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Algebraic Schouten Solitons Associated to Perturbed Canonical Connection and Perturbed Kobayashi-Nomizu Connection on Three-Dimensional Lorentzian Lie Groups

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Abstract: In this paper, we study algebraic Schouten solitons concerning the perturbed canonical connection and the perturbed Kobayashi Nomizu connection on three-dimensional Lorentzian Lie groups and give the complete classification of the algebraic Schouten solitons with product structure on three-dimensional Lorentzian Lie groups.

Keywords: Pseudo-Riemannian metric; algebraic Schouten solitons; three-dimensional Lorentzian Lie groups

1. Introduction

Einstein metrics are fundamental in differential geometry and physics and have been widely studied in (pseudo)-Riemannian geometry. The Riemannian manifolds (M, g) are called Ricci soliton if there exists a smooth vector field X and a real constant λ satisfying,

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1}$$

Ric is the Ricci tensor, and $\mathcal{L}_X g$ denotes the Lie derivative of g concerning the vector field X. It is a natural generalization of Einstein metric, a fixed solution of Hamilton Ricci flow equation up to diffeomorphism and scaling. The theory of Ricci flow was developed by Hamilton [1], and it was applied by Perelman [2] to prove the long well-open problem Poincare conjecture. Ricci flow, Ricci soliton have applications in Physics [3,4]. Especially, Lauret introduced the concept of algebraic Ricci soliton as a natural generalization of the Einstein metric in the Riemannian case [5]. Onda and Batat extended the idea to the pseudo-Riemannian Lie group and obtained a complete classification of algebraic Ricci solitons of three-dimensional Lorentzian Lie groups in [6], the author also proved that, in the pseudo-Riemannian setting, Ricci solitons need not be algebraic Ricci solitons. Motivated by Lauret's research, Wears derived the algebraic T-solitons and established the connection between T-solitons and algebraic T-solitons. In [7], Wears gave a complete classification of algebraic T-solitons on three-dimensional unimodular Lie groups using Milnor frames. In [8], a generalized Ricci soliton on three-dimensional Lie groups was defined, which could be considered a Schouten soliton.

In [9], Etayo and Santamaria studied some affine connections on manifolds with a product or complex structure. Motivated by this research, mathematicians started to study Ricci soliton associated with different affine connections. For example, in [10], Wang classified affine Ricci solitons associated with canonical connections, perturbed canonical connection, Kobayashi-Nomizu connection, and perturbed Kobayashi-Nomizu connection on three-dimensional Lorentzian Lie groups with some product structure. In [8,11], Azami investigated the affine generalized Ricci solitons concerning the perturbed canonical connection and the perturbed Kobayashi Nomizu connection on three-dimensional Lorentzian Lie groups. For more results related to Ricci solitons, see [12–15]. In this paper, we investigate algebraic Schouten solitons associated with the perturbed canonical connection and the perturbed Kobayashi-Nomizu connection. Next, we derive the detailed classification of algebraic Schouten solitons related to these connections on three-dimensional Lorentzian Lie groups.

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This paper is organized as follows. In section 2, we give some fundamental concepts for Lie groups, specifically relating to the perturbed canonical connection and the perturbed Kobayashi Nomizu connection. In section 3, we classify algebraic Schouten solitons associated with the perturbed canonical connection and the perturbed KobayashiNomizu connection on three-dimensional Lorentzian Lie groups with a product structure.

2. Preliminaries

Milnor surveyed old and new results on left-invariant Riemannian metrics on Lie groups, especially three-dimensional unimodular Lie groups, which are entirely classified in [16]. In [17], Rahmani classified three-dimensional unimodular Lie groups equipped with a left-invariant Lorentzian metric. Moreover, the non-unimodular cases were solved in [18,19]. Throughout this paper, we shall by $\{G_i\}_{i=1}^7$ denote the connected three-dimensional Lie groups equipped with left-invariant Lorentzian metrics and $\{\mathfrak{g}_i\}_{i=1}^7$ as their Lie algebras (see [6]). Let J be a product structure on G_i and is defined by

$$Je_1 = e_1, \quad Je_2 = e_2, \quad Je_3 = e_3. \tag{2}$$

Then $J^2 = id$ and $g(Je_i, Je_i) = g(e_i, e_i)$. Canonical connection and Kobayashi-Nomizu connection [9] are defined by

$$\begin{aligned} \nabla^0_X Y &= \nabla_X Y - \frac{1}{2}(\nabla_X J)JY, \\ \nabla^1_X Y &= \nabla^0_X Y - \frac{1}{4}[(\nabla_Y J)JX - (\nabla_{JY} J)X]. \end{aligned} \tag{3}$$

On G_i, $i = 1, 2, ..., 7$, the perturbed canonical connection and the perturbed Kobayashi-Nomizu connection is defined by

$$\begin{aligned} \overline{\nabla}^0_X Y &= \nabla^0_X Y + \rho e_3^\flat(X) e_3^\flat(Y) e_3, \\ \overline{\nabla}^1_X Y &= \nabla^1_X Y + \rho e_3^\flat(X) e_3^\flat(Y) e_3, \end{aligned} \tag{4}$$

where ρ is a non-zero constant. Then

$$\overline{\nabla}^0_{e_i} e_j = \nabla^0_{e_i} e_j, \quad \overline{\nabla}^0_{e_3} e_3 = \rho e_3, \quad \overline{\nabla}^1_{e_i} e_j = \nabla^1_{e_i} e_j, \quad \overline{\nabla}^1_{e_3} e_3 = \rho e_3, \tag{5}$$

for $i, j \in \{1, 2\}$. Now, we define the Ricci curvature as follow

$$R^i(X, Y)Z = [\overline{\nabla}^i_X, \overline{\nabla}^i_Y]Z - \overline{\nabla}^i_{[X,Y]} Z, \quad i \in \{0, 1\}. \tag{6}$$

The Ricci tensor of (G_i, g) with respect to $\overline{\nabla}^i$ are defined by

$$\rho^i(X, Y) = -g(R(X, e_1)Y, e_1) - g(R(X, e_2)Y, e_2) + g(R(X, e_3)Y, e_3), \quad i \in \{0, 1\}, \tag{7}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis, with e_3 timelike. Let

$$\tilde{\rho}^i(X, Y) = \frac{\rho^i(X, Y) + \rho^i(Y, X)}{2}, \quad i \in \{0, 1\}. \tag{8}$$

Then, the Ricci operator $\widetilde{Ric}^i$ with respect to $\overline{\nabla}^i$ is given by

$$\tilde{\rho}^i(X, Y) = g(\widetilde{Ric}^i(X), Y), \quad i \in \{0, 1\}. \tag{9}$$

One can define the Schouten tensor with the expression as follow

$$S^i(e_i, e_j) = \tilde{\rho}^i(e_i, e_j) - s^i \lambda_0 g(e_i, e_j), \quad i \in \{0, 1\}, \tag{10}$$

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where s^i denotes the scalar curvature and λ_0 is a real number. Refer to [20], we have

$$s^i = \tilde{\rho}^i(e_1, e_1) + \tilde{\rho}^i(e_2, e_2) - \tilde{\rho}^i(e_3, e_3), \quad i \in \{0, 1\}. \tag{11}$$

Definition 1. *(G_i, g, J) is called the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$, if it satisfies*

$$\widetilde{Ric}^0 = (s^0\lambda_0 + c)Id + D^0, \tag{12}$$

where c is a real number, and D^0 is a derivation of $\mathfrak{g}_i$, i.e.,

$$D^0[X, Y] = [D^0X, Y] + [X, D^0Y] \; for \; X, Y \in \mathfrak{g}_i. \tag{13}$$

(G_i, g, J) is called the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$, if it satisfies

$$\widetilde{Ric}^1 = (s^1\lambda_0 + c)Id + D^1, \tag{14}$$

where c is a real number, and D^1 is a derivation of $\mathfrak{g}_i$, i.e.,

$$D^1[X, Y] = [D^1X, Y] + [X, D^1Y] \; for \; X, Y \in \mathfrak{g}_i. \tag{15}$$

3. Algebraic Schouten Solitons Associated with the Perturbed Canonical Connections on Three-Dimensional Lorentzian Lie Groups

This section presents the curvature property for the perturbed canonical connection corresponding to three-dimensional Lorentzian Lie groups. Then, we fully classify algebraic Schouten soliton associated with the perturbed canonical connection on Lie groups G_i, $i = 1, \ldots, 7$.

3.1. Algebraic Schouten Soliton of G_1

By [6], we have the following Lie algebra of G_1 satisfies

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \quad [e_1, e_3] = -\alpha e_1 - \beta e_2, \quad [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \tag{16}$$

where $\alpha \neq 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_1$, with e_3 timelike. From [21], we have the following lemma.

