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Density Formula in Malliavin Calculus by Using Stein's Method and Diffusions

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Abstract: Let G be a random variable of functionals of an isonormal Gaussian process X defined on some probability space. An explicit formula for the density of G is obtained by Nourdin and Viens (2009) [Density formula and concentration inequalities with Malliavin calculus, *Elec. J. of Probab.*, **14**(78). 2287-2309]. In this paper, unlike previous studies, we will use Stein's method for invariant measures of diffusions to obtain the density formula of G. Using this, we will show that the diffusion coefficient of a Itô diffusion with an invariant measure having a density can be expressed as in terms of operators in Malliavin calculus.

Keywords: Malliavin calculus; Stein's method; density function; standard normal random variable; Itô diffusion

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1. Introduction

Let $X = \{X(h), h \in \mathfrak{H}\}$, where $\mathfrak{H}$ is a real separable Hilbert space, be an isonormal Gaussian process defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and let G be a random variable of functionals of an isonormal Gaussian process X.

The following formula on the density of a random variable G is a well-known fact of the Malliavin calculus: if $\frac{DG}{\|DG\|_{\mathfrak{H}}}$ belongs to the domain of divergence operator δ, then the law of G has a continuous and bounded density p_G given by

$$p_G(x) = \mathbb{E}\left[\mathbf{1}_{\{G>x\}}\delta\left(\frac{DG}{\|DG\|_{\mathfrak{H}}}\right)\right] \quad \text{for all} \quad x \in \mathbb{R}.$$

Several examples are detailed in Section 2.1.1 of Nualart's book [1] (or [2]). Nourdin and Viens (2009) prove a new general formula for p_G which does not refer to divergence operator δ. For a random variable $G \in \mathbb{D}^{1,2}$ with $\mathbb{E}[G] = 0$, where $\mathbb{D}^{1,2}$ is the domain of the Malliavin derivative operator D with respect to X such that the Malliavin derivative DG of G is a random element belonging in $\mathfrak{H}$ with $\mathbb{E}[\|DG\|^2_{\mathfrak{H}}] < \infty$, we define the function g_G by

$$g_G(x) = \mathbb{E}[\langle DG, -DL^{-1}G\rangle_{\mathfrak{H}}|G = x]. \tag{1}$$

The operator L appearing in (1) is the so-called generator of the Ornstein-Uhlenbeck semigroup and L^{-1} is its pseudo-inverse. For details, see the section 2. It is well known that g_G is non-negative on the support of the law of G (see Proposition 3.9 in [3]).

Under some general conditions on a random variable G, Nourdin and Viens (2009) obtain the new formula of the density p_G for the law of G provided it exists. A precise statement is given in the following theorem:

Theorem 1. [Nourdin and Viens] *The law of G admits a density (with respect to Lebesgue measure), say p_G, if and only if the random variable $g_G(G)$ is strictly positive almost surely. In this case, the support of p_G, denoted by $supp(p_G)$, is a closed interval of $\mathbb{R}$ containing zero and, for almost all $x \in supp(p_G)$,*

$$p_G(x) = \frac{\mathbb{E}[|G|]}{2g_G(x)} \exp\left(-\int_0^x \frac{y}{g_G(y)} dy \right). \tag{2}$$

Assume that the density p satisfies the following conditions: it is continuous, bounded, with $\int_l^u x^2 p(x)dx < \infty$. Let us set an interval $I = (l, u)$ $(-\infty \le l < u \le \infty)$. Then

$$\begin{cases} p(x) > 0 & \text{if } x \in I \\ p(x) = 0 & \text{if } x \in I^c \end{cases}.$$

We define a continuous function b on I such that there exists $e \in (l, u)$ satisfying

$$\begin{cases} b(x) > 0 & \text{if } x \in (l, e) \\ b(x) < 0 & \text{if } x \in (e, u) \end{cases},$$

bp is bounded on I and

$$\int_l^u b(x)p(x)dx = 0.$$

Define

$$a(x) = \frac{2}{p(x)} \int_l^x b(y)p(y)dy. \tag{3}$$

Then the diffusion with the invariant density p has the Stochastic Differential Equation (SDE) with the form

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dW_t, \tag{4}$$

where W is a standard Brownian motion.

In this paper, we derive the new density formula of a random variable G, that satisfies appropriate conditions related to Malliavin calculus, from the following equation: for every $z \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(G \le z) - \mathbb{P}(F \le z) \quad &= \quad \mathbb{E}\left[\tilde{h}'_z(G) \left(\frac{1}{2} a(G) + \langle -DL^{-1} b(G), DG \rangle_{\mathfrak{H}} \right) \right] \\ &\quad + \mathbb{E}[b(G)]\mathbb{E}[\tilde{h}_z(G)], \end{aligned} \tag{5}$$

where F is a random variable with the invariant density p and $\tilde{h}_z$ is a solution to the *Stein's equation* (for detailed explanation of *Stein's method*, see [4–6]). Also we will show that the diffusion coefficient a of SDE (4) can be written in an explicit form like (1) if the random variable G in (5) with its value on I has a density p and satisfies $b(G) \in L^2(\Omega)$. .

The rest of this paper is organized as follows. Section 2 reviews some basic notations, and the contents of Malliavin calculus. In section 3, we will briefly discuss the construction of a diffusion process with an invariant density p and then describe our main results. Finally, as an application of our main results, in Section 4, we give some examples.

