. The concepts of fuzzy sublattices and fuzzy ideals of a lattice were presented in [5]. Bhakas and Das [6,7] introduced the concept of (α, β)-fuzzy subgroup. Shabir and Ali [8,9] studied $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, two-sided, interior) ideals of semigroups and characterized regular, intra-regular and semisimple semigroups by the properties of these fuzzy ideals.

In this paper, $(\in_\gamma, \in_\gamma \vee q_\delta)-$fuzzy ideals of a lattice is defined, and sublattices are characterized by the properties of the fuzzy ideals.

2. Preliminaries

In this section we recall some notations and terminology which will be used in the sequel.

The notions of lattice, sublattice and ideal of a lattice are well known and can be found in the books [10,11]. Recall that a subset I of a lattice L is an *ideal* in L if

(i) $a, b \in I \Longrightarrow a \vee b \in I$,

(ii) $a \in I, b \in L$ and $b \leq a \Longrightarrow b \in I$.

We denote by $(L, \vee, \wedge)$ (or simply by L) a lattice, by I an ideal in L, and by $\mathfrak{I}(L)$ the set of all ideals of L. Notice that $(\mathfrak{I}(L), \subseteq, \cup, \cap)$ is a distributive lattice. $([0,1], \vee, \wedge)$ is a complete lattice, where [0,1] is the unit segment of real numbers, and $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$.

2.1. Fuzzy Sublattices (Ideals) of a Lattice

A fuzzy subset μ of L is a function $\mu : L \longrightarrow [0,1]$. This function is called a membership function. The set of all fuzzy subsets of L is denoted by $\mathfrak{P}(L)$. Let $\mu, \lambda \in \mathfrak{P}(L)$. The fuzzy subsets $\mu \wedge \lambda$ and $\mu \vee \lambda$ are defined as follows:

$$(\mu \wedge \lambda)(x) = \min\{\mu(x), \lambda(x)\}; \quad (\mu \vee \lambda)(x) = \max\{\mu(x), \lambda(x)\}.$$

We say that $\mu \leq \lambda$, if $\mu(x) \leq \lambda(x)$, for all $x \in L$.

If $\{\mu_s\}_{i \in S}$ is a family of fuzzy subsets of L, then for all $x \in L$, we define:

$$(\bigcap_{s \in S} \mu_s)(x) = \bigwedge_{i \in I}\{\mu_i(x)\}; \quad (\bigcup_{i \in I} \mu_i)(x) = \bigvee_{i \in I}\{\mu_i(x)\}.$$

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Definition 2.1 ([12]). *Let μ be a fuzzy subset in L. Then μ is called a* fuzzy sublattice *of L if for all $x, y \in L$,*

(i) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,

or equivalently

$\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in L$.

Definition 2.2 ([12]). *Let μ be a fuzzy sublattice of L. Then μ is called a* fuzzy ideal *of L if $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y \in L$.*

Let L be a lattice and χ_I be the characteristic function of a subset I of L. The χ_I is a fuzzy sublattice if and only if I is a sublattice.

Definition 2.3. For $\alpha \in [0,1]$, the set $\mu_\alpha = \{x \in L, \mu(x) \geq \alpha\}$ is called *$\alpha-$level subset* of μ.

For any fuzzy subset μ of L, the set $\{x \in L, \mu(x) > 0\}$ is called the support of μ, and is denoted by $supp\mu$.

Definition 2.4. *Let $t \in [0,1]$ and $x \in L$. A fuzzy set μ in L is defined by*

$$\mu(y) = \begin{cases} t, & \text{if } x = y; \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in L$, is called a fuzzy point *and denoted by x_t, where the point x is called its support point, and t is called its value.*

2.2. $(\in, \in \vee q)$-Fuzzy Sublattice and Ideal of a Lattice

A fuzzy point x_t is said to belong to (resp., be quasi-coincident with) a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $\mu(x) \geq t$ or $x_t q \mu$, then we write $x_t(\in \vee q)\mu$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Definition 2.5 ([13]). *A fuzzy subset μ of a lattice L is said to be an $(\in, \in \vee q)$-*fuzzy sublattice *of L if for all $t, r \in (0,1]$ and $x, y \in L$,*

(i) $x_t, y_r \in \mu$ implies $(x \vee y)_{t \wedge r}(\in \vee q)\mu$.

(ii) $x_t, y_r \in \mu$ implies $(x \wedge y)_{t \wedge r}(\in \vee q)\mu$.