Lemma 1 ([21])**.** *The Ricci tensor of (G_1, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} -(\alpha^2 + \frac{1}{2}\beta^2) & 0 & \frac{1}{4}\alpha\beta \\ \alpha\beta & -(\alpha^2 + \beta^2) & \frac{1}{2}(\alpha^2 + \alpha\rho) \\ \frac{1}{4}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix}. \tag{17}$$

Theorem 1. *(G_1, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if it satisfies $\alpha \neq 0$, $\beta = \rho = 0$, $c = \frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0$.*

Proof. Using (17), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \frac{1}{2}\beta^2) & 0 & -\frac{1}{4}\alpha\beta \\ \alpha\beta & -(\alpha^2 + \frac{1}{2}\beta^2) & -\frac{1}{2}(\alpha^2 + \alpha\rho) \\ \frac{1}{4}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{18}$$

Therefore, the scalar curvature can be obtained as $s^0 = -(2\alpha^2 + \beta^2)$. We can write D^0 as

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$$\begin{cases} D^0 e_1 = -(\alpha^2 + \frac{\beta^2}{2} - (2\alpha^2 + \beta^2)\lambda_0 + c)e_1 - \frac{\alpha\beta}{4} e_3, \\ D^0 e_2 = -(\alpha^2 + \frac{\beta^2}{2} - (2\alpha^2 + \beta^2)\lambda_0 + c)e_2 - \frac{\alpha^2 + \alpha\rho}{2} e_3, \\ D^0 e_3 = \frac{\alpha\beta}{4} e_1 + \frac{\alpha^2}{2} e_2 + ((2\alpha^2 + \beta^2)\lambda_0 - c)e_3. \end{cases} \quad (19)$$

Hence, by (13), there exists an algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ can be established if it satisfies the following system of equations

$$\begin{cases} \alpha(\alpha^2 + \frac{\beta^2}{2} - (2\alpha^2 + \beta^2)\lambda_0 + c) - \frac{\alpha\beta^2}{2} - \frac{\alpha^3 + \alpha^2\rho}{2} = 0, \\ 5\alpha^2\beta + 2\alpha\beta\rho = 0, \\ \beta(2\alpha^2 + \beta^2 - (2\alpha^2 + \beta^2)\lambda_0 + c) - \frac{\alpha^2\beta}{4} = 0, \\ \alpha((2\alpha^2 + \beta^2)\lambda_0 - c) - \frac{\alpha^3}{2} = 0, \\ \beta((2\alpha^2 + \beta^2)\lambda_0 - c) = 0, \\ \beta((2\alpha^2 + \beta^2)\lambda_0 - c) - \frac{\alpha^2\beta}{2} = 0. \end{cases} \quad (20)$$

Since $\alpha \neq 0$, the last two equations of the system (20) yields $\beta = 0$. Then, (20) reduce to

$$\begin{cases} \frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0 + c - \frac{\alpha\rho}{2} = 0, \\ -\frac{\alpha^2}{2} + 2\alpha^2\lambda_0 - c = 0. \end{cases} \quad (21)$$

Then we have $\rho = 0$, $c = \frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0$. □

3.2. *Algebraic Schouten Soliton of G_2*

By [6], we have the following Lie algebra of G_2 satisfies

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 - \gamma e_3, \quad [e_2, e_3] = \alpha e_1, \quad (22)$$

where $\gamma \neq 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_2$, with e_3 timelike. From [21], we have the following lemma.

Lemma 2 ([21]). *The Ricci tensor of (G_2, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} -(\gamma^2 + \frac{1}{2}\alpha\beta) & 0 & -\frac{1}{2}\gamma\rho \\ 0 & -(\gamma^2 + \frac{\alpha\beta}{2}) & \frac{1}{2}\beta\gamma - \frac{1}{4}\alpha\gamma \\ 0 & \frac{1}{2}\beta\gamma - \frac{1}{4}\alpha\gamma & 0 \end{pmatrix}. \quad (23)$$

Theorem 2. *(G_2, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if it satisfies $\alpha = \beta = 0$, $\gamma \neq 0$, and $c = -\gamma^2 + 2\gamma^2\lambda_0$.*

Proof. Using (23), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\gamma^2 + \frac{1}{2}\alpha\beta) & 0 & -\frac{1}{2}\gamma\rho \\ 0 & -(\gamma^2 + \frac{1}{2}\alpha\beta) & -\frac{1}{2}\beta\gamma + \frac{1}{4}\alpha\gamma \\ 0 & \frac{1}{2}\beta\gamma - \frac{1}{4}\alpha\gamma & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \quad (24)$$

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Therefore, the scalar curvature can be obtained as $s^0 = -(2\gamma^2 + \alpha\beta)$. We can write D^0 as

$$\begin{cases} D^0 e_1 = -(\gamma^2 + \frac{\alpha\beta}{2} - (2\gamma^2 + \alpha\beta)\lambda_0 + c)e_1 + \frac{\gamma\rho}{2} e_3, \\ D^0 e_2 = -(\gamma^2 + \frac{\alpha\beta}{2} - (2\gamma^2 + \alpha\beta)\lambda_0 + c)e_2 - (\frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4})e_3, \\ D^0 e_3 = (\frac{\beta\gamma}{2} - \frac{\alpha\gamma}{4})e_2 + ((2\gamma^2 + \alpha\beta)\lambda_0 - c)e_3. \end{cases} \tag{25}$$

Hence, by (13) , there exists an algebraic Schouten soliton associated with the perturbed canonical connections $\overline{\nabla}^0$ can be established if it satisfies the following system of equations

$$\begin{cases} \gamma(\gamma^2 + \frac{\alpha\beta}{2} - (2\gamma^2 + \alpha\beta)\lambda_0 + c) - \beta(\beta\gamma - \frac{\alpha\gamma}{2}) = 0, \\ \alpha\gamma\rho = 0, \\ \beta(2\gamma^2 + \frac{\alpha\beta}{2} - (2\gamma^2 + \alpha\beta)\lambda_0 + c) + \gamma(\beta\gamma - \frac{\alpha\gamma}{2}) = 0, \\ \beta(-(2\gamma^2 + \alpha\beta)\lambda_0 + c) + \gamma(\beta\gamma - \frac{\alpha\gamma}{2}) = 0, \\ \alpha(-(2\gamma^2 + \alpha\beta)\lambda_0 + c) = 0. \end{cases} \tag{26}$$

Since $\gamma \neq 0$, the second equation of (26) yields $\alpha = 0$, or $\rho = 0$. Let $\alpha = 0$. In this case, (26) becomes

$$\begin{cases} \gamma(\gamma^2 - \beta^2 - 2\gamma^2\lambda_0 + c) = 0, \\ \beta(3\gamma^2 - 2\gamma^2\lambda_0 + c) = 0, \\ \beta(\gamma^2 - 2\gamma^2\lambda_0 + c) = 0. \end{cases} \tag{27}$$

The last two equations above leads to $\beta = 0$ and $c = -\gamma^2 + 2\gamma^2\lambda_0$. Now, let $\rho = 0$. The last two equations in (26) give $\beta\gamma - \frac{1}{2}\alpha\gamma = 0$, substitute into the first two equations in (26) yields $\gamma^2 = 0$, which is a contradiction. □

3.3. *Algebraic Schouten Soliton of G_3*

By [6], we have the following Lie algebra of G_3 satisfies

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1, \tag{28}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_3$, with e_3 timelike. From [21], we have the following lemma.

Lemma 3 ([21])**.** *The Ricci tensor of (G_3, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} -\frac{1}{2}\gamma(\alpha + \beta - \gamma) & 0 & 0 \\ 0 & -\frac{1}{2}\gamma(\alpha + \beta - \gamma) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

Theorem 3. *(G_3, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if it satisfies one of the following conditions:*

1. $\alpha = \beta = \gamma = 0$, *for all* c;
2. $\alpha \neq 0, \beta = \gamma = 0, c = 0$;
3. $\alpha = \gamma = 0, \beta \neq 0, c = 0$;
4. $\alpha \neq 0, \beta \neq 0, \gamma = 0, c = 0$;
5. $\alpha = \beta = 0, \gamma \neq 0, c = -\gamma^2 + \gamma^2\lambda_0$;

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6. $\alpha = \gamma \neq 0, \beta = 0, c = 0$;
7. $\alpha = 0, \beta \neq 0, \gamma \neq 0$, *and* $\beta = \gamma, c = 0$;
8. $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$, *and* $\alpha + \beta = \gamma, c = 0$.