2. Preliminaries

In this section, we present some basic facts about Malliavin operators defined on spaces of random elements that are functionals of possibly infinite-dimensional Gaussian fields. For a more detailed explanation, see [1,7]. Suppose that $\mathfrak{H}$ is a real separable Hilbert space with a scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. Let $X = \{X(h), h \in \mathfrak{H}\}$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}$. For every $n \ge 1$, let $\mathbb{H}_n$ be the nth Wiener

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chaos of X, that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_n(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. We define a linear isometric mapping $I_n : \mathfrak{H}^{\odot n} \to \mathbb{H}_n$ by $I_n(h^{\otimes n}) = n!H_n(X(h))$, where $\mathfrak{H}^{\odot n}$ is the symmetric tensor product. It is well known that any square integrable random variable $F \in L^2(\Omega, \mathfrak{F}, \mathbb{P})$ ($\mathfrak{F}$ denotes the σ-field generated by X) can be expanded into a series of multiple stochastic integrals:

$$F = \sum_{q=0}^{\infty} I_q(f_q),$$

where $f_0 = \mathbb{E}[F]$, the series converges in L^2, and the functions $f_q \in \mathfrak{H}^{\odot q}$ are uniquely determined by F.

Let $\mathfrak{S}$ be the class of smooth and cylindrical random variables F of the form

$$F = f(X(\varphi_1), \cdots, X(\varphi_n)), \tag{6}$$

where $n \geq 1$, $f \in C_b^{\infty}(\mathbb{R}^n)$ and $\varphi_i \in \mathfrak{H}$, $i = 1, \cdots, n$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(\varphi_1), \cdots, X(\varphi_n))\varphi_i. \tag{7}$$

We denote by $\mathbb{D}^{l,p}$ the closure of its associated smooth random variable class with respect to the norm

$$\|F\|_{l,p}^p = \mathbb{E}(|F|^p) + \sum_{k=1}^{l} \mathbb{E}(\|D^k F\|_{\mathfrak{H}^{\otimes k}}^p).$$

We denote by δ the adjoint of the operator D, also called the *divergence operator*. The domain of δ, denoted by $\mathrm{Dom}(\delta)$, is an element $u \in \mathbb{L}^2(\Omega; \mathfrak{H})$ such that

$$|\mathbb{E}(< D^l F, u >_{\mathfrak{H}^{\otimes l}})| \leq C(\mathbb{E}|F|^2)^{1/2} \quad \text{for all } F \in \mathbb{D}^{l,2}.$$

If $u \in \mathrm{Dom}(\delta)$, then $\delta(u)$ is the element of $L^2(\Omega)$ defined by the duality relationship

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathfrak{H}}] \quad \text{for every } F \in \mathbb{D}^{1,2}.$$

Recall that $F \in L^2(\Omega)$ can be expanded as $F = \mathbb{E}[F] + \sum_{q=1}^{\infty} P_q F$, where p_q is the projection operator $L^2(\Omega)$ to the qth Wiener chaos $\mathbb{H}_n$. The operator L is defined through the projection operator P_q, $q = 0, 1, 2 \ldots$, as $L = \sum_{q=0}^{\infty} -qP_q$, and is called the *infinitesimal generator of the Ornstein-Uhlhenbeck semigroup*. The relationship between the operator D, δ, and L is given as follows: $\delta DF = -LF$, that is, for $F \in L^2(\Omega)$ the statement $F \in \mathrm{Dom}(L)$ is equivalent to $F \in \mathrm{Dom}(\delta D)$ (i.e, $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}(\delta)$), and in this case $\delta DF = -LF$. For any $F \in L^2(\Omega)$, we define the operator L^{-1}, which is the *pseudo-inverse* of L, as $L^{-1}F = \sum_{q=1}^{\infty} \frac{1}{q} P_q F$. Note that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and $LL^{-1}F = F - \mathbb{E}[F]$ for all $F \in L^2(\Omega)$.

3. Diffusion Process with Invariant Measures and Main Results

In this section, we will give the construction of a diffusion process with an invariant measure, and present our main results in this paper.

3.1. Diffusion Process with Invariant Measures

In this section, we will briefly describe the construction of a diffusion process with an invariant measure μ having a density p with respect to the Lebesgue measure (for more details, see [8,9]).

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Let F be a random variable with a probability measure μ on $I = (l, u)$ $(-\infty \le l < u \le \infty)$ with a density p which is continuous, bounded, strictly positive on I and $\mathbb{E}[F^2] < \infty$. Let b be a continuous function on I such that there exists $e \in (l, u)$ that satisfies $b(x) > 0$ for $e \in (l, u)$ and $b(x) < 0$ for $e \in (l.u)$. Moreover, the function bp is bounded on I and

$$\mathbb{E}[b(F)] = 0. \tag{8}$$

For $x \in I$, define

$$a(x) = \frac{2}{p(x)} \int_l^x b(y)p(y)dy. \tag{9}$$

Then the diffusion coefficient a in (9) is strictly positive for all $x \in (l, u)$, and also satisfies $\mathbb{E}[a(F)] < \infty$. The equation (9) implies that, for some $c \in I$,

$$p(x)a(x) = p(c)a(c) \exp\left(\int_c^x \frac{2b(y)}{a(y)} dy \right). \tag{10}$$

Then the following SDE

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dB_t, \tag{11}$$

has a unique ergodic Markovian weak solution with the invariant density p.