*μ is called an $(\in, \in \vee q)$-*fuzzy ideal *of L if μ is an $(\in, \in \vee q)$-fuzzy sublattice of L and*

(iii) $x_t \in \mu$ and $y \leq x$ implies $y_t(\in \vee q)\mu$.

Theorem 2.1 (([13]). *Conditions (i)-(iii) in the above definition are equivalent to the following conditions, respectively:*

(1) $\mu(x) \wedge \mu(y) \wedge 0.5 \leq \mu(x \vee y)$,

(2) $\mu(x) \wedge \mu(y) \wedge 0.5 \leq \mu(x \wedge y)$,

(3) $y \leq x$ *implies* $\mu(x) \wedge 0.5 \leq \mu(y)$

for all $x, y \in L$.

Remark 2.1. *1) ([14]) Not every fuzzy subset of L need be $(\in, \in \vee q)$-fuzzy sublattice of L.*

2) ([14]) Not every $(\in, \in \vee q)$-fuzzy sublattice of L need be $(\in, \in \vee q)$-fuzzy ideal of L.

3) ([14]) Any fuzzy ideal of L is an $(\in, \in \vee q)$-fuzzy ideal of L.

4) ([13]) There is an $\in, \in \vee q)$-fuzzy ideal of L, which is not a fuzzy ideal.

Theorem 2.2 ([13]). *Let μ is a fuzzy subset of L. $\mu_t \neq \emptyset$ is a sublattice (ideal) of L, for all $0 < t \leq 0.5$, if and only if μ is an $(\in, \in \vee q)$-fuzzy sublattice (ideal) of L.*

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Theorem 2.3 ([14]). *A non-empty subset I of L is an ideal of L if and only if the characteristic function χ_I is an $(\in, \in \vee q)$-fuzzy ideal of L.*

3. (α, β)-Fuzzy Sublattices and Ideals of a Lattice

This section gives information about (α, β)-fuzzy sublattices for $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ and $\alpha \neq \in_\gamma \wedge q_\delta$ (see the next definition).

Definition 3.1 ([8]). *Let $\gamma, \delta, t \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point x_t and a fuzzy subset μ of L, we define:*

1) $x_t \in_\gamma \mu$ if $\mu(x) \geq t > \gamma$;
2) $x_t q_\delta \mu$ if $\mu(x) + t > 2\delta$;
3) $x_t(\in_\gamma \vee q_\delta)\mu$ If $x_t \in_\gamma \mu$ or $x_t q_\delta \mu$;
4) $x_t(\in_\gamma \wedge q_\delta)\mu$ If $x_t \in_\gamma \mu$ and $x_t q_\delta \mu$;
5) $x_t \overline{\alpha}$ if $x_t \alpha \mu$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

In this paper, we will consider $\gamma, \delta \in [0,1]$, where $\gamma < \delta$ and $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ and $\alpha \neq \in_\gamma \wedge q_\delta$. It is worth noting that if $x \in L$ and $t \in [0,1]$ such that $x_t \in_\gamma \wedge q_\delta \mu$, then $\mu(x) \geq t > \gamma$ and $\mu(x) + t > 2\delta$. Hence $2\delta < \mu(x) + t \leq \mu(x) + \mu(x) = 2\mu(x)$. Consequently, $\mu(x) > \delta$.

Definition 3.2. *A fuzzy subset μ of a lattice L is said to be an (α, β)-*fuzzy sublattice *of L if for all $t, r \in (0,1]$ and $x, y \in L$ the following hold;*

(i) $x_t, y_r \alpha \mu$ implies $(x \vee y)_{t \wedge r} \beta \mu$.
(ii) $x_t, y_r \alpha \mu$ implies $(x \wedge y)_{t \wedge r} \beta \mu$.

*μ is called an (α, β)-*fuzzy ideal *of L if μ is an (α, β)-fuzzy sublattice of L and*

(iii) $x_t \alpha \mu$ and $y \leq x$ implies $y_t \beta \mu$.