Proof. Using (29), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\gamma(\alpha+\beta-\gamma) & 0 & 0 \\ 0 & -\frac{1}{2}\gamma(\alpha+\beta-\gamma) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{30}$$

The scalar curvature is given by $s^0 = -\gamma(\alpha+\beta-\gamma)$. We can express D^0 as

$$\begin{cases} D^0 e_1 = -(\frac{1}{2}\gamma(\alpha+\beta-\gamma) - \gamma(\alpha+\beta-\gamma)\lambda_0 + c)e_1, \\ D^0 e_2 = -(\frac{1}{2}\gamma(\alpha+\beta-\gamma) - \gamma(\alpha+\beta-\gamma)\lambda_0 + c)e_2, \\ D^0 e_3 = (\gamma(\alpha+\beta-\gamma)\lambda_0 - c)e_3. \end{cases} \tag{31}$$

Therefore, by (13) there exists an algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ can be established if it satisfies the following system of equations

$$\begin{cases} \gamma(\gamma(\alpha+\beta-\gamma) - \gamma(\alpha+\beta-\gamma)\lambda_0 + c) = 0, \\ \beta(\gamma(\alpha+\beta-\gamma)\lambda_0 - c) = 0, \\ \alpha(\gamma(\alpha+\beta-\gamma)\lambda_0 - c) = 0. \end{cases} \tag{32}$$

Assuming that $\gamma = 0$, we have cases (1)-(4). Next, let $\gamma \neq 0$, then $c = -\gamma(\alpha+\beta-\gamma) + \gamma(\alpha+\beta-\gamma)\lambda_0$. Meanwhile, if $\alpha = 0$, we have cases (5)-(7); if $\alpha \neq 0$, for case (8), system (32) hold. □

3.4. *Algebraic Schouten Soliton of G_4*

By [6], we have the following Lie algebra of G_4 satisfies

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \quad [e_1, e_3] = e_3 - \beta e_2, \quad [e_2, e_3] = \alpha e_1, \tag{33}$$

where $\eta = 1$ *or* -1. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_4$, with e_3 timelike. From [21], we have the following lemma.

Lemma 4 ([21]). *The Ricci tensor of (G_4, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} (2\eta-\beta)(\frac{\alpha}{2}+\eta) - 1 & 0 & \frac{\rho}{2} \\ 0 & (2\eta-\beta)(\frac{\alpha}{2}+\eta) - 1 & (\frac{\alpha}{4}+\frac{\eta}{2}-\frac{\beta}{2}) \\ 0 & (\frac{\alpha}{4}+\frac{\eta}{2}-\frac{\beta}{2}) & 0 \end{pmatrix}. \tag{34}$$

Theorem 4. *(G_4, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if it satisfies $\alpha = 0, \beta = \eta, \rho \neq 0, c = 0$.*

Proof. Using (34), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} (2\eta-\beta)(\frac{\alpha}{2}+\eta) - 1 & 0 & -\frac{\rho}{2} \\ 0 & (2\eta-\beta)(\frac{\alpha}{2}+\eta) - 1 & -(\frac{\alpha}{4}+\frac{\eta}{2}-\frac{\beta}{2}) \\ 0 & (\frac{\alpha}{4}+\frac{\eta}{2}-\frac{\beta}{2}) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{35}$$

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Therefore, the scalar curvature can be obtained as $s^0 = 2(2\eta - \beta)(\frac{1}{2}\alpha + \eta) - 2$. We can write D^0 as

$$\begin{cases} D^0 e_1 = ((2\eta - \beta)(\frac{\alpha}{2} + \eta) - 1 - (2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 - c)e_1 - \frac{\rho}{2} e_3, \\ D^0 e_2 = ((2\eta - \beta)(\frac{\alpha}{2} + \eta) - 1 - (2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 - c)e_1 e_2 - (\frac{\alpha}{4} + \frac{\eta}{2} - \frac{\beta}{2})e_3, \\ D^0 e_3 = (\frac{\alpha}{4} + \frac{\eta}{2} - \frac{\beta}{2})e_2 - ((2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 + c)e_3. \end{cases} \tag{36}$$

Hence, by (13), there exists an algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ can be established if it satisfies

$$\begin{cases} \frac{\alpha\rho}{2} = 0, \\ (2\eta - \beta)(\frac{\alpha}{2} + \eta) - 1 - (2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 - c + (\eta - \beta)(\frac{\alpha}{2} + \eta - \beta) = 0, \\ (2\eta - \beta)(2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2 - (2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 - c) - (\frac{\alpha}{2} + \eta - \beta) = 0, \\ \beta((2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 + c) - (\frac{\alpha}{2} + \eta - \beta) = 0, \\ \alpha((2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0 + c) = 0. \end{cases} \tag{37}$$

Consider that $\alpha = 0$, we get

$$\begin{cases} (2\eta - \beta)\eta - 1 - (2\eta(2\eta - \beta) - 2)\lambda_0 - c + (\eta - \beta)(\eta - \beta) = 0, \\ (2\eta - \beta)(2(2\eta - \beta)\eta - 2 - (2\eta(2\eta - \beta) - 2)\lambda_0 - c) - (\eta - \beta) = 0, \\ \beta((2\eta(2\eta - \beta) - 2)\lambda_0 + c) - (\eta - \beta) = 0. \end{cases} \tag{38}$$

Then we obtain $\alpha = 0, \beta = \eta, \rho \neq 0, c = 0$.

Next, let $\rho = 0$ and $\alpha \neq 0$, then $c = -(2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2)\lambda_0$. Meanwhile, we have

$$\begin{cases} (2\eta - \beta)(\frac{\alpha}{2} + \eta) - 1 = 0, \\ (2\eta - \beta)(2(2\eta - \beta)(\frac{\alpha}{2} + \eta) - 2) = 0. \end{cases} \tag{39}$$

This is a contradiction. □

3.5. Algebraic Schouten Soliton of G_5

By [6], we have the following Lie algebra of G_5 satisfies

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \tag{40}$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_5$, with e_3 timelike. From [21], we have the following lemma.

Lemma 5 ([21]). *The Ricci tensor of (G_5, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

Theorem 5. *(G_5, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if $c = 0$.*

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Proof. Using (41), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{42}$$

Therefore, the scalar curvature can be obtained as $s^0 = 0$. We can write D^0 as

$$\begin{cases} D^0 e_1 = -ce_1, \\ D^0 e_2 = -ce_2, \\ D^0 e_3 = -ce_3. \end{cases} \tag{43}$$

Hence, by (13), there exists an algebraic Schouten soliton associated with the perturbed canonical connection can be established if $c = 0$. □

3.6. Algebraic Schouten Soliton of G_6

By [6], we have the following Lie algebra of G_6 satisfies

$$\begin{aligned} [e_1, e_2] &= \alpha e_2 + \beta e_3, \\ [e_1, e_3] &= \gamma e_2 + \delta e_3, \\ [e_2, e_3] &= 0, \end{aligned} \tag{44}$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_6$, with e_3 timelike. From [21], we have the following lemma.

Lemma 6 ([21]). *The Ricci tensor of (G_6, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 & 0 & \frac{1}{2}\delta\rho \\ 0 & \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 & \frac{1}{2}(-\alpha\gamma + \frac{1}{2}\delta(\beta-\gamma)) \\ 0 & \frac{1}{2}(-\alpha\gamma + \frac{1}{2}\delta(\beta-\gamma)) & 0 \end{pmatrix}. \tag{45}$$

Theorem 6. *(G_6, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if it satisfies one of the following conditions:*

1. *$\alpha = \beta = \gamma = 0$, $\delta \neq 0$, $c = -s\lambda_0$;*
2. *$\alpha \neq 0$, $\beta \neq 0$, $\gamma = \delta = 0$, and $\beta^2 = 2\alpha^2$, $c = -s\lambda_0$;*
3. *$\alpha \neq 0$, $\beta = \gamma = \delta = 0$, $c = -\alpha^2 - s\lambda_0$;*
4. *$\alpha \neq 0$, $\beta = \gamma = 0$, $\delta \neq 0$, and $\alpha = \delta$, $c = -\alpha^2 - s\lambda_0$;*
5. *$\alpha \neq 0$, $\beta = \gamma = 0$, $\delta \neq 0$, and $\alpha \neq \delta$, $c = -\alpha^2 - s\lambda_0$.*

Proof. Using (45), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 & 0 & -\frac{1}{2}\delta\rho \\ 0 & \frac{1}{2}\beta(\beta-\gamma) - \alpha^2 & -\frac{1}{2}(-\alpha\gamma + \frac{1}{2}\delta(\beta-\gamma)) \\ 0 & \frac{1}{2}(-\alpha\gamma + \frac{1}{2}\delta(\beta-\gamma)) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{46}$$