Let $\mathcal{C}_0(I) = \{f : I \to \mathbb{R} |\ f$ is continuous on I vanishing at the boundary of $I\}$. For $f \in \mathcal{C}_0(I)$, define

$$h_f(x) = \int_0^x \tilde{h}_f(y)dy,$$

where

$$\tilde{h}_f(x) = \frac{2\int_l^x (f(y) - \mathbb{E}[f(F)])p(y)dy}{a(x)p_F(x)}.$$

Then h_f satisfies the following *Stein's equation*:

$$\begin{aligned} f(x) - \mathbb{E}[f(F)] &= b(x)h_f'(x) + \frac{1}{2}a(x)h_f''(x) \\ &= b(x)\tilde{h}_f(x) + \frac{1}{2}a(x)\tilde{h}_f'(x), \end{aligned} \tag{12}$$

where F is a random variable with a probability measure μ as its law.

3.2. Main Results

Before describing our main result in this paper, we begin with the following simple result, given in Theorem 2.9.1 in [7].

Lemma 1. *Suppose that $F, G \in \mathbb{D}^{1,2}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable with bounded derivative (or when g is only almost everywhere differentiable, one needs G to have an absolutely continuous). Then*

$$\mathbb{E}[Fg(G)] = \mathbb{E}[F]\mathbb{E}[g(G)] + \mathbb{E}[g'(G)\langle DF, -DL^{-1}G\rangle_{\mathfrak{H}}]. \tag{13}$$

Let us set

$$g_{b(G)}(x) = \mathbb{E}[\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\rangle_{\mathfrak{H}} | G = x]. \tag{14}$$

Similar to the proof of Proposition 3.9 in [3], we will show that $g_{b(G)}(x)$ is non-negative almost everywhere with respect to the law of G.

Proposition 1. *Let $G \in \mathbb{D}^{1,2}$. Then we have that $g_{b(G)}(x) \ge 0$ for almost everywhere with respect to the law of G, say $H_G(x) = \mathbb{P}(G \le x)$.*

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Proof: Let q be a smooth non-negative real function. Define

$$Q(x) = \begin{cases} \int_\beta^x q(y)dy & \text{if } x \geq \beta \\ -\int_x^\beta q(y)dy & \text{if } x < \beta, \end{cases}$$

where $\beta \in \mathbb{R}$ is a constant that satisfies $b(x) - \mathbb{E}[b(G)] > 0$ for $\beta \in (l, u)$ and $b(x) - \mathbb{E}[b(G)] < 0$ for $\beta \in (l.u)$. Since $Q(x) \geq 0$ for $x \geq \beta$ and $Q(x) < 0$ for $x < \beta$, we have $\mathbb{E}[(b(G) - \mathbb{E}[b(G)])Q(G)] \geq 0$. An application of Lemma 13 yields that

$$\begin{aligned} \mathbb{E}[(b(G) - \mathbb{E}[b(G)])Q(G)] &= \mathbb{E}[\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\rangle_{\mathfrak{H}}] \\ &= \int_{-\infty}^{\infty} g_{b(G)}(x)q(x)dH_G(x) \geq 0. \end{aligned} \tag{15}$$

By an approximation of the function q, we can show that for all Borel measurable set $B \in \mathfrak{B}(\mathbb{R})$, we have

$$\int_B g_{b(G)}(x)q(x)dH_G(x) \geq 0.$$

This obviously implies that $g_{b(G)}(x) \geq 0$ for almost everywhere with respect to the law of G. □

Lemma 2. *If the random variable $g_{b(G)}(G)$ is strictly positive almost surely, then the law of G has a density with respect to Lebesgue measure, say p_G.*

Proof: By a similar argument to the proof of Theorem 3.1 in [10], we have that, for any Borel set $B \in \mathfrak{B}(\mathbb{R})$ and any $n \geq 1$,

$$\begin{aligned} &\mathbb{E}\left[(b(G)\mathbb{E}[b(G)])\int_{-\infty}^{G} \mathbf{1}_{B\cap[-n,n]}(x)dx\right] \\ &= \mathbb{E}\Big[(b(G)\mathbb{E}[b(G)])\mathbf{1}_{B\cap[-n,n]}(G)g_{b(G)}(G)\Big]. \end{aligned} \tag{16}$$

The same argument as for the case of $b(G) = G$ in the proof of Theorem 3.1 in [10] shows that the law of G has a density. □

An explicit formula for the density is the following statement:

Theorem 2. *Let F be a random variable having the law μ, and let G be a random variable in $\mathbb{D}^{1,2}$ with $b(G) \in \mathbb{L}^2(\Omega)$. Assume that the random variable $g_{b(G)}(G)$ is strictly positive almost surely and*

$$\|b\tilde{h}_f\|_\infty \leq C\|f\|_\infty = \sup_{x\in I} |f(x)| < \infty. \tag{17}$$

In this case, the support of p_G, denoted by $supp(p_G)$, is a closed interval of $\mathbb{R}$ and, for almost all $x \in supp(p_G)$,

$$p_G(x) = \frac{p_G(\beta)g_{b(G)}(\beta)}{g_{b(G)}(x)} \exp\left(-\int_\beta^x \frac{b(y) - \mathbb{E}[b(G)]}{g_{b(G)}(y)}dy\right) \tag{18}$$

for some $\beta \in supp(p_G)$.