Theorem 3.1. *If $2\delta = 1 + \gamma$ and μ is an (α, β)-fuzzy sublattice (ideal) of L, then $\mu_\gamma = \{x \in L : \mu(x) > \gamma\}$ is a sublattice (ideal) of L.*

Proof. Suppose $x, y \in \mu_\gamma$ and $x \wedge y \notin \mu_\gamma$ or $x \vee y \notin \mu_\gamma$. Then

$$\mu(x \wedge y) \leq \gamma \quad \text{or} \quad \mu(x \vee y) \leq \gamma.$$

Case 1. $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$.
Define $t = \mu(x)$ and $r = \mu(y)$. Hence $x_{\mu(x)} \alpha \mu$ and $y_{\mu(y)} \alpha \mu$. It follows

$$\mu(x \wedge y) \leq \gamma < \mu(x) \wedge \mu(y) \Longrightarrow (x \wedge y)_{\mu(x) \wedge \mu(y)} \overline{\in_\gamma} \mu \quad \text{or}$$
$$\mu(x \vee y) \leq \gamma < \mu(x) \wedge \mu(y) \Longrightarrow (x \vee y)_{\mu(x) \wedge \mu(y)} \overline{\in_\gamma} \mu,$$

and

$$\mu(x \wedge y) + \mu(x) \wedge \mu(y) \leq \gamma + \mu(x) \wedge \mu(y) \leq \gamma + 1 = 2\delta \Longrightarrow (x \wedge y)_{\mu(x) \wedge \mu(y)} \overline{q_\delta} \mu \quad \text{or}$$
$$\mu(x \vee y) + \mu(x) \wedge \mu(y) \leq \gamma + \mu(x) \wedge \mu(y) \leq \gamma + 1 = 2\delta \Longrightarrow (x \vee y)_{\mu(x) \wedge \mu(y)} \overline{q_\delta} \mu.$$

So, we have

$$(x \wedge y)_{\mu(x) \wedge \mu(y)} \overline{\beta} \mu \quad \text{or} \quad (x \vee y)_{\mu(x) \wedge \mu(y)} \overline{\beta} \mu.$$

This contradicts our assumptions.

Case 2. $\alpha = q_\delta$ and $t = 1$.
Then then

$$\mu(x) + 1 > \gamma + 1 = 2\delta \Longrightarrow x_1 q_\delta \mu \quad \text{and} \quad \mu(y) + 1 > \gamma + 1 = 2\delta \Longrightarrow y_1 q_\delta \mu.$$

Hence,

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$$\mu(x \wedge y) + 1 \leq \gamma + 1 = 2\delta \Longrightarrow (x \wedge y)_1 \overline{q_\delta} \mu,$$
$$\mu(x \vee y) + 1 \leq \gamma + 1 = 2\delta \Longrightarrow (x \vee y)_1 \overline{q_\delta} \mu.$$

Also

$$\mu(x \wedge y) \leq \gamma < 1 \Longrightarrow (x \wedge y)_1 \overline{\in_\gamma} \mu \text{ or}$$
$$\mu(x \vee y) \leq \gamma < 1 \Longrightarrow (x \vee y)_1 \overline{\in_\gamma} \mu.$$

Therefore, $(x \wedge y)_1 \overline{\beta} \mu$ or $(x \vee y)_1 \overline{\beta} \mu$. This is again a contradiction.

Thus μ_γ is a sublattice of L. □

The following theorem characterizes a sublattice (ideal) of a lattice.

Theorem 3.2. *Let $2\delta = 1 + \gamma$ and A be a nonempty subset of L. Then A is a sublattice (ideal) of L if and only if the fuzzy subset μ of L defined by*

$$\mu(x) = \begin{cases} t_1 \in [\delta, 1], & \text{if } x \in A; \\ t_2 \in [0, \gamma], & \text{otherwise} \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.

Proof. $(\Rightarrow)$ If A is a sublattice of L, then the following cases can occur.

Case 1. $\alpha = \in_\gamma$.

We have

$$x_t \in_\gamma \mu \Longrightarrow \mu(x) \geq t > \gamma \text{ and } y_r \in_\gamma \mu \Longrightarrow \mu(y) \geq r > \gamma.$$

Therefore, according to the definition of,μ, we conclude $x, y \in A$. Since A is a sublattice, $x \wedge y \in A$ and $x \vee y \in A$, therefore, $\mu(x \wedge y) \geq \delta$ and $\mu(x \vee y) \geq \delta$. Now we have two possibilities (i) and (ii) below.

(i) $t \wedge r \leq \delta$.

Then

$$\mu(x \wedge y) \geq \delta \geq t \wedge r > \gamma \text{ and } \mu(x \vee y) \geq \delta \geq t \wedge r > \gamma$$

which implies

$$(x \wedge y)_{t \wedge r} \in_\gamma \mu \text{ and } (x \vee y)_{t \wedge r} \in_\gamma \mu.$$

(ii) $t \wedge r > \delta$.