Therefore, the scalar curvature can be obtained as $s^0 = \beta(\beta-\gamma) - 2\alpha^2$. We can write D^0 as

$$\begin{cases} D^0 e_1 = (\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - (\beta(\beta-\gamma) - 2\alpha^2)\lambda_0 - c)e_1 - \frac{\delta\rho}{2}e_3, \\ D^0 e_2 = (\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - (\beta(\beta-\gamma) - 2\alpha^2)\lambda_0 - c)e_2 - \frac{1}{2}(-\alpha\gamma + \frac{1}{2}\delta(\beta-\gamma))e_3, \\ D^0 e_3 = \frac{1}{2}(-\alpha\gamma + \frac{1}{2}\delta(\beta-\gamma))e_2 - ((\beta(\beta-\gamma) - 2\alpha^2)\lambda_0 + c)e_3. \end{cases} \tag{47}$$

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Hence, by (13), there exists an algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ can be established if it satisfies the following system of equations

$$\begin{cases} \alpha(\frac{1}{2}\beta(\beta-\gamma)-\alpha^2-(\beta(\beta-\gamma)-2\alpha^2)\lambda_0-c)+\frac{1}{2}(\beta+\gamma)(-\alpha\gamma+\frac{1}{2}\delta(\beta-\gamma))=0, \\ \beta(\beta(\beta-\gamma)-2\alpha^2-(\beta(\beta-\gamma)-2\alpha^2)\lambda_0-c)+\frac{1}{2}(\delta-\alpha)(-\alpha\gamma+\frac{1}{2}\delta(\beta-\gamma))=0, \\ \gamma((\beta(\beta-\gamma)-2\alpha^2)\lambda_0+c)+\frac{1}{2}(\delta-\alpha)(-\alpha\gamma+\frac{1}{2}\delta(\beta-\gamma))=0, \\ \delta(\frac{1}{2}\beta(\beta-\gamma)-\alpha^2-(\beta(\beta-\gamma)-2\alpha^2)\lambda_0-c)+\frac{1}{2}(\beta+\gamma)(-\alpha\gamma+\frac{1}{2}\delta(\beta-\gamma))=0. \end{cases} \tag{48}$$

Assume first $\alpha=0$, then we have $\delta\neq 0$, $\beta=\gamma=0$, and $c=0$. Therefore, the case (1) hold.

Consider $\alpha\neq 0$, and let $\delta=0$. Since $\alpha\gamma-\beta\delta=0$, we have $\gamma=0$. Then (48) reduce to

$$\begin{cases} \alpha(\frac{1}{2}\beta^2-\alpha^2-(\beta^2-2\alpha^2)\lambda_0-c)=0, \\ \beta(\beta^2-2\alpha^2-(\beta^2-2\alpha^2)\lambda_0-c)=0. \end{cases} \tag{49}$$

If $\beta\neq 0$, then $\beta^2=2\alpha^2$ and $c=0$; if $\beta=0$ then $c=-\alpha^2+2\alpha^2\lambda_0$. Therefore, the case (2) and (3) holds.

Next, let $\delta\neq 0$. The first and the fourth equations of the system (48) give $(\alpha-\delta)(\frac{1}{2}\beta(\beta-\gamma)-\alpha^2-s\lambda_0-c)=0$, which yields $\alpha=\delta$ or $\alpha\neq\delta$. Let $\alpha=\delta$, since $\alpha\gamma-\beta\delta=0$ we have $\beta=\gamma$. In this case, the system (48) becomes

$$\begin{cases} \alpha(-\alpha^2+2\alpha^2\lambda_0-c)+\frac{1}{2}(\beta+\gamma)(-\alpha\gamma+\frac{1}{2}\delta(\beta-\gamma))=0, \\ \beta(-2\alpha^2+2\alpha^2\lambda_0-c)=0, \\ \gamma(2\alpha^2\lambda_0-c)=0. \end{cases} \tag{50}$$

If $\beta=\gamma=0$, then we have $c=-\alpha^2+2\alpha^2\lambda_0$ and case (4) is true; if $\beta\neq 0$, the last two equations of the system (50) gives $2\alpha^2=0$, which is a contradiction.

If $\alpha\neq\delta$, the first and the fourth equations of the system (48) gives $\frac{1}{2}\beta(\beta-\gamma)-\alpha^2-(\beta(\beta-\gamma)-2\alpha^2)\lambda_0-c=0$. In this case, system (48) reduce to

$$\begin{cases} -\alpha\gamma+\frac{1}{2}\delta(\beta-\gamma)=0, \\ \beta(\beta(\beta-\gamma)-2\alpha^2-(\beta(\beta-\gamma)-2\alpha^2)\lambda_0-c)=0, \\ \gamma((\beta(\beta-\gamma)-2\alpha^2)\lambda_0+c)=0. \end{cases} \tag{51}$$

The second equation in the system (51) yields $\beta+\gamma=0$. Since $\alpha\gamma-\beta\delta=0$ we deduce that $\beta=\gamma=0$. Therefore, the case (5) hold. □

3.7. Algebraic Schouten Soliton of G_7

By [6], we have the following Lie algebra of G_7 satisfies

$$\begin{aligned} [e_1,e_2]&=-\alpha e_1-\beta e_2-\beta e_3, \\ [e_1,e_3]&=\alpha e_1+\beta e_2+\beta e_3, \\ [e_2,e_3]&=\gamma e_1+\delta e_2+\delta e_3, \end{aligned} \tag{52}$$

where $\alpha+\delta\neq 0$ and $\alpha\gamma=0$. $\{e_1,e_2,e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_7$, with e_3 timelike. From [21], we have the following lemma.

Lemma 7 ([21]). *The Ricci tensor of (G_7, g, J) associated with the perturbed canonical connection $\overline{\nabla}^0$ is given by*

$$\tilde{\rho}^0(e_i, e_j) = \begin{pmatrix} -(\alpha^2 + \frac{1}{2}\beta\gamma) & 0 & -\frac{1}{2}(\alpha\gamma + \frac{1}{2}\gamma\delta - \beta\rho) \\ 0 & -(\alpha^2 + \frac{1}{2}\beta\gamma) & \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma + \delta\rho) \\ -\frac{1}{2}(\alpha\gamma + \frac{1}{2}\gamma\delta) & \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma) & 0 \end{pmatrix}. \tag{53}$$

Theorem 7. *(G_7, g, J) is the algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ if it satisfies one of the following conditions:*

1. $\alpha \neq 0, \beta = \gamma = 0, \delta = 0, \rho \neq 0, c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$;
2. $\alpha \neq 0, \beta = \gamma = 0, \delta \neq 0, \rho = 0, c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$;
3. $\alpha = \gamma = 0, \delta \neq 0, \rho = 0,$ *and* $c = 0$.

Proof. Using (53), we have

$$\widetilde{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \frac{1}{2}\beta\gamma) & 0 & \frac{1}{2}(\alpha\gamma + \frac{1}{2}\gamma\delta - \beta\rho) \\ 0 & -(\alpha^2 + \frac{1}{2}\beta\gamma) & -\frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma + \delta\rho) \\ -\frac{1}{2}(\alpha\gamma + \frac{1}{2}\gamma\delta) & \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{54}$$

Therefore, the scalar curvature can be obtained as $s^0 = -(2\alpha^2 + \beta\gamma)$. We can write D^0 as

$$\begin{cases} D^0 e_1 = -(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_1 + \frac{1}{2}(\alpha\gamma + \frac{1}{2}\gamma\delta - \beta\rho)e_3, \\ D^0 e_2 = -(\alpha^2 + \frac{1}{2}\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_2 - \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma + \delta\rho)e_3, \\ D^0 e_3 = -\frac{1}{2}(\alpha\gamma + \frac{1}{2}\gamma\delta)e_1 + \frac{1}{2}(\alpha^2 + \frac{1}{2}\beta\gamma)e_2 + ((2\alpha^2 - \beta\gamma)\lambda_0 - c)e_3. \end{cases} \tag{55}$$