Proof: Obviously, using (10) shows that the function $\tilde{h}_f$ can be written as

$$\begin{aligned} \tilde{h}_f(x) &= \frac{2}{p_F(\beta)a(\beta)} \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right) \\ &\quad \times \int_l^x (f(y) - \mathbb{E}[f(F)])p_F(y)dy. \end{aligned} \tag{19}$$

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Let us set $H_F(x) = \mathbb{P}(F \leq z)$. If $f(x) = \mathbf{1}_{(-\infty,z]}(x)$ for $z \in \mathbb{R}$, we write $h_f = h_z$ and $\tilde{h}_f = \tilde{h}_z$. Then the function $\tilde{h}_z$ can be written as

$$\begin{aligned} \tilde{h}_z(x) &= \frac{2}{p_F(\beta)a(\beta)} \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right) \\ &\quad \times \begin{cases} H_F(z)[1-H_F(x)] & \text{if } x \geq z \\ H_F(x)[1-H_F(z)] & \text{if } x < z. \end{cases} \end{aligned} \tag{20}$$

From (20), it follows that for $x \geq z$,

$$\begin{aligned} \tilde{h}'_z(x) &= \frac{2}{p_F(\beta)a(\beta)} \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right) \\ &\quad \times \left\{\left(-\frac{2b(x)}{a(x)}\right) H_F(z)[1-H_F(x)] - p_F(x)H_F(z)\right\}. \end{aligned} \tag{21}$$

For $x < z$,

$$\begin{aligned} \tilde{h}'_z(x) &= \frac{2}{p_F(\beta)a(\beta)} \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right) \\ &\quad \times \left\{\left(-\frac{2b(x)}{a(x)}\right) H_F(x)[1-H_F(z)] + p_F(x)[1-H_F(z)]\right\}. \end{aligned} \tag{22}$$

If $f(x) = \mathbf{1}_{(-\infty,z]}(x)$ for $x \in I$, we take $f_n \in \mathcal{C}_0(I)$ such that $\{f_n\}$ is an increasing sequence and $f_n(x) \to f(x)$ for all $x \in I$. Obviously, by the dominated convergence theorem, we have that, as $n \to \infty$,

$$\tilde{h}_{f_n}(x) \to \tilde{h}_z(x) \;\text{ and }\; \tilde{h}'_{f_n}(x) \to \tilde{h}'_z(x) \;\text{ for all }\; x \in I. \tag{23}$$

The bound of (17) yields that, for all $n \geq 1$,

$$\|b\tilde{h}_{f_n}\|_\infty \leq C\|f_n\|_\infty \leq 1. \tag{24}$$

Combining (10) with the bound in (17), we also get, for all $n \geq 1$,

$$\|a\tilde{h}'_{f_n}\|_\infty \leq C\|f_n\|_\infty \leq 1. \tag{25}$$

From (12), it follows that, for $f_n \in \mathcal{C}_0(I)$,

$$\mathbb{E}[f_n(G)] - \mathbb{E}[f_n(F)] = \mathbb{E}[b(G)\tilde{h}_{f_n}(G)] + \mathbb{E}\left[\frac{1}{2}a(G)\tilde{h}'_{f_n}(G)\right]. \tag{26}$$

Due to the bounds of (24) and (25), the dominated convergence theorem can be applied to (26), which gives the following limit value:

$$\begin{aligned} \mathbb{P}(G \leq z) - \mathbb{P}[(F \leq z) &= \mathbb{E}\Big[(b(G) - \mathbb{E}[b(G)])\tilde{h}_z(G)\Big] \\ &\quad + \mathbb{E}[b(G)]\mathbb{E}[]\tilde{h}_z(G)] + \mathbb{E}\left[\frac{1}{2}a(G)\tilde{h}'_z(G)\right]. \end{aligned} \tag{27}$$

Applying (13) in Lemma 1 to the first expectation in (27), we obtain that

$$\begin{aligned} \mathbb{P}(G \le z) - \mathbb{P}[(F \le z) &= \mathbb{E}\left[\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\rangle_{\mathfrak{H}} \tilde{h}'_z(G)\right] \\ &\quad + \mathbb{E}\left[\frac{1}{2}a(G)\tilde{h}'_z(G)\right] + \mathbb{E}[b(G)]\mathbb{E}[\tilde{h}_z(G)] \\ &= \mathbb{E}\left[\tilde{h}'_z(G)\mathbb{E}[\langle -DL^{-1}(b(G) - \mathbb{E}[b(G)]), DG\rangle_{\mathfrak{H}}|G]\right] \\ &\quad + \mathbb{E}\left[\frac{1}{2}a(G)\tilde{h}'_z(G)\right] + \mathbb{E}[b(G)]\mathbb{E}[\tilde{h}_z(G)]. \end{aligned} \tag{28}$$

Differentiating the both sides in (28) yields that

$$\begin{aligned} p_G(z) - p_F(z) &= \frac{\partial}{\partial z}\int_{-\infty}^{\infty} \tilde{h}'_z(x)\left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \\ &\quad + \mathbb{E}[b(G)]\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \tilde{h}_z(x)p_G(x)dx. \end{aligned} \tag{29}$$

Next, we concentrate on the computations of two integrals in (29). Using (21) and (22) gives that

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty} \tilde{h}'_z(x)\left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx := J_1(z) + J_2(z),$$

where

$$\begin{aligned} J_1(z) &= \frac{\partial}{\partial z}\int_{-\infty}^{z} \tilde{h}'_z(x)\left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \\ J_2(z) &= \frac{\partial}{\partial z}\int_{z}^{\infty} \tilde{h}'_z(x)\left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \end{aligned}$$