Then

$$\mu(x \wedge y) + t \wedge r > \delta + \delta = 2\delta \text{ and } \mu(x \vee y) + t \wedge r > \delta + \delta = 2\delta$$

which implies

$$(x \wedge y)_{t \wedge r} q_\delta \mu \text{ and } (x \vee y)_{t \wedge r} q_\delta \mu.$$

We conclude from (i) and (ii) that

$$(x \wedge y)_{t \wedge r} \in_\gamma \vee q_\delta \mu \text{ and } (x \vee y)_{t \wedge r} \in_\gamma \vee q_\delta \mu.$$

Therefore, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice.

Case 2. $\alpha = q_\delta$.

In this case we have

$$x_t q_\delta \mu \Longrightarrow \mu(x) + t > 2\delta \Longrightarrow \mu(x) > 2\delta - t > 2\delta - 1 = \gamma \text{ and}$$
$$y_r q_\delta \mu \Longrightarrow \mu(y) + r > 2\delta \Longrightarrow \mu(y) > 2\delta - r > 2\delta - 1 = \gamma.$$

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As in Case 1 we conclude $x, y \in A$. Since A is a sublattice, $x \wedge y \in A$ and $x \vee y \in A$, hence $\mu(x \wedge y) \geq \delta$ and $\mu(x \vee y) \geq \delta$. We have again two possibiities.

(i) $t \wedge r \leq \delta$.

Then

$$\mu(x \wedge y) \geq \delta \geq t \wedge r > \gamma \text{ and } \mu(x \vee y) \geq \delta \geq t \wedge r > \gamma$$

which implies $(x \wedge y)_{t \wedge r} \in_\gamma \mu$ and $(x \vee y)_{t \wedge r} \in_\gamma \mu$.

(ii) $t \wedge r > \delta$.

Now we have

$$\mu(x \wedge y) + t \wedge r > \delta + \delta = 2\delta \text{ and } \mu(x \vee y) + t \wedge r > \delta + \delta = 2\delta$$

which implies $(x \wedge y)_{t \wedge r} q_\delta \mu$ and $(x \vee y)_{t \wedge r} q_\delta \mu$.

From (i) and (ii) it follows $(x \wedge y)_{t \wedge r} \in_\gamma \vee q_\delta \mu$ and $(x \vee y)_{t \wedge r} \in_\gamma \vee q_\delta \mu$, i.e., in Case 2 we also conclude ghat μ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy sublattice.

Case 3. $\alpha = \in_\gamma \vee q_\delta$.

We have

$$x_t \in_\gamma \mu \Longrightarrow \mu(x) \geq t > \gamma \text{ and}$$
$$y_r q_\delta \mu \Longrightarrow \mu(y) + r > 2\delta \Longrightarrow \mu(y) > 2\delta - r > 2\delta - 1 = \gamma.$$

As in Cases 1 and 2 we conclude $x \wedge y \in A$ and $x \vee y \in A$, hence $\mu(x \wedge y) \geq \delta$ and $\mu(x \vee y) \geq \delta$. Then we obtain $(x \wedge y)_{t \wedge r} \in_\gamma \vee q_\delta \mu$ and $(x \vee y)_{t \wedge r} \in_\gamma \vee q_\delta \mu$. Therefore, in this case we finally have that μ is a $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy sublattice.

($\Leftarrow$) Let μ be an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy sublattice. Then it follows from $A = \mu_\gamma = \{x \in L : \mu(x) > \gamma\}$ and Theorems 2.3 and 3.1 that A is a sublattice of L. □

Corollary 3.1. *Let $2\delta = 1 + \gamma$, and let A be a nonempty subset of L . Then A is a sublattice (ideal) of L if and only if the fuzzy subset χ_I of L is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.*

Theorem 3.3. *The following assertions are satisfied:*

(1) *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L;*

(2) *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ideal of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of L.*

Proof. We can use the fact $x_t \in_\gamma \mu$ imploes $x_t \in_\gamma \vee q_\delta \mu$ to prove this theorem. □

Theorem 3.4. *Let L be a lattice Then:*

(1) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L;*

(2) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ideal of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of L.*

Proof. We prove only (1) because the proof of (2) is quite similar. Let μ be a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L. Let $x, y \in L$ and $t, r \in (\gamma, 1]$ be such that

$$x_t \in_\gamma \mu \text{ , } y_r \in_\gamma \mu \Longrightarrow \mu(x) \geq t > \gamma \text{ and } \mu(y) \geq r > \gamma.$$