Hence, by (13), there exists an algebraic Schouten soliton associated with the perturbed canonical connection $\overline{\nabla}^0$ can be established if it satisfies the following system of equations

$$\begin{cases} \alpha(\frac{1}{2}\alpha^2 + \frac{\beta\gamma}{4} - \frac{1}{2}\delta\rho - (2\alpha^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}\gamma(\alpha\gamma + \frac{\gamma\delta}{2} - \beta\rho) - \frac{1}{2}\beta(\alpha\gamma + \frac{\gamma\delta}{2}) = 0, \\ \beta(\alpha^2 + \frac{\beta\gamma}{2} - \frac{1}{2}\delta\rho - (2\alpha^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}\delta(\alpha\gamma + \frac{\gamma\delta}{2} - \beta\rho) = 0, \\ \beta(\alpha^2 + \frac{\beta\gamma}{2} - \delta\rho - (2\alpha^2 + \beta\gamma)\lambda_0 + c) + \frac{1}{2}(\alpha - \delta)(\alpha\gamma + \frac{\gamma\delta}{2} - \beta\rho) = 0, \\ \alpha(\frac{1}{2}\alpha^2 + \frac{\beta\gamma}{4} - (2\alpha^2 + \beta\gamma)\lambda_0 + c) - \frac{1}{2}\beta(\alpha\gamma + \frac{\gamma\delta}{2}) = 0, \\ \beta(\alpha^2 + \frac{\beta\gamma}{2} - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \beta(\frac{1}{2}\alpha^2 + \frac{\beta\gamma}{4} - \delta\rho - (2\alpha^2 + \beta\gamma)\lambda_0 + c) + \frac{1}{2}\alpha(\alpha\gamma + \frac{\gamma\delta}{2} - \beta\rho) = 0, \\ \gamma((2\alpha^2 + \beta\gamma)\lambda_0 - c) - \frac{1}{2}(\alpha - \delta)(\alpha\gamma + \frac{\gamma\delta}{2}) = 0, \\ \delta(\frac{1}{2}\alpha^2 + \frac{\beta\gamma}{4} - (2\alpha^2 + \beta\gamma)\lambda_0 + c) + \frac{1}{2}\beta(\alpha\gamma + \frac{\gamma\delta}{2}) = 0, \\ \delta(\frac{1}{2}\alpha^2 + \frac{\beta\gamma}{4} - \delta\rho - (2\alpha^2 + \beta\gamma)\lambda_0 + c) + \frac{1}{2}\beta(\alpha\gamma + \frac{\gamma\delta}{2}) + \frac{1}{2}\gamma(\alpha\gamma + \frac{\gamma\delta}{2} - \beta\rho) = 0. \end{cases} \tag{56}$$

Since $\alpha\gamma = 0$ and $\alpha + \delta \neq 0$. Assume that $\alpha \neq 0, \gamma = 0$. In this case, (56) becomes,

$$\begin{cases} \alpha(\alpha^2 - 2\alpha^2\lambda_0 + c) - \frac{1}{2}\alpha(\alpha^2 + \delta\rho) = 0, \\ \beta(\alpha^2 - 2\alpha^2\lambda_0 + c) = 0, \\ \beta(\alpha^2 - \delta\rho - 2\alpha^2\lambda_0 + c) - \frac{1}{2}(\alpha - \delta)\beta\rho = 0, \\ \frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0 + c = 0, \\ \beta(\alpha^2 - 2\alpha^2\lambda_0 + c) = 0, \\ \beta(\alpha^2 - \frac{1}{2}\delta\rho - 2\alpha^2\lambda_0 + c) - \frac{1}{2}\alpha\beta\rho = 0, \\ \delta(\frac{1}{2}\alpha^2 - 2\alpha^2\lambda_0 + c) = 0, \\ \delta(\frac{1}{2}\alpha^2 - \frac{1}{2}\delta\rho - 2\alpha^2\lambda_0 + c) = 0. \end{cases} \tag{57}$$

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The fourth and the fifth equations of the system (57) give $\beta = 0$, then we have $\delta\rho = 0$. If $\delta = 0$ then $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$; if $\rho = 0$ then $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$. Therefore, the cases (1)-(2) holds.

If we assume $\alpha = \gamma = 0$, then $\delta \neq 0$. In this case, (56) becomes,

$$\begin{cases} \beta c = 0, \\ \beta c - \dfrac{1}{2}\beta\delta\rho = 0, \\ \delta c = 0, \\ \delta c - \dfrac{1}{2}\delta^2\rho = 0. \end{cases} \tag{58}$$

Then we have $\rho = 0$, and $c = 0$. Therefore, the case (3) hold. □

4. Algebraic Schouten Solitons Associated with the Perturbed Kobayashi-Nomizu Connections on Three-Dimensional Lorentzian Lie Groups

In this section, we present the curvature property for the perturbed Kobayashi-Nomizu connection corresponding to three-dimensional Lorentzian Lie groups. Then, we fully classify algebraic Schouten soliton associated with the Kobayashi-Nomizu canonical connection on Lie groups G_i, $i = 1, \ldots, 7$.

4.1. Algebraic Schouten Soliton of G_1

By [6], we have the following Lie algebra of G_1 satisfies

$$[e_1, e_2] = \alpha e_1 - \beta e_3, \quad [e_1, e_3] = -\alpha e_1 - \beta e_2, \quad [e_2, e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \tag{59}$$

where $\alpha \neq 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_1$, with e_3 timelike. Then from [21], we have the following lemma. From [21], we have the following lemma.

Lemma 8 ([21])**.** *The Ricci tensor of (G_1, g, J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\tilde{\nabla}}^1$ is given by*

$$\tilde{\rho}^1(e_i, e_j) = \begin{pmatrix} -(\alpha^2+\beta^2) & \alpha\beta & -\frac{1}{2}\alpha\beta \\ \alpha\beta & -(\alpha^2+\beta^2) & \frac{1}{2}(\alpha^2+\alpha\rho) \\ -\frac{1}{2}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix}. \tag{60}$$

Theorem 8. *(G_1, g, J) is the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\tilde{\nabla}}^1$ if it satisfies $\alpha \neq 0$, $\beta = \rho = 0$, $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$.*

Proof. Using (60), we have

$$\widetilde{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2+\beta^2) & \alpha\beta & \frac{1}{2}\alpha\beta \\ \alpha\beta & -(\alpha^2+\beta^2) & -\frac{1}{2}(\alpha^2+\alpha\rho) \\ -\frac{1}{2}\alpha\beta & \frac{1}{2}\alpha^2 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{61}$$

Therefore, the scalar curvature can be obtained as $s^1 = -2(\alpha^2+\beta^2)$. We can write D^1 as

$$\begin{cases} D^1 e_1 = -(\alpha^2+\beta^2 - 2(\alpha^2+\beta^2)\lambda_0 + c)e_1 + \alpha\beta e_2 + \dfrac{\alpha\beta}{2}e_3, \\ D^1 e_2 = \alpha\beta e_1 - (\alpha^2+\beta^2 - 2(\alpha^2+\beta^2)\lambda_0 + c)e_2 - \dfrac{\alpha^2+\alpha\rho}{2}e_3, \\ D^1 e_3 = -\dfrac{\alpha\beta}{2}e_1 + \dfrac{\alpha^2}{2}e_2 + (2(\alpha^2+\beta^2)\lambda_0 - c)e_3. \end{cases} \tag{62}$$

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Hence, by (15) an algebraic Schouten soliton exists associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if it satisfies the following system of equations

$$\begin{cases} \alpha(\alpha^2+\beta^2-2(\alpha^2+\beta^2)\lambda_0+c)=\dfrac{\alpha^3+\alpha^2\rho}{2}-\alpha\beta^2, \\ \alpha^2\beta-\alpha\beta\rho=0, \\ \beta(2\alpha^2+2\beta^2-2(\alpha^2+\beta^2)\lambda_0+c)-\alpha^2\beta=0, \\ \alpha(2(\alpha^2+\beta^2)\lambda_0-c)-2\alpha\beta^2-\dfrac{\alpha^3}{2}=0, \\ \beta(2(\alpha^2+\beta^2)\lambda_0-c)-2\alpha^2\beta=0, \\ \beta(2(\alpha^2+\beta^2)\lambda_0-c)-\alpha^2\beta=0. \end{cases} \tag{63}$$

Since $\alpha \neq 0$, the last two equations of the (63) yields $\beta = 0$. In this case, (63) reduce to

$$\begin{cases} \dfrac{1}{2}\alpha^2-2\alpha^2\lambda_0+c=\dfrac{\alpha\rho}{2}, \\ \dfrac{\alpha^2}{2}-2\alpha^2\lambda_0+c=0. \end{cases} \tag{64}$$

Then, we have $\rho = 0$, $c = -\frac{1}{2}\alpha^2 + 2\alpha^2\lambda_0$. □

4.2. Algebraic Schouten Soliton of G_2

By [6], we have the following Lie algebra of G_2 satisfies

$$[e_1,e_2]=\gamma e_2-\beta e_3, \quad [e_1,e_3]=-\beta e_2-\gamma e_3, \quad [e_2,e_3]=\alpha e_1, \tag{65}$$

where $\gamma \neq 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_2$, with e_3 timelike. From [21], we have the following lemma.