Obviously, we write $J_1(z) = J_{11}(z) + J_{12}(z)$,

$$\begin{aligned} J_{11}(z) &= \tilde{h}'_z(z)\left\{g_{b(G)}(z) + \frac{1}{2}a(z)\right\}p_G(z), \\ J_{12}(z) &= \int_{-\infty}^{z} \frac{\partial}{\partial z}\tilde{h}'_z(x)\left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx. \end{aligned}$$

For J_{12}, we first differentiate $\tilde{h}'_z(x)$ with respect to z. For $x < z$,

$$\begin{aligned} \frac{\partial}{\partial z}\tilde{h}'_z(x) &= \frac{2}{p_F(c)a(c)}\exp\left(-\int_c^x \frac{2b(y)}{a(y)}dy\right) \\ &\quad \times \left\{\left(\frac{2b(x)}{a(x)}\right)H_F(x)p_F(z) - p_F(x)p_F(z)\right\}. \end{aligned} \tag{30}$$

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By (22) and (30), we get

$$\begin{aligned}
J_{11}(z) &= \frac{2}{p_F(c)a(c)}\exp\left(-\int_c^z \frac{2b(y)}{a(y)}dy\right) \\
&\quad \times \left\{\left(-\frac{2b(z)}{a(z)}\right)H_F(z)[1-H_F(z)] + p_F(z)[1-H_F(z)]\right\} \\
&\quad \times \left\{g_{b(G)}(z) + \frac{1}{2}a(z)\right\}p_G(z), \qquad (31)\\
J_{12}(z) &= \frac{2}{p_F(c)a(c)}\int_{-\infty}^z \exp\left(-\int_c^x \frac{2b(y)}{a(y)}dy\right) \\
&\quad \times \left\{\left(\frac{2b(x)}{a(x)}\right)H_F(x)p_F(z) - p_F(x)p_F(z)\right\} \\
&\quad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx. \qquad (32)
\end{aligned}$$

For $x \geq z$,

$$\begin{aligned}
\frac{\partial}{\partial z}\tilde{h}'_z(x) &= \frac{2}{p_F(\beta)a(\beta)}\exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right) \\
&\quad \times \left\{\left(-\frac{2b(x)}{a(x)}\right)p_F(z)[1-H_F(x)] - p_F(x)p_F(z)\right\}. \qquad (33)
\end{aligned}$$

On the other hand, we write $J_2(z) = J_{21}(z) + J_{22}(z)$, where

$$\begin{aligned}
J_{21}(z) &= -\tilde{h}'_z(z)\left\{g_{b(G)}(z) + \frac{1}{2}a(z)\right\}p_G(z), \\
J_{22}(z) &= \int_z^\infty \frac{\partial}{\partial z}\tilde{h}'_z(x)\left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx.
\end{aligned}$$

From (32). we have that

$$\begin{aligned}
J_{21}(z) &= -\frac{2}{p_F(\beta)a(\beta)}\exp\left(-\int_c^z \frac{2b(y)}{a(y)}dy\right) \\
&\quad \times \left\{\left(-\frac{2b(z)}{a(z)}\right)H_F(z)[1-H_F(z)] - p_F(z)H_F(z)\right\} \\
&\quad \times \left\{g_{b(G)}(z) + \frac{1}{2}a(z)\right\}p_G(z), \qquad (34)\\
J_{22}(z) &= \frac{2}{p_F(\beta)a(\beta)}\int_z^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right) \\
&\quad \times \left\{\left(-\frac{2b(x)}{a(x)}\right)[1-H_F(x)]p_F(z) - p_F(x)p_F(z)\right\} \\
&\quad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx. \qquad (35)
\end{aligned}$$

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From (20), the differentiation of the second integral in (29) can be easily calculated as follows:

$$
\begin{aligned}
&\mathbb{E}[b(G)]\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\tilde{h}_z(x)p_G(x)dx \\
=\;&\mathbb{E}[b(G)]\frac{\partial}{\partial z}\int_{-\infty}^{z}\tilde{h}_z(x)p_G(x)dx \\
&+\mathbb{E}[b(G)]\frac{\partial}{\partial z}\int_{z}^{\infty}\tilde{h}_z(x)p_G(x)dx \\
=\;&\frac{2\mathbb{E}[b(G)]}{p_F(\beta)a(\beta)}\left\{-p_F(z)\int_{-\infty}^{z}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)H_F(x)p_G(x)dx\right. \\
&\left.+(1-H_F(z))\exp\left(-\int_\beta^z\frac{2b(y)}{a(y)}dy\right)H_F(z)p_G(z)\right\} \\
&+\frac{2\mathbb{E}[b(G)]}{p_F(\beta)a(\beta)}\left\{p_F(z)\int_{z}^{\infty}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)(1-H_F(x))p_G(x)dx\right. \\
&\left.-H_F(z)\exp\left(-\int_\beta^z\frac{2b(y)}{a(y)}dy\right)(1-H_F(z))p_G(z)\right\} \\
=\;&\frac{2p_F(z)\mathbb{E}[b(G)]}{p_F(\beta)a(\beta)}\left\{-\int_{-\infty}^{z}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)H_F(x)p_G(x)dx\right. \\
&\quad\left.+\int_{z}^{\infty}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)(1-H_F(x))p_G(x)dx\right\}.
\end{aligned}
\tag{36}
$$