Then we have

$$(x \wedge y)_{t \wedge r} \overline{\in_\gamma \vee q_\delta} \mu \Longrightarrow \mu(x \wedge y) < t \wedge r, \;\; \mu(x \wedge y) + (t \wedge r) \leq 2\delta \Longrightarrow \mu(x \wedge y) < \delta;$$
$$(x \vee y)_{t \wedge r} \overline{\in_\gamma \vee q_\delta} \mu \Longrightarrow \mu(x \vee y) < t \wedge r, \;\; \mu(x \vee y) + (t \wedge r) \leq 2\delta \Longrightarrow \mu(x \vee y) < \delta.$$

Further,

$$\mu(x \wedge y)(\mu(x \vee y)) < t \wedge r \text{ and } \gamma < t \wedge r \leq \mu(x) \wedge \mu(y)$$

imply

$$\mu(x \wedge y)(\mu(x \vee y)) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta.$$

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Choose $k \in (\gamma, 1]$ such that

$$2\delta - \mu(x \wedge y) \vee \gamma > k \geq 2\delta - \mu(x) \wedge \mu(y) \wedge \delta.$$

Then we have

$$\begin{aligned} 2\delta \quad & -\mu(x \wedge y) \geq 2\delta - \mu(x \wedge y) \vee \gamma > k \geq (2\delta - \mu(x)) \wedge (2\delta - \mu(y)) \wedge \delta \\ \Longrightarrow \quad & \mu(x) + k \geq 2\delta, \mu(y) + k \geq 2\delta, \mu(x \wedge y) + k < 2\delta \text{ and } \mu(x \wedge y) < \delta \leq k \\ \Longrightarrow \quad & x_k q_\delta \mu, y_k q_\delta \mu, (x \wedge y)_k \overline{\in_\gamma \vee q_\delta} \mu. \end{aligned}$$

This contradicts our assumption. Therefore,

$$(x \wedge y)_k \in_\gamma \vee q_\delta \mu.$$

Similarly, we prove

$$(x \vee y)_k \in_\gamma \vee q_\delta \mu.$$

So, we conclude that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L. $\square$

The above theorem shows that each (α, β)-fuzzy sublattice (ideal) of L is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L, and each $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.

4. $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Sublattice and Ideal of a Lattice

In this section we study $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattices and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of a lattice L and characterize sublattices and ideals of L in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattices and ideals of L.

Theorem 4.1. *For any fuzzy sublattice μ of a lattice L and for all $x, y, z \in L$ and $t, r \in (\gamma, 1]$ the following hold:*

(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L if and only if $x_t \in_\gamma \mu$, $y_r \in_\gamma \mu$ implies $\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ and $\mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$.*
(2) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$[-fuzzy ideal of L if and only if $x_t \in_\gamma \mu$ implies $\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \delta$ and $\mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \delta$.*

Proof. We prove (1); the proof of (2) is similar.

($\Rightarrow$) Let μ be a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L and suppose, to the contrary, that there are $x, y \in L$ such that

$$\mu(x \wedge y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta \quad \text{or} \quad \mu(x \vee y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta.$$

If $\mu(x \wedge y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$, then we choose $t \in (\gamma, 1]$ such that $\mu(x \wedge y) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta$, which implies

$$\mu(x) \geq t > \gamma, \;\; \mu(y) \geq t > \gamma, \;\; \mu(x \wedge y) < t \;\text{ and }\; \mu(x \wedge y) + t < \delta + \delta = 2\delta.$$

It follows from here

$$x_t \in_\gamma \mu \,, y_t \in_\gamma \mu, (x \wedge y)_t \overline{\in_\gamma} \mu \;\text{ and }\; (x \wedge y)_t \overline{q_\delta} \mu,$$

i.e. $(x \wedge y)_t \overline{\in_\gamma \vee q_\delta} \mu$.

Similarly, we prove $(x \vee y)_t \overline{\in_\gamma \vee q_\delta} \mu$. However, the last two conclusions are in contradiction with our assumption.