Lemma 9 ([21])**.** *The Ricci tensor of (G_2, g, J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ is given by*

$$\tilde{\rho}^1(e_i,e_j)=\begin{pmatrix} -(\gamma^2+\beta^2) & 0 & -\frac{1}{2}\gamma\rho \\ 0 & -(\gamma^2+\alpha\beta) & -\frac{1}{2}\alpha\gamma \\ 0 & -\frac{1}{2}\alpha\gamma & 0 \end{pmatrix}. \tag{66}$$

Theorem 9. *(G_2, g, J) is the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ if it satisfies $\alpha = \beta = 0$, $\gamma \neq 0$, and $c = -\gamma^2 + 2\gamma^2\lambda_0$.*

Proof. Using (66), we have

$$\widetilde{Ric}^1\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}=\begin{pmatrix} -(\gamma^2+\beta^2) & 0 & \frac{1}{2}\gamma\rho \\ 0 & -(\gamma^2+\alpha\beta) & \frac{1}{2}\alpha\gamma \\ 0 & -\frac{1}{2}\alpha\gamma & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{67}$$

Therefore, the scalar curvature can be obtained as $s^1 = -(2\gamma^2 + \beta^2 + \alpha\beta)$. We can write D^1 as

$$\begin{cases} D^1e_1=-(\gamma^2+\beta^2-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)e_1+\dfrac{1}{2}\gamma\rho e_3, \\ D^1e_2=-(\gamma^2+\alpha\beta-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)e_2+\dfrac{1}{2}\alpha\gamma e_3, \\ D^1e_3=-\dfrac{1}{2}\alpha\gamma e_2+((2\gamma^2+\beta^2+\alpha\beta)\lambda_0-c)e_3. \end{cases} \tag{68}$$

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Hence, by (15), there exists an algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if it satisfies the following system of equations

$$\begin{cases} \gamma(\gamma^2+\beta^2-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)+\alpha\beta\gamma=0, \\ \beta(2\gamma^2+\beta^2+\alpha\beta-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)-\alpha\gamma^2=0, \\ \alpha\gamma\rho=0, \\ \beta(-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)-\alpha\gamma^2=0. \end{cases} \tag{69}$$

Since $\gamma\neq 0$, the third equation in (69) yields $\alpha=0$ or $\rho=0$. Let $\alpha=0$. In this case, (69) becomes

$$\begin{cases} \gamma(\gamma^2+\beta^2-(2\gamma^2+\beta^2)\lambda_0+c)=0, \\ \beta(2\gamma^2+\beta^2+\alpha\beta-(2\gamma^2+\beta^2)\lambda_0+c)=0, \\ \beta(-(2\gamma^2+\beta^2)\lambda_0+c)=0. \end{cases} \tag{70}$$

Assume $\beta\neq 0$, then the first and the last equations above leads to $\beta^2+\gamma^2=0$, which is a contradiction. Then we have $\beta=0$ and $c=-\gamma^2+2\gamma^2\lambda_0$.

Next, let $\rho=0$. If $\beta=0$, then the last equation in (69) give $\alpha\gamma^2$, which is a contradiction. If $\beta\neq 0$, the second and the last equations in (69) yields $\beta(2\gamma^2+\beta^2+\alpha\beta)=0$. In this case, (69) reduce to

$$\begin{cases} \gamma(\gamma^2+\beta^2-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)+\alpha\beta\gamma=0, \\ \beta(-(2\gamma^2+\beta^2+\alpha\beta)\lambda_0+c)-\alpha\gamma^2=0, \\ 2\gamma^2+\beta^2+\alpha\beta=0. \end{cases} \tag{71}$$

The first two equation above yields $\alpha=-\beta$ and replacing it in the last equation, we get $\gamma^2=0$, which is a contradiction. □

4.3. *Algebraic Schouten Soliton of G_3*

By [6], we have the following Lie algebra of G_3 satisfies

$$[e_1,e_2]=-\gamma e_3, \quad [e_1,e_3]=-\beta e_2, \quad [e_2,e_3]=\alpha e_1, \tag{72}$$

where $\{e_1,e_2,e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_3$, with e_3 timelike. From [21], we have the following lemma.

Lemma 10 ([21]). *The Ricci tensor of (G_3,g,J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ is given by*

$$\tilde{\rho}^1(e_i,e_j)=\begin{pmatrix} -\beta\gamma & 0 & 0 \\ 0 & -\alpha\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{73}$$

Theorem 10. *(G_3,g,J) is the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ if it satisfies one of the following conditions:*

1. *$\alpha=\beta=\gamma=0$, for all c;*
2. *$\alpha\neq 0$, $\beta=\gamma=0$, $c=0$;*
3. *$\alpha=\gamma=0$, $\beta\neq 0$, $c=0$;*
4. *$\alpha\neq 0$, $\beta\neq 0$, $\gamma=0$, $c=0$;*
5. *$\alpha=\beta=0$, $\gamma\neq 0$, $c=0$;*
6. *$\alpha\neq 0$, $\gamma\neq 0$, $\beta=0$, and $\alpha=\gamma$, $c=-\alpha^2+\alpha^2\lambda_0$;*
7. *$\alpha=0$, $\beta\neq 0$, $\gamma\neq 0$, $c=-\beta^2+\beta^2\lambda_0$.*

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Proof. Using (73), we have

$$\widetilde{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\beta\gamma & 0 & 0 \\ 0 & -\alpha\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{74}$$

Therefore, the scalar curvature can be obtained as $s^1 = -(\alpha\gamma + \beta\gamma)$. We can write D^1 as

$$\begin{cases} D^1 e_1 = -(\beta\gamma - (\alpha\gamma + \beta\gamma)\lambda_0 + c)e_1, \\ D^1 e_2 = -(\alpha\gamma - (\alpha\gamma + \beta\gamma)\lambda_0 + c)e_2, \\ D^1 e_3 = ((\alpha\gamma + \beta\gamma)\lambda_0 - c)e_3. \end{cases} \tag{75}$$

Hence, by (15) there exists an algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if it satisfies the following system of equations

$$\begin{cases} \gamma(\alpha\gamma + \beta\gamma - (\alpha\gamma + \beta\gamma)\lambda_0 + c) = 0, \\ \beta(\beta\gamma - \alpha\gamma - (\alpha\gamma + \beta\gamma)\lambda_0 + c) = 0, \\ \alpha(\alpha\gamma - \beta\gamma - (\alpha\gamma + \beta\gamma)\lambda_0 + c) = 0. \end{cases} \tag{76}$$

Assume that $\gamma = 0$, then we have cases (1)-(4). Now, we assume that $\gamma \neq 0$ then $c = -\alpha\gamma - \beta\gamma + (\alpha\gamma + \beta\gamma)\lambda_0$. Meanwhile, if $\beta = 0$ we have cases (5) and (6); if $\beta \neq 0$, for case (7), system (76) hold. □

4.4. Algebraic Schouten Soliton of G_4

By [6], we have the following Lie algebra of G_4 satisfies

$$[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \quad [e_1, e_3] = e_3 - \beta e_2, \quad [e_2, e_3] = \alpha e_1, \tag{77}$$

where $\eta = 1$ *or* -1. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_4$, with e_3 timelike. From [21], we have the following lemma.

Lemma 11 ([21]). *The Ricci tensor of (G_4, g, J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ is given by*

$$\tilde{\rho}^1(e_i, e_j) = \begin{pmatrix} -(1 + (\beta - 2\eta)\beta) & 0 & \frac{1}{2}\rho \\ 0 & -(1 + (\beta - 2\eta)\alpha) & \frac{1}{2}\alpha \\ 0 & \frac{1}{2}\alpha & 0 \end{pmatrix}. \tag{78}$$

Theorem 11. *(G_4, g, J) is not the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$.*

Proof. Using (78), we have

$$\widetilde{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(1 + (\beta - 2\eta)\beta) & 0 & -\frac{1}{2}\rho \\ 0 & -(1 + (\beta - 2\eta)\alpha) & -\frac{1}{2}\alpha \\ 0 & \frac{1}{2}\alpha & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{79}$$

Therefore, the scalar curvature can be obtained as $s^1 = -(2 + (\beta - 2\eta)(\beta + \alpha))$. We can write D^1 as

$$\begin{cases} D^1 e_1 = -(1+(\beta-2\eta)\beta-(2+(\beta-2\eta)(\beta+\alpha))\lambda_0+c)e_1-\frac{1}{2}\rho e_3, \\ D^1 e_2 = -(1+(\beta-2\eta)\beta-(2+(\beta-2\eta)(\beta+\alpha))\lambda_0+c)e_2-\frac{1}{2}\alpha e_3, \\ D^1 e_3 = \frac{1}{2}\alpha e_2+((2+(\beta-2\eta)(\beta+\alpha))\lambda_0-c)e_3. \end{cases} \tag{80}$$