Combining (31), (32), (34), (35) and (36) yields that, for $z\in\mathbb{R}$,

$$
\begin{aligned}
p_G(z)-p_F(z) =\;& \frac{2p_F(z)}{p_F(\beta)a(\beta)}\exp\left(-\int_\beta^z\frac{2b(y)}{a(y)}dy\right)\left\{g_{b(G)}(z)+\frac{1}{2}a(z)\right\}p_G(z) \\
&+\frac{2p_F(z)}{p_F(\beta)a(\beta)}\left[\int_{-\infty}^{z}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)\left(\frac{2b(x)}{a(x)}\right)H_F(x)\right. \\
&\quad\times\left\{g_{b(G)}(x)+\frac{1}{2}a(x)\right\}p_G(x)dx \\
&\quad+\int_{z}^{\infty}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)\left(-\frac{2b(x)}{a(x)}\right)[1-H_F(x)] \\
&\quad\left.\times\left\{g_{b(G)}(x)+\frac{1}{2}a(x)\right\}p_G(x)dx\right] \\
&-\frac{2p_F(z)}{p_F(\beta)a(\beta)}\int_{-\infty}^{\infty}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)p_F(x) \\
&\quad\times\left\{g_{b(G)}(x)+\frac{1}{2}a(x)\right\}p_G(x)dx \\
&+\frac{2p_F(z)\mathbb{E}[b(G)]}{p_F(\beta)a(\beta)}\left\{-\int_{-\infty}^{z}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)H_F(x)p_G(x)dx\right. \\
&\left.+\int_{z}^{\infty}\exp\left(-\int_\beta^x\frac{2b(y)}{a(y)}dy\right)(1-H_F(x))p_G(x)dx\right\}.
\end{aligned}
\tag{37}
$$

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Substituting p_F in (10) for p_F in the right-hand side of the equation (37), we get

$$\begin{aligned}
& p_G(z) - p_F(z) \\
= & \frac{2}{a(z)}\left\{g_{b(G)}(z) + \frac{1}{2}a(z)\right\}p_G(z) \\
& + \frac{2}{a(z)}\exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right)\Bigg[\int_{-\infty}^z \exp\left(-\int_c^x \frac{2b(y)}{a(y)}dy\right)\left(\frac{2b(x)}{a(x)}\right)H_F(x) \\
& \qquad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \\
& \qquad + \int_z^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)\left(-\frac{2b(x)}{a(x)}\right)[1 - H_F(x)] \\
& \qquad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx\Bigg] \\
& - \frac{2}{a(z)}\exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right)\int_{-\infty}^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)p_F(x) \\
& \qquad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \\
& + \frac{2\mathbb{E}[b(G)]}{a(z)}\exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right)\left\{-\int_{-\infty}^z \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)H_F(x)p_G(x)dx\right. \\
& \qquad \left. + \int_z^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)(1 - H_F(x))p_G(x)dx\right\}.
\end{aligned} \tag{38}$$

From the formula of p_F in (10) and (38), we obtain that, for some $\beta \in supp(p_G)$,

$$\begin{aligned}
& -g_{b(G)}(z)p_G(z) \\
= & \frac{p_F(\beta)a(\beta)}{2}\exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right) \\
& + \exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right)\Bigg\{\int_{-\infty}^z \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)\left(\frac{2b(x)}{a(x)}\right)H_F(x) \\
& \qquad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \\
& \qquad + \int_z^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)\left(-\frac{2b(x)}{a(x)}\right)[1 - H_F(x)] \\
& \qquad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx\Bigg\} \\
& - \exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right)\int_{-\infty}^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)p_F(x) \\
& \qquad \times \left\{g_{b(G)}(x) + \frac{1}{2}a(x)\right\}p_G(x)dx \\
& + \mathbb{E}[b(G)]\exp\left(\int_\beta^z \frac{2b(y)}{a(y)}dy\right)\left\{-\int_{-\infty}^z \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)\right. \\
& \qquad \times H_F(x)p_G(x)dx \\
& \qquad \left. + \int_z^\infty \exp\left(-\int_\beta^x \frac{2b(y)}{a(y)}dy\right)(1 - H_F(x))p_G(x)dx\right\}.
\end{aligned} \tag{39}$$

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Differentiating the equation (39) with respect to z proves that

$$\begin{aligned}\frac{\partial}{\partial z}g_{b(G)}(z)p_G(z) &= \frac{2b(z)}{a(z)}g_{b(G)}(z)p_G(z) \\ &\quad -\left(\frac{2b(z)}{a(z)}\right)\left\{g_{b(G)}(z)+\frac{1}{2}a(z)\right\}p_G(z)-\mathbb{E}[b(G)]p_G(z) \\ &= -(b(z)-\mathbb{E}[b(G)])p_G(z). \end{aligned} \tag{40}$$

This equation (40) proves that, for almost all $z \in \text{supp}(p_G)$,

$$g_{b(G)}(z)p_G(z) = -\int_{-\infty}^{z}(b(x)-\mathbb{E}[b(G)])p_G(x)dx. \tag{41}$$

From (40) and (41), it follows that, for almost all $z \in \text{supp}(p_G)$,

$$\frac{\frac{d}{dz}(g_{b(G)}(z)p_G(z))}{g_{b(G)}(z)p_G(z)} = -\frac{b(z)-\mathbb{E}[b(G)]}{g_{b(G)}(z)} \tag{42}$$