($\Leftarrow$) Let

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$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta,\ \mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta,$$
$$\text{and}\ \ x_t \in_\gamma \mu,\ y_r \in_\gamma \mu.$$

Assume, to the contrary, that

$$(x \wedge y)_{t\wedge r}\overline{\in_\gamma \vee q_\delta}\mu \ \text{ or } \ (x \vee y)_{t\wedge r}\overline{\in_\gamma \vee q_\delta}\mu.$$

Consider the first case, $(x \wedge y)_{t\wedge r}\overline{\in_\gamma \vee q_\delta}\mu$. Then

$$(x \wedge y)_{t\wedge r}\overline{\in_\gamma}\mu \ \text{ and } \ (x \wedge y)_{t\wedge r}\overline{q_\delta}\mu.$$

From here we have the following implications:

$$\begin{aligned}&\mu(x \wedge y) < t \wedge r \ \text{ and } \ \mu(x \wedge y) + t \wedge r \leq 2\delta \\ \Longrightarrow\ &\mu(x \wedge y) < \delta \ \text{ and } \ \mu(x \wedge y) < t \wedge r \leq \mu(x) \wedge \mu(y) \\ \Longrightarrow\ &\mu(x \wedge y) < t \wedge r \leq \mu(x) \wedge \mu(y) \wedge \delta, \gamma < \delta \\ \Longrightarrow\ &\mu(x \wedge y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta.\end{aligned}$$

Similarly one proves $\mu(x \vee y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$. The last two conclusions contradict the assumption. □

Theorem 4.2. *Let μ be a fuzzy subset of a lattice L and $2\delta = 1 + \gamma$. Then $\mu_t \neq \emptyset$ is a sublattice (ideal) of L for all $t \in (\gamma, \delta]$ if and only if μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.*

Proof. ($\Leftarrow$) Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L, and $x, y \in \mu_t$. Then we have

$$\mu(x) \geq t > \gamma \text{ and } \mu(y) \geq t > \gamma \Longrightarrow \mu(x) \wedge \mu(y) \geq t.$$

Also, by the above theorem, we have

$$\begin{aligned}\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta &\Longrightarrow \mu(x \wedge y) \vee \gamma \geq t \wedge \delta = t \\ &\Longrightarrow \mu(x \wedge y) \geq t > \gamma \Longrightarrow x \wedge y \in \mu_t,\end{aligned}$$

and

$$\begin{aligned}\mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta &\Longrightarrow \mu(x \vee y) \vee \gamma \geq t \wedge \delta = t \\ &\Longrightarrow \mu(x \vee y) \geq t > \gamma \Longrightarrow x \vee y \in \mu_t.\end{aligned}$$

Therefore, μ_t is a sublattice of L.

($\Rightarrow$) Let μ_t be a sublattice of L for all $t \in (\gamma, \delta]$. We will prove that

$$\mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \ \text{ and } \ \mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta.$$

Let $x_t, y_t \in_\gamma \mu$ and $\mu(x \wedge y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$. We choose $t \in (\gamma, \delta]$ such that $\mu(x \wedge y) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta$. This implies

$$\mu(x) \geq t > \gamma,\ \mu(y) \geq t > \gamma \ \text{ and } \ \mu(x \wedge y) < t,$$

which finally implies $x, y \in \mu_t$, but $x \wedge y \notin \mu_t$. This is a contradiction showing that $\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ holds.

Similarly, we can prove that $\mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$, which mens that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L. □

Definition 4.1. *Let μ be a fuzzy subset of L. For all $r \in (\gamma, 1]$ we define:*

$$\begin{aligned}\mu_r^\delta &= \{x \in L : x_r q_\delta \mu\} = \{x \in L : \mu(x) + r > 2\delta\}; \\ [\mu]_r^\delta &= \{x \in L : x_r \in_\gamma \vee q_\delta \mu\} = \mu_r \cup \mu_r^\delta.\end{aligned}$$

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Theorem 4.3. *Let μ be a fuzzy subset of L and $2\delta = 1+\gamma$. If $\mu_r^\delta \neq \emptyset$ is a sublattice (ideal) of L, for all $r \in (\gamma, 1]$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.*

Also, for $r \in (\delta, 1]$, if μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L, then $\mu_r^\delta \neq \emptyset$ is a sublattice (ideal) of L.

Proof. Let $\mu_r^\delta \neq \emptyset$ be a sublattice of L. We prove that

$$\mu(x\wedge y)\vee\gamma \geq \mu(x)\wedge\mu(y)\wedge\delta \text{ and } \mu(x\vee y)\vee\gamma \geq \mu(x)\wedge\mu(y)\wedge\delta.$$

We consider the following two cases:

Case 1. $r \in (\gamma, \delta]$.