Hence, by (15) there exists an algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if it satisfies the following system of equations

$$\begin{cases} \alpha\rho = 0, \\ 1+(\beta-2\eta)\beta-(2+(\beta-2\eta)(\beta+\alpha))\lambda_0+c = \alpha(\eta-\beta), \\ (2\eta-\beta)(2+(\beta-2\eta)(\alpha+\beta)-(2+(\beta-2\eta)(\beta+\alpha))\lambda_0+c) = -\alpha, \\ \beta((\beta-2\eta)(\beta-\alpha)-(2+(\beta-2\eta)(\beta+\alpha))\lambda_0+c) = \alpha, \\ \alpha((\beta-2\eta)(\alpha-\beta)-(2+(\beta-2\eta)(\beta+\alpha))\lambda_0+c) = 0. \end{cases} \tag{81}$$

Assume first that $\alpha = 0$, we have

$$\begin{cases} 1+(\beta-2\eta)\beta-\beta(2+(\beta-2\eta))\lambda_0+c = 0, \\ \beta((\beta-2\eta)\beta-\beta(2+(\beta-2\eta))\lambda_0+c) = 0. \end{cases} \tag{82}$$

This is a contradiction.

If $\rho = 0$, and $\alpha \neq 0$, we have

$$\begin{cases} 1+(\beta-2\eta)\beta-(\beta-2\eta)(\alpha-\beta) = \alpha(\eta-\beta), \\ (2\eta-\beta)(2+(\beta-2\eta)(\alpha+\beta)-(\beta-2\eta)(\alpha-\beta)) = -\alpha, \\ \beta((\beta-2\eta)(\beta-\alpha)-(\beta-2\eta)(\alpha-\beta)) = \alpha. \end{cases} \tag{83}$$

Which have no solution. □

4.5. Algebraic Schouten Soliton of G_5

By [6], we have the following Lie algebra of G_5 satisfies

$$[e_1, e_2] = 0, \quad [e_1, e_3] = \alpha e_1 + \beta e_2, \quad [e_2, e_3] = \gamma e_1 + \delta e_2, \tag{84}$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_5$, with e_3 timelike. From [21], we have the following lemma.

Lemma 12 ([21])**.** *The Ricci tensor of (G_5, g, J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ is given by*

$$\tilde{\rho}^1(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{85}$$

Theorem 12. *(G_5, g, J) is the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ if $c = 0$.*

Proof. Using (85), we have

$$\widetilde{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{86}$$

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Therefore, the scalar curvature can be obtained as $s^1 = 0$. We can write D^1 as

$$\begin{cases} D^1 e_1 = -ce_1, \\ D^1 e_2 = -ce_2, \\ D^1 e_3 = -ce_3. \end{cases} \tag{87}$$

Hence, by (15), there exists an algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if $c = 0$. □

4.6. Algebraic Schouten Soliton of G_6

By [6], we have the following Lie algebra of G_6 satisfies

$$\begin{aligned} [e_1, e_2] &= \alpha e_2 + \beta e_3, \\ [e_1, e_3] &= \gamma e_2 + \delta e_3, \\ [e_2, e_3] &= 0, \end{aligned} \tag{88}$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_6$, with e_3 timelike.

Lemma 13 ([21]). *The Ricci tensor of (G_6, g, J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ is given by*

$$\tilde{\rho}^1(e_i, e_j) = \begin{pmatrix} -(\alpha^2 + \beta\gamma) & 0 & \frac{1}{2}\delta\rho \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{89}$$

Theorem 13. *(G_6, g, J) is the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ if it satisfies one of the following conditions:*

1. $\alpha = \beta = \gamma = 0, \delta \neq 0, c = -s\lambda_0$;
2. $\alpha = \beta = 0, \gamma \neq 0, \delta \neq 0, c = -s\lambda_0$;
3. $\alpha \neq 0, \beta = \gamma = \delta = 0, c = -\alpha^2 - s\lambda_0$;
4. $\alpha \neq 0, \beta = \gamma = 0, \delta \neq 0, c = -\alpha^2 - s\lambda_0$.

Proof. Using (89), we have

$$\widetilde{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2 + \beta\gamma) & 0 & -\frac{1}{2}\delta\rho \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{90}$$

Therefore, the scalar curvature can be obtained as $s^1 = -(2\alpha^2 + \beta\gamma)$. We can write D^1 as

$$\begin{cases} D^1 e_1 = -(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_1 - \dfrac{\delta\rho}{2} e_3, \\ D^1 e_2 = -(\alpha^2 - (2\alpha^2 + \beta\gamma)\lambda_0 + c)e_2, \\ D^1 e_3 = ((2\alpha^2 + \beta\gamma)\lambda_0 - c)e_3. \end{cases} \tag{91}$$

Hence, by (15) there exists an algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if it satisfies the following system of equations

$$\begin{cases} \alpha(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \beta(2\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \gamma(\beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0, \\ \delta(\alpha^2 + \beta\gamma - (2\alpha^2 + \beta\gamma)\lambda_0 + c) = 0. \end{cases} \tag{92}$$

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Assume first $\alpha = 0$, since $\alpha + \delta \neq 0$ and $\alpha\gamma - \beta\delta = 0$, we have $\beta = 0$ and $\delta \neq 0$. In this case, (92) becomes

$$\begin{cases} \gamma c = 0, \\ \delta c = 0. \end{cases} \tag{93}$$

If $\gamma = 0$, then $c = 0$. Therefore, the case (1) hold. If $\gamma \neq 0$, then $c = 0$. In this case, the system (92) holds.

If $\alpha \neq 0$, consider that $\delta = 0$. Similarly, we have $\gamma = 0$, then (92) reduce to

$$\begin{cases} \alpha(\alpha^2 - 2\alpha^2\lambda_0 + c) = 0, \\ \beta(2\alpha^2 - 2\alpha^2\lambda_0 + c) = 0, \end{cases} \tag{94}$$

which leads to $\beta = 0$ and $c = -\alpha^2 + 2\alpha^2\lambda_0$.

If $\delta \neq 0$, then the second and the third equations of the system (92) gives $\alpha^2\beta = \beta\gamma^2 = 0$, which provides $\beta = \gamma = 0$. Therefore, the case (4) hold. □

4.7. Algebraic Schouten Soliton of G_7

By [6], we have the following Lie algebra of G_7 satisfies

$$\begin{aligned} [e_1, e_2] &= -\alpha e_1 - \beta e_2 - \beta e_3, \\ [e_1, e_3] &= \alpha e_1 + \beta e_2 + \beta e_3, \\ [e_2, e_3] &= \gamma e_1 + \delta e_2 + \delta e_3, \end{aligned} \tag{95}$$

where $\alpha + \delta \neq 0$ and $\alpha\delta = 0$. $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of $\mathfrak{g}_7$, with e_3 timelike.

Lemma 14 ([21])**.** *The Ricci tensor of (G_7, g, J) associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ is given by*

$$\tilde{\rho}^1(e_i, e_j) = \begin{pmatrix} -\alpha^2 & \frac{1}{2}(\beta\delta - \alpha\beta) & \beta(\alpha + \delta + \frac{\rho}{2}) \\ \frac{1}{2}(\beta\delta - \alpha\beta) & -(\alpha^2 + \beta^2 + \beta\gamma) & \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2 + \delta\rho) \\ \beta(\alpha + \delta) & \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2) & 0 \end{pmatrix}. \tag{96}$$

Theorem 14. *(G_7, g, J) is the algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ if it satisfies one of the following conditions:*

1. *$\alpha \neq 0, \beta = 0, \gamma = 0, \alpha = \delta$ and $\rho = -4\alpha$, $c = -\frac{3}{2}\delta^2 + 2\alpha^2\lambda_0$,*
2. *$\alpha = \beta = \gamma = 0, \delta \neq 0$, and $\rho = -4\delta$, $c = -\delta^2$.*

Proof. Using (96), we have

$$\widetilde{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\alpha^2 & \frac{1}{2}(\beta\delta - \alpha\beta) & -\beta(\alpha + \delta + \frac{\rho}{2}) \\ \frac{1}{2}(\beta\delta - \alpha\beta) & -(\alpha^2 + \beta^2 + \beta\gamma) & -\frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2 + \delta\rho) \\ \beta(\alpha + \delta) & \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2) & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{97}$$