Hence

$$\frac{d}{dz}\log(g_{b(G)}(z)p_G(z)) = -\frac{b(z)-\mathbb{E}[b(G)]}{g_{b(G)}(z)} \tag{43}$$

By integrating both sides of (43) from $\beta \in \text{supp}(p_G)$ to z, we have

$$\log(g_{b(G)}(z)p_G(z)) = \log(g_{b(G)}(\beta)p_G(\beta)) - \int_{\beta}^{z}\frac{b(x)-\mathbb{E}[b((G))]}{g_{b(G)}(x)}dx. \tag{44}$$

The above equation (44) proves that, for almost all $z \in \text{supp}(p_G)$,

$$p_G(z) = \frac{g_{b(G)}(\beta)p_G(\beta)}{g_{b(G)}(z)}\exp\left(-\int_{\beta}^{z}\frac{b(x)-\mathbb{E}[b(G)]}{g_{b(G)}(x)}dx\right). \tag{45}$$

□

When a random variable G is general, it is not easy to find an explicit computation of $g_{b(G)}((x))$. In particular, when $\langle -DL^{-1}(b(G)-\mathbb{E}[b(G)]), DG\rangle_{\mathfrak{H}}$ is not measurable with respect to the σ-field generated by G, there are cases where it is impossible to compute the expectation. Using the above Theorem 2, we derive the explicit form of $g_{b(G)}(x)$. The following theorem corresponds to Theorem 2 in [9].

Theorem 3. *A random variable $G \in \mathbb{D}^{1,2}$, taking its value on I, has the distribution μ and satisfies that $\mathbb{E}[b(G)^2] < \infty$ if and only if $\mathbb{E}[b(G)] = 0$ and*

$$g_{b(G)}(x) = -\frac{1}{2}a(x) \ \textit{for all } x \in I. \tag{46}$$

Proof: Suppose that $\mathbb{E}[b(G)] = 0$ and the equation (46) holds true. Let p_F be a density of an invariant measure μ corresponding to a solution of SDE (11). Then substituting $-\frac{1}{2}a(x)$ in (46) instead of $g_{b(G)}(x)$ in (18) gives that

$$\begin{aligned} p_G(x) &= \frac{p_G(\beta)g_{b(G)}(\beta)}{g_{b(G)})(x)}\exp\left(-\int_{\beta}^{x}\frac{b(y)}{g_{b(G)}(y)}dy\right) \\ &= \frac{p_G(\beta)a(\beta)}{a(x)}\exp\left(\int_{\beta}^{x}\frac{2b(y)}{a(y)}dy\right). \end{aligned} \tag{47}$$

Combining (10) and (47), we get

$$p_G(x) = \frac{p_G(\beta)}{p_F(\beta)} p_F(x). \tag{48}$$

This equation (48) shows that $supp(p_G) = supp(p_F)$ Hence integrating both sides of (48) over $I = (l, u)$ yields that

$$\frac{p_G(\beta)}{p_F(\beta)} = 1,$$

which implies that $p_G = p_F$ on I. If $p_G = p_F$ on I, then $\mathbb{E}[b(G)] = 0$. From (9) and (41), it follows that

$$\begin{aligned} a(x) &= \frac{2\int_l^x b(y)p_F(y)dy}{p_F(x)} \\ &= \frac{2\int_l^x b(y)p_G(y)dy}{p_G(x)} \\ &= -2g_{b(G)}(z), \end{aligned}$$

which gives that (46) holds. □

4. Examples

In this section, two examples will be given where invariant measures have the standard Gaussian and uniform distribution, respectively.

4.1. The standard Gaussian distribution

When μ is the standard Gaussian distribution, then the coefficients in (12) are given by $a(x) = 2$ and $b(x) = -x$, and $u = \infty$ and $l = -\infty$. Then we have, from (20), that

$$\begin{aligned} h_z &= e^{\frac{x^2}{2}} \int_{-\infty}^x [\mathbf{1}_{(-\infty,z]}(y)) - \Phi(z)] e^{-x^2 2} dy \\ &= \begin{cases} \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(x)(1-\Phi(z)) & \text{if } x \le z \\ \sqrt{2\pi} e^{\frac{x^2}{2}} \Phi(z)(1-\Phi(x)) & \text{if } x > z, \end{cases} \end{aligned} \tag{49}$$

where $\Phi(z) = \mathbb{P}(Z \le z)$. We have, from (21), that for $x > z$, taking $\beta = 0$

$$\begin{aligned} \tilde{h}_z'(x) &= \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(z)(1-\Phi(x)) \\ &\quad -\sqrt{2\pi} e^{\frac{x^2}{2}} p_F(x)\Phi(z) \\ &= [\sqrt{2\pi} x e^{\frac{x^2}{2}} (1-\Phi(x)) - 1]\Phi(z), \end{aligned} \tag{50}$$

and for $x < z$,

$$\begin{aligned} \tilde{h}_z'(x) &= \sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(x)[1-\Phi(z)] \\ &\quad +\sqrt{2\pi} e^{\frac{x^2}{2}} p_F(x)[1-\Phi(z)] \\ &= [\sqrt{2\pi} x e^{\frac{x^2}{2}} \Phi(x) + 1][1-\Phi(z)]. \end{aligned} \tag{51}$$