If $\mu(x\wedge y)\vee\gamma < \mu(x)\wedge\mu(y)\wedge\delta$, then we define $r = \mu(x)\wedge\mu(y)\wedge\delta$, and obtain

$$\mu(x\wedge y)\vee\gamma < r \Longrightarrow \mu(x\wedge y) < r \Longrightarrow \mu(x\wedge y) + r < 2r < 2\delta \Longrightarrow \wedge y \notin \mu_r^\delta.$$

Similarly we get that if $\mu(x\vee y)\vee\gamma < \mu(x)\wedge\mu(y)\wedge\delta$, then $x\vee y \notin \mu_r^\delta$. Therefore, we have a contradiction.

Case 2. $r \in (\delta, 1]$.

If $\mu(x\wedge y)\vee\gamma < \mu(x)\wedge\mu(y)\wedge\delta$, then

$$2\delta - (\mu(x\wedge y)\vee\gamma) > 2\delta - (\mu(x)\wedge\mu(y)\wedge\delta) \Longrightarrow$$
$$(2\delta - \mu(x\wedge y))\wedge(2\delta-\gamma) > (2\delta-\mu(x))\vee(2\delta-\mu(y))\vee\delta.$$

Take now $r \in (\delta, 1]$ such that

$$(2\delta-\mu(x))\vee(2\delta-\mu(y))\vee\delta < r \leq (2\delta-\mu(x\wedge y))\wedge(2\delta-\gamma).$$

Thus we have

$$2\delta - \mu(x) < r \Longrightarrow \mu(x) + r > 2\delta \Longrightarrow x \in \mu_r^\delta,$$
$$2\delta - \mu(y) < r \Longrightarrow \mu(y) + r > 2\delta \Longrightarrow y \in \mu_r^\delta,$$
$$2\delta - \mu(x\wedge y) \geq r \Longrightarrow \mu(x\wedge y) + r \leq 2\delta \Longrightarrow x\wedge y \notin \mu_r^\delta.$$

We have obtained a contradiction which shows $\mu(x\wedge y)\vee\gamma \geq \mu(x)\wedge\mu(y)\wedge\delta$.

Similarly, one obtains $\mu(x\wedge y)\vee\gamma \geq \mu(x)\wedge\mu(y)\wedge\delta$.

Conversely, let $r \in (\delta, 1]$. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L, and $x, y \in \mu_r^\delta$. Then we have

$$\mu(x) + r > 2\delta \Longrightarrow \mu(x) > 2\delta - r \geq 2\delta - 1 = \gamma \text{ and}$$
$$\mu(y) + r > 2\delta \Longrightarrow \mu(y) > 2\delta - r \geq 2\delta - 1 = \gamma.$$

It follows from here

$$\mu(x\wedge y)\vee\gamma \geq \mu(x)\wedge\mu(y)\wedge\delta > (2\delta-r)\wedge(2\delta-r)\wedge\delta.$$

Since $\delta \leq r \leq 1$ we have

$$\delta \geq 2\delta - r \geq 2\delta - 1 \text{ and } (2\delta-r)\wedge(2\delta-r)\wedge\delta = 2\delta - r,$$

which implies

$$\mu(x\wedge y) > 2\delta - r \Longrightarrow \mu(x\wedge y) + r > 2\delta$$

which means $x\wedge y \in \mu_r^\delta$.

Similarly, we obtain $x\vee y \in \mu_r^\delta$. Thus μ_r^δ is a sublattice of L. □

Theorem 4.4. *Let μ be a fuzzy subset of L and $2\delta = 1+\gamma$. Then $[\mu]_r^\delta \neq \emptyset$ is a sublattice (ideal) of L, for all $t \in (\gamma, 1]$, if and only if μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.*

Proof. ($\Leftarrow$) Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L, and $x, y \in [\mu]_r^\delta$. Then we have

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$$\mu(x) + r > 2\delta \Longrightarrow \mu(x) \geq 2\delta - r \geq 2\delta - 1 = \gamma \text{ or } \mu(x) \geq r > \gamma, \text{ and}$$
$$\mu(y) + r > 2\delta \Longrightarrow \mu(y) \geq 2\delta - r \geq 2\delta - 1 = \gamma \text{ or } \mu(y) \geq r > \gamma.$$

Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L, we have

$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \text{ and}$$
$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta.$$

We consider the following two cases:

Case 1. $r \in (\gamma, \delta]$.

In this case we have

$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge r \wedge \delta = r \Longrightarrow \mu(x \wedge y) \vee \gamma \geq r > \gamma$$

which means $(x \wedge y)_r \in_\gamma \mu$, i.e., $x \wedge y \in [\mu]_r^\delta$.

Similarly, we can prove $x \vee y \in [\mu]_r^\delta$. Thus $[\mu]_r^\delta$ is a sublattice of L.