Therefore, the scalar curvature can be obtained as $s^1 = -(2\alpha^2 + \beta^2 + \beta\gamma)$. We can write D^1 as

$$\begin{cases} D^1 e_1 = -(\alpha^2 + s^1\lambda_0 + c)e_1 + \frac{1}{2}(\beta\delta - \alpha\beta)e_2 - \beta(\alpha + \delta + \frac{\rho}{2})e_3, \\ D^1 e_2 = \frac{1}{2}(\beta\delta - \alpha\beta)e_1 - (\alpha^2 + \beta^2 + \beta\gamma + s^1\lambda_0 + c)e_2 - \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2 + \delta\rho)e_3, \\ D^1 e_3 = \beta(\alpha + \delta)e_1 + \frac{1}{2}(\beta\gamma + \alpha\delta + 2\delta^2)e_2 - (s^1\lambda_0 + c)e_3. \end{cases} \tag{98}$$

Hence, by (15) there exists an algebraic Schouten soliton associated with the perturbed Kobayashi-Nomizu connection $\overline{\nabla}^1$ can be established if it satisfies the following system of equations

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$$\begin{cases} \alpha(\alpha^2+\beta^2+\beta\gamma+s^1\lambda_0+c)+\beta\gamma(\alpha+\delta+\frac{\rho}{2})+\frac{1}{2}\beta(\beta\delta-\alpha\beta)-\frac{1}{2}\alpha(\beta\gamma+\alpha\delta+2\delta^2+\delta\rho)=0, \\ \beta(\alpha^2+s^1\lambda_0+c)-\frac{1}{2}\beta\delta\rho+\frac{1}{2}\alpha(\beta\delta-\alpha\beta)+\beta\delta(\alpha+\delta+\frac{\rho}{2})=0, \\ \beta(2\alpha^2+\beta^2+\beta\gamma+s^1\lambda_0+c)-\beta(\beta\gamma+\alpha\delta+2\delta^2+\delta\rho)+(\beta-\alpha)\delta(\alpha+\delta+\frac{\rho}{2})=0, \\ \alpha(s^1\lambda_0+c)+\frac{1}{2}(\beta-\gamma)(\beta\delta-\alpha\beta)+\beta^2(\alpha+\delta)+\frac{1}{2}\alpha(\beta\gamma+\alpha\delta+2\delta^2)=0, \\ \beta(-\beta^2-\beta\gamma+s^1\lambda_0+c)+\frac{1}{2}(\alpha-\delta)(\beta\delta-\alpha\beta)+\beta(\beta\gamma+\alpha\delta+2\delta^2)=0, \\ \beta(\alpha^2+s^1\lambda_0+c)-\frac{1}{2}\beta\delta\rho-\alpha\beta(\alpha+\delta+\frac{\rho}{2})-\frac{1}{2}\delta(\beta\delta-\alpha\beta)=0, \\ \gamma(\beta^2+\beta\gamma+s^1\lambda_0+c)+\frac{1}{2}\beta(\delta-\alpha)(3\delta+\alpha)=0, \\ \delta(s^1\lambda_0+c)+\frac{1}{2}(\gamma-\beta)(\beta\delta-\alpha\beta)+\frac{1}{2}\delta(\beta\gamma+\alpha\delta+2\delta^2)-\beta^2(\alpha+\delta)=0, \\ \delta(\alpha^2+\beta^2+\beta\gamma+s^1\lambda_0+c)-\beta\gamma(\alpha+\delta+\frac{\rho}{2})-\frac{1}{2}\delta(\beta\gamma+\alpha\delta+2\delta^2+\delta\rho)=\frac{1}{2}\beta^2(3\delta+\alpha). \end{cases} \tag{99}$$

Throughout the proof, recall that $\alpha+\delta\neq 0$ and $\alpha\gamma=0$. Assume first $\alpha\neq 0$, $\gamma=0$. In this case, (99) becomes

$$\begin{cases} \alpha(\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)+\frac{1}{2}\beta(\beta\delta-\alpha\beta)-\frac{1}{2}\alpha(\beta\gamma+\alpha\delta+2\delta^2+\delta\rho)=0, \\ \beta(\alpha^2-(2\alpha^2+\beta^2)\lambda_0+c)-\frac{1}{2}\beta\delta\rho+\frac{1}{2}\alpha(\beta\delta-\alpha\beta)+\beta\delta(\alpha+\delta+\frac{\rho}{2})=0, \\ \beta(2\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)-\beta(\alpha\delta+2\delta^2+\delta\rho)+(\beta-\alpha)\delta(\alpha+\delta+\frac{\rho}{2})=0, \\ \alpha(-(2\alpha^2+\beta^2)\lambda_0+c)+\frac{1}{2}\beta(\beta\delta-\alpha\beta)+\beta^2(\alpha+\delta)+\frac{1}{2}\alpha(\alpha\delta+2\delta^2)=0, \\ \beta(-\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)+\frac{1}{2}(\alpha-\delta)(\beta\delta-\alpha\beta)+\beta\alpha\delta+2\delta^2)=0, \\ \beta(\alpha^2-(2\alpha^2+\beta^2)\lambda_0+c)-\frac{1}{2}\beta\delta\rho-\alpha\beta(\alpha+\delta+\frac{\rho}{2})-\frac{1}{2}\delta(\beta\delta-\alpha\beta)=0, \\ \frac{1}{2}\beta(\delta-\alpha)(3\delta+\alpha)=0, \\ \delta(-(2\alpha^2+\beta^2)\lambda_0+c)-\frac{1}{2}\beta(\beta\delta-\alpha\beta)+\frac{1}{2}\delta(\alpha\delta+2\delta^2)-\beta^2(\alpha+\delta)=0, \\ \delta(\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)-\frac{1}{2}\delta(\alpha\delta+2\delta^2+\delta\rho)=\frac{1}{2}\beta^2(3\delta+\alpha). \end{cases} \tag{100}$$

The seventh equation of (100) yields $\beta=0$, or $\alpha=\delta$ or $\alpha=-3\delta$. Let $\beta=0$. In this case, (100) becomes

$$\begin{cases} \alpha(\alpha^2-2\alpha^2\lambda_0+c)-\frac{1}{2}\alpha(\alpha\delta+2\delta^2+\delta\rho)=0, \\ \alpha\delta(\alpha+\delta+\frac{\rho}{2})=0, \\ \alpha(-2\alpha^2\lambda_0+c)+\frac{1}{2}\alpha(\alpha\delta+2\delta^2)=0, \\ \delta(-2\alpha^2\lambda_0+c)+\frac{1}{2}\delta(\alpha\delta+2\delta^2)=0, \\ \delta(\alpha^2-2\alpha^2\lambda_0+c)-\frac{1}{2}\delta(\alpha\delta+2\delta^2+\delta\rho)=0. \end{cases} \tag{101}$$

Then, we have $\alpha=\delta$ and $\rho=-4\alpha$. Therefore, the case (1) hold. Now, let $\alpha=\delta$, then the second and the sixth equations of (100) yield $\alpha+\delta+\frac{\rho}{2}=0$. In this case, the first and the third equations in (100) becomes

$$\begin{cases} \alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c+\dfrac{1}{2}\alpha\delta=0, \\ 2\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c+\alpha\delta=0. \end{cases} \tag{102}$$

We have $\alpha=-\frac{\delta}{2}$, which is a contradiction. Finally, let $\alpha=-3\delta$, again the second and the sixth equations of (100) yields $\rho=0$. The last two equations in (100) give

$$\begin{cases} \delta(-(2\alpha^2+\beta^2)\lambda_0+c)-\dfrac{1}{2}\delta^3=0, \\ \delta(\alpha^2+\beta^2-(2\alpha^2+\beta^2)\lambda_0+c)+\dfrac{1}{2}\delta^3=0. \end{cases} \tag{103}$$

This is a contradiction.

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Assume that $\alpha = \gamma = 0$, since $\alpha + \delta \neq 0$, we have $\delta \neq 0$. In this case, (100) becomes

$$\begin{cases} \beta^2\delta = 0, \\ \beta\delta(\delta + \frac{\rho}{2}) = 0, \\ \beta(-\beta^2 - \beta^2\lambda_0 + c) + \frac{3}{2}\beta\delta^2 = 0, \\ \beta(-\beta^2\lambda_0 + c) - \frac{1}{2}\beta\delta\rho = 0, \\ \delta(-\beta^2\lambda + c) - \frac{3}{2}\beta^2\delta + \delta^3 = 0, \\ \delta(\beta^2 - \beta^2\lambda_0 + c) - \frac{1}{2}\delta(2\delta^2 + \delta\rho) = 0. \end{cases} \tag{104}$$

We have $\beta = 0$, $\rho = -4\delta$. Therefore, the case (2) hold. □

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