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If $G \in \mathbb{D}^{1,2}$ and the random variable $g_{-G}(G)$ is strictly positive almost surely, then the density p_G of G can be obtained, with $\beta = 0$, by

$$\begin{aligned} p_G(z) &= \frac{g_{-G}(0)p_G(0)}{g_{-G}(z)} \exp\left(\int_0^z \frac{x}{g_{-G}(x)}dx\right) \\ &= \frac{g_G(0)p_G(0)}{g_G(z)} \exp\left(-\int_0^z \frac{x}{g_G(x)}dx\right). \end{aligned} \tag{52}$$

Since $\mathbb{E}[G] = 0$, we get, from (41), that

$$\begin{aligned} g_G(0)p_G(0) &= -g_{-G}(0)p_G(0) \\ &= -\int_l^0 xp_G(x)dx \\ &= \frac{1}{2}\mathbb{E}[|G|]. \end{aligned} \tag{53}$$

Substituting (53) into (52), we have

$$p_G(z) = \frac{\mathbb{E}[|G|]}{2g_G(z)} \exp\left(-\int_0^z \frac{x}{g_G(x)}dx\right),$$

which is the density (18) in Theorem 1. If $g_G = (z) = 1$,

$$\begin{aligned} p_G(z) &= \frac{\mathbb{E}[|G|]}{2} \exp\left(-\int_0^z xdx\right) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \end{aligned}$$

which implies that Theorem 3 holds.

4.2. The Uniform Distribution

When μ is the uniform distribution, i.e., $F \sim \mathcal{U}([0,1])$, then the coefficients in (12) are given by

$$a(x) = x(1-x) \text{ and } b(x) = -\left(x - \frac{1}{2}\right) \text{ for } x \in (0,1).$$

From (20), we have that

$$\begin{aligned} \tilde{h}_z(x) &= \frac{2}{p_F(1/2)a(1/2)} \exp\left(\int_{1/2}^x \frac{(2y-1))}{y(1-y)}dy\right) \\ &\quad \times [(x \wedge z) - zx] \\ &= 8\exp\left(\int_{1/2}^x \frac{(2y-1))}{y(1-y)}dy\right) \times [(x\wedge z) - zx] \\ &= \frac{2x}{1-x} \times \begin{cases} z(1-x) & \text{if } x \geq z \\ x(1-z) & \text{if } x < z, \end{cases}. \end{aligned} \tag{54}$$

Then the density of G is given by

$$p_G(x) = \frac{p_G(\beta) g_{b(G)}(\beta)}{g_{b(G)}(x)} \exp\left(- \int_\beta^x \frac{b(y) - \mathbb{E}[b(G)]}{g_{b(G)}(y)} dy \right). \tag{55}$$

Taking $\beta = \mathbb{E}[G]$, then

$$p_G(x) = \frac{p_G(\mathbb{E}[G]) g_{b(G)}(\mathbb{E}[G])}{g_{b(G)}(x)} \exp\left(- \int_{\mathbb{E}[G]}^x \frac{b(y) - \mathbb{E}[b(G)]}{g_{b(G)}(y)} dy \right). \tag{56}$$

The relation (41) gives that

$$p_G(\mathbb{E}[G]) g_{b(G)}(\mathbb{E}[G]) = -\frac{1}{2}\mathbb{E}[|G - \mathbb{E}[G]|].$$

Hence (56) can be written as

$$\begin{aligned} p_G(x) &= \frac{\mathbb{E}[|G - \mathbb{E}[G]|]}{-2g_{-G}(x)} \exp\left(\int_{\mathbb{E}[G]}^x \frac{y - \mathbb{E}[G]}{g_{-G}(y)} dy \right) \\ &= \frac{\mathbb{E}[|G - \mathbb{E}[G]|]}{2g_G(x)} \exp\left(- \int_{\mathbb{E}[G]}^x \frac{y - \mathbb{E}[G]}{g_G(y)} dy \right). \end{aligned} \tag{57}$$

Putting $\mathbb{E}[G] = 0$, we know, from (57), that the density p_G is identical to the density in Theorem 1. If $g_G(x) = \frac{1}{2}x(1-x)$ for $x \in (0,1)$, a direct computation yields that

$$\begin{aligned} p_G(x) &= \frac{1}{4x(1-x)} \exp\left(- \int_{1/2}^x \frac{y - \frac{1}{2}}{\frac{1}{2}y(1-y)} dy \right) \\ &= \frac{1}{4x(1-x)} \exp\left(- \int_{1/2}^x \left\{ \frac{-2(1-y)}{y(1-y)} + \frac{1}{y(1-y)} \right\} dy \right) \\ &= \frac{1}{4x(1-x)} \exp\left(\log x + \log(1-x) + \log 4 \right) \\ &= \mathbf{1}_{[0,1]}(x), \end{aligned}$$

which implies that Theorem 3 holds true.

5. Conclusions and Future Works

When a random variable F follows the invariant measure μ that has a density p_F, and a random variable $G \in \mathbb{D}^{1,2}$ also allows a density p_G, this paper find an explicit formula of the density p_G based on the coefficients in the diffusion associated with the density p_F. The significant feature of our works is that it shows that the density p_G can be obtained by connecting the diffusion with the invariant measure, and that if $g_{b(G)}$ is equal to the diffusion coefficient, Theorem 2 in [9] can be easily proven.

Future works will be carried out in two directions: (1) Using the results worked in this paper, we plan to derive a density formula associated with an Edgeworth expansion with general terms given in [11]. (2) In the case when G is a random variable belonging to a fixed Wiener chaos, we will obtain a more rigorous formula than the formula obtained in the previous works.

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