Case 2. $r \in (\delta, 1]$.

Then $\delta < r \leq 1$ implies $\delta > 2\delta - r \geq 2\delta - 1$. Let $x, y \in [\mu]_r^\delta$. We have

$$\mu(x) + r > 2\delta \Longrightarrow \mu(x) > 2\delta - r \geq 2\delta - 1 = \gamma \text{ or } \mu(x) \geq r > \gamma, \text{ and}$$
$$\mu(y) + r > 2\delta \Longrightarrow \mu(y) > 2\delta - r \geq 2\delta - 1 = \gamma \text{ or } \mu(y) \geq r > \gamma.$$

We conclude from here

$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta > (2\delta - r) \wedge (2\delta - r) \wedge \delta = 2\delta - r,$$
$$\text{i.e. } \mu(x \wedge y) > 2\delta - r \Longrightarrow \mu(x \wedge y) + r > 2\delta,$$

or

$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta > r \wedge r \wedge \delta = \delta,$$
$$\text{i.e. } \mu(x \wedge y) + r > \delta + r > 2\delta.$$

So, we conclude that $(x \wedge y)_r q_\delta \mu$, which implies $x \wedge y \in [\mu]_r^\delta$.

Similarly, we prove that $x \vee y \in [\mu]_r^\delta$. Thus $[\mu]_r^\delta$ is a sublattice of L.

($\Rightarrow$) Let $[\mu]_r^\delta \neq \emptyset$ be a sublattice of L. We prove that

$$\mu(x \wedge y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \text{ and } \mu(x \vee y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta.$$

Let, to the contrary, $\mu(x \wedge y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$. Then choose $r \in (\gamma, 1]$ such that

$$\mu(x \wedge y) \vee \gamma < r \leq \mu(x) \wedge \mu(y) \wedge \delta.$$

Then

$$\mu(x) \geq r > \gamma, \mu(y) \geq r > \gamma, \ \mu(x \wedge y) < r \text{ and } \mu(x \wedge y) + r < \delta + \delta = 2\delta,$$

which implies

$$x_r \in_\gamma \mu, \ y_r \in_\gamma \mu, \ (x \wedge y)_r \overline{\in_\gamma} \mu, \text{ and } (x \wedge y)_r \overline{q_\delta} \mu.$$

This means $(x \wedge y)_r \overline{\in_\gamma \vee q_\delta} \mu$.

Similarly, we get $(x \vee y)_r \overline{\in_\gamma \vee q_\delta} \mu$. The last two conclusions contradict the assumption. □

Theorem 4.5. *Let λ and μ be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattices (ideals) of L. Then $\lambda \cap \mu$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.*

Proof. For any $x, y \in L$, we have

$$(\lambda \cap \mu)(x \wedge y) \vee \gamma = (\lambda(x \wedge y) \wedge \mu(x \wedge y)) \vee \gamma = (\lambda(x \wedge y) \vee \gamma) \wedge (\mu(x \wedge y) \vee \gamma).$$

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By Theorems 2.3–3.3, we have

$$\begin{aligned}(\lambda(x \wedge y) \vee \gamma) \wedge (\mu(x \wedge y) \vee \gamma) &\geq (\lambda(x) \wedge \lambda(y) \wedge \delta) \wedge (\mu(x) \wedge \mu(y) \wedge \delta) \\ &= (\lambda(x) \wedge \mu(x)) \wedge (\lambda(y) \wedge \mu(y)) \wedge \delta \\ &= (\lambda \cap \mu(x)) \wedge (\lambda \cap \mu(y)) \wedge \delta,\end{aligned}$$

which means

$$(\lambda \cap \mu)(x \wedge y) \vee \gamma \geq (\lambda \cap \mu)(x) \wedge (\lambda \cap \mu)(y) \wedge \delta.$$

Similarly, we can prove

$$(\lambda \cap \mu)(x \vee y) \vee \gamma \geq (\lambda \cap \mu)(x) \wedge (\lambda \cap \mu)(y) \wedge \delta.$$

Hence, $\lambda \cap \mu$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice of L. □

Remark 4.1. *If $\{\mu_s\}_{s\in S}$, S is an index set, is a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattices (ideals) of L, then $\bigcap_{s\in S} \mu_s$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattice (ideal) of L.*

5. Conclusions

We introduced and studied (α, β)-fuzzy sublattices and (α, β)-fuzzy ideals of a lattice. In particular, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy sublattices and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals are considered and characterized. we hope that this study can be extended to other mathematical structures.